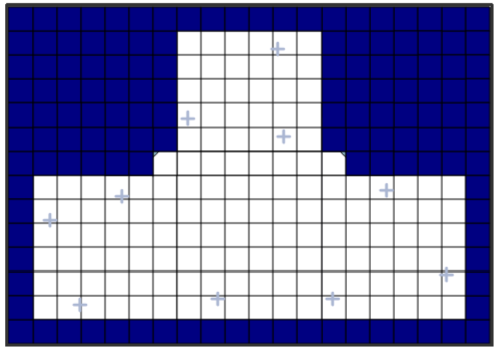
Three Major Map Models

Grid-Based:

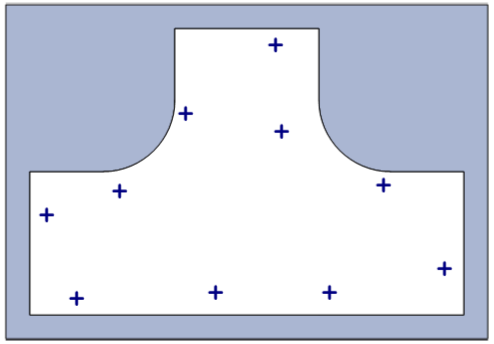
Collection of discretized obstacle/free-space pixels



Elfes, Moravec, Thrun, Burgard, Fox, Simmons, Koenig, Konolige, etc.

Feature-Based:

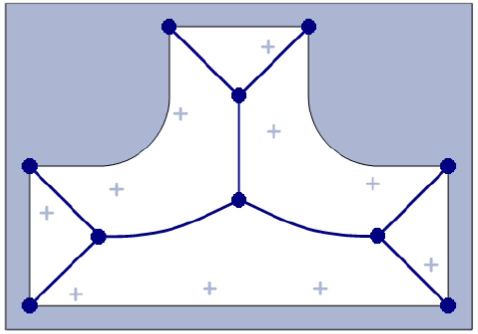
Collection of landmark locations and correlated uncertainty



Smith/Self/Cheeseman, Durrant–Whyte, Leonard, Nebot, Christensen, etc.

Topological:

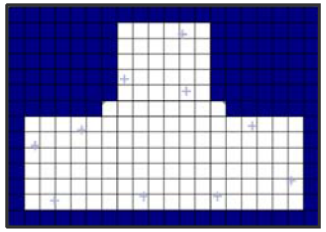
Collection of nodes and their interconnections

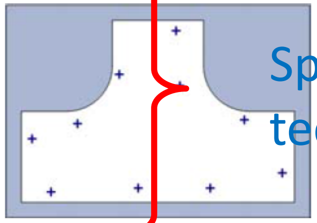


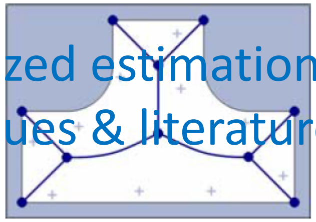
Kuipers/Byun, Chong/Kleeman, Dudek, Choset, Howard, Mataric, etc.

Three Major Map Models

	Grid-Based	Feature-Based	Topological
Resolution vs. Scale	Discrete localization	Arbitrary localization	Localize to nodes
Computational Complexity	Grid size and resolution	Landmark covariance (N^2)	Minimal complexity
Exploration Strategies	Frontier-based exploration	No inherent exploration	Graph exploration





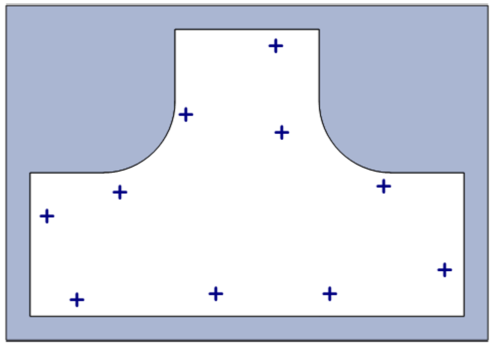


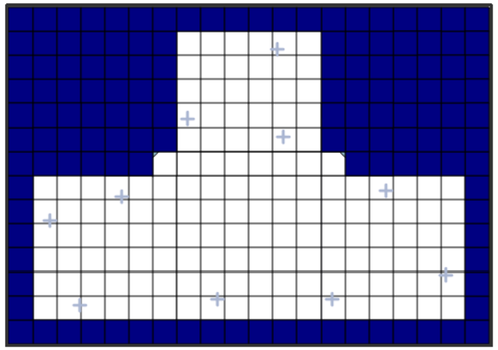
Specialized estimation techniques & literatures

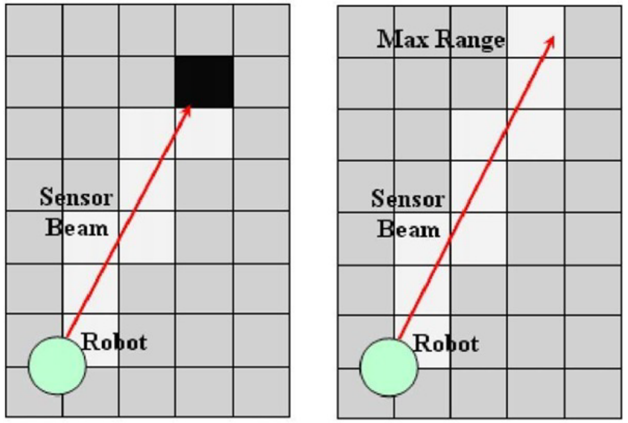
Gmapping

Occupancy Grid: ""map" is a grid of "cells": $\{x^m_{i,j}\}$

- $x^m_{i,j} = 0$ if cell (i,j) is empty; $x^m_{i,j} = 1$ if cell (i,j) is occupied
- $p\left(x^r_{k+1}, \{x^m_{i,j}\}_{k+1} \middle| x^r_{1:k}, \{x^m_{i,j}\}_k, y_{1:k+1}\right)$ (estimate cell occupancy probability)







(a) (b)

Gmapping:

- Uses a *Rao-Blackwellized* particle filter for estimator
- Actually computes $p\left(x^r_{1:T}, \{x^m_{i,j}\} \middle| x^r_{1:k}, x^m_k, y_{1:k+1}\right)$

Axioms of Set-Based Probability

Probability Space:

- Let Ω be a set of experimental outcomes (e.g., roll of dice)

$$\Omega = \{A_1, A_2, \dots, A_N\}$$

  - the A_i are "elementary events" and subsets of Ω are termed "events"
  - Empty set $\{\emptyset\}$ is the "impossible event"
  - $S=\{\Omega\}$ is the "certain event"
- A probability space $(\Omega, \mathrm{F}, \mathrm{P})$
  - F = set of subsets of Ω, or "events", P assigns probabilities to events

Probability of an Event—the Key Axioms:

- Assign to each A_i a number, P(A_i), termed the "probability" of event A_i
- P(A_i) must satisfy these axioms
  1. P(A_i) ≥ 0
  2. P(S) = 1
  3. If events $A,B \in \Omega$ are "mutually exclusive," or disjoint, elements or events ($A \cap B = \{\emptyset\}$), then

$$\mathrm{P}(A \cup B) = P(A) + P(B)$$

Axioms of Set-Based Probability

As a result of these three axioms and basic set operations (e.g., DeMorgan's laws, such as $\overline{A \cup B} = \overline{A} \cap \overline{B}$)

- P({∅})=0
- P(A) = 1-P($\overline{A}$) $\Rightarrow$ $\mathrm{P(A)} + \mathrm{P}(\overline{A}) = 1$, where $\overline{A}$ is complement of A
- If $A_1, A_2, \ldots, A_N$ mutually disjoint

$$\mathrm{P}(A_1 \cup A_1 \cup \cdots \cup A_N) = P(A_1) + P(A_1) + \cdots + P(A_N)$$

For Ω an infinite, but countable, set we add the "Axiom of infinite additivity"

3(b). If $A_1, A_2, \ldots$ are mutually exclusive,

$$\mathrm{P}(A_1 \cup A_1 \cup \cdots) = P(A_1) + P(A_1) + \cdots$$

We assume that all countable sets of events satisfy Axioms 1, 2, 3, 3(b)

But we need to model uncountable sets…

Continuous Random Variables (CRVs)

Let $\Omega = \mathbb{R}$ (an uncountable set of events)

- *Problem:* it is not possible to assign probabilities to subsets of $\mathbb{R}$ which satisfy the above Axioms

- *Solution:*
  - let "events" be intervals of $\mathbb{R}$: $A = \{x \mid x_l \leq x \leq x_u\}$, and their ***countable*** unions and intersections.
  - Assign probabilities to these events

    $$\mathrm{P}(x_l \leq x \leq x_u) = \textit{Probability that } x \textit{ takes values in } [x_l, x_u]$$

  - x is a *"continuous random variable* (CRV).

Some basic properties of CRVs

- If x is a CRV in $[L, U]$, then $\mathrm{P}(L \leq x \leq U) = 1$
- If y in $[L, U]$, then $\mathrm{P}(L \leq y \leq x) = 1 - \mathrm{P}(y \leq x \leq U)$

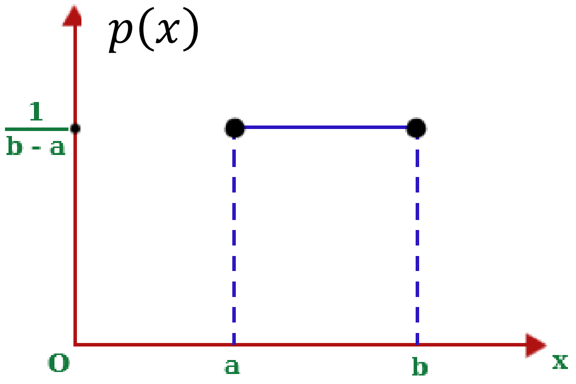
Probability Density Function (pdf)

$$p(x_l \leq x \leq x_u) \equiv \int_{x_l}^{x_u} p(x)dx$$

E.g.

– *Uniform Probability pdf:*

$$p(x) = \frac{1}{b-a}$$



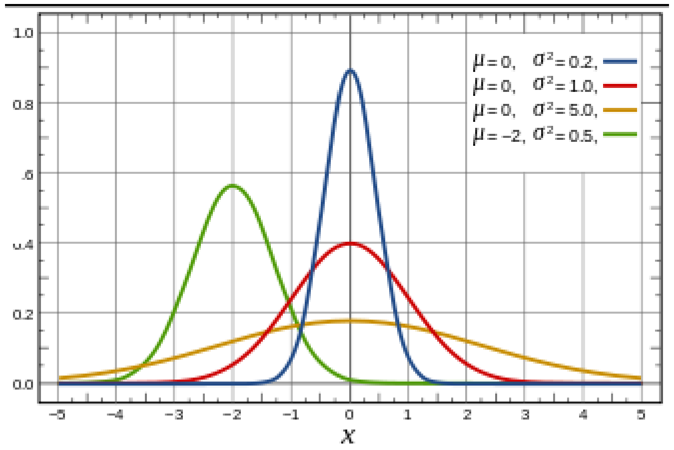
– *Gaussian (Normal) pdf:*

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ =*"mean" of pdf*

σ = *"standard deviation"*

$p(x)$



Most of our Estimation theory will be built on the Gaussian distribution

Expectation

Expectation: (key for estimation)

– Let x be a CRV with distribution $p(x)$. The expected value (or mean) of x is

$$E[x] = \int_{-\infty}^{\infty} x p(x) dx \qquad E[g(x)] = \int_{-\infty}^{\infty} g(x) p(x) dx$$

Mean Square: $$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x) dx$$

Variance: $$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx \qquad \mu(x) = E[x]$$

Random Processes (continued)

A stochastic system whose state is characterized by a time evolving CRV, x(t), t ε [0,T].

- At each t, x(t) is a CRV
- x(t) is the "state" of the random process, which can be characterized by

$$\text{P}[x_l \le x(t) \le x_u] = \int_{-\infty}^{\infty} p(x,t)dx$$

Random Processes can also be characterized by:

- Joint probability function

Joint probability density function

$$\text{P}[x_{1l} \le x(t_1) \le x_{1u}; x_{2l} \le x(t_2) \le x_{2u}] = \int_{x_{1l}}^{x_{1u}} \int_{x_{2l}}^{x_{2u}} \underbrace{p(x_1, x_2, t_1, t_2)}_{\text{Joint probability density function}} \, dx_1 dx_2$$

- A random process x(t) is ***Stationary*** if $p(x,t+\tau)=p(x,t)$ for all τ

- Correlation Function

Correlation function

$$E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} x_1 \, x_2 \, p(x_1, x_2, t_1, t_2) \, dx_1 dx_2 \equiv \underbrace{\rho(t_1, t_2)}_{\text{Correlation function}}$$

Joint and Conditional Probability

- $P(X = x \; and \; Y = y) \; = \; P(x, y)$

- If X and Y are independent then

$$P(x, y) \; = \; P(x) \, P(y)$$

- $P(x \mid y)$ is the probability of x given y

$$P(x \mid y) \; = \; P(x, y) \, / \, P(y)$$
$$P(x, y) \; = \; P(x \mid y) \, P(y)$$

- If X and Y are independent then

$$P(x \mid y) \; = \; P(x)$$

Conditional independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

Equivalent to

- $P(x|z) = P(x|z, y)$
- $P(y|z) = P(y|z, x)$

Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

$$P(x|y) = \sum p(x|y,z) p(z|y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

$$p(x|y) = \int p(x|y,z) p(z|y) dz$$

Bayes Formula

$$P(x, y) = P(x \mid y) P(y) = P(y \mid x) P(x)$$

$$\Rightarrow$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

Normalization

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

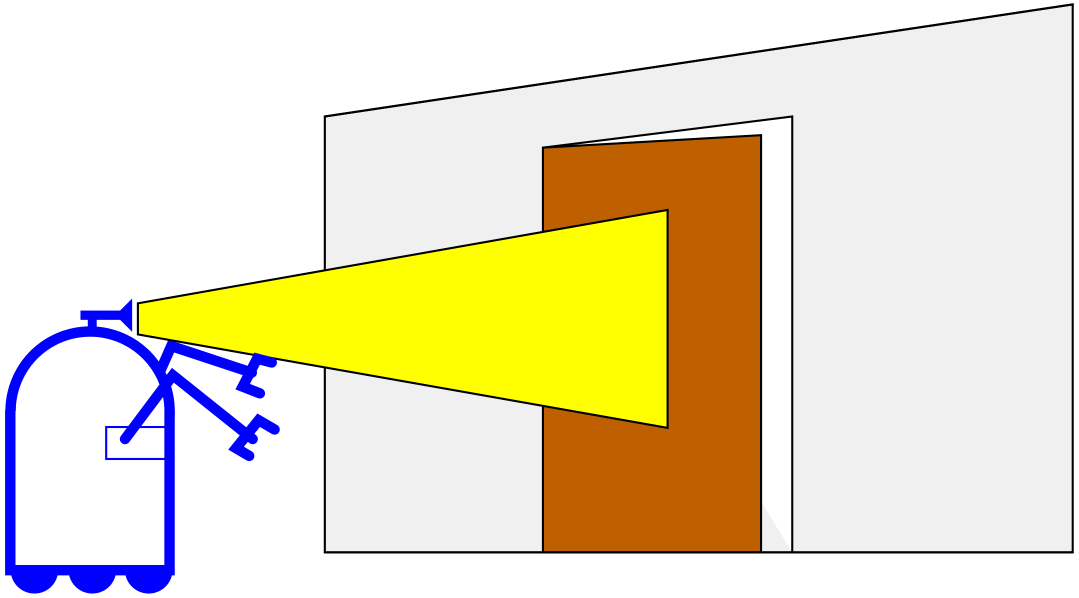
$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Simple Example

- Suppose robot measures *z*
- What is *P(open|z)?*



- *P(open|z)* is diagnostic.
- *P(z|open)* is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:
  - Causal knowledge can come from a *frequentist* approach
  - Causal knowledge can come from a model.
- $P(open|z) = \frac{P(z|open)P(open)}{P(z)}$