Three Major Map Models

**Grid-Based:**
Collection of discretized obstacle/free-space pixels

Elfes, Moravec, Thrun, Burgard, Fox, Simmons, Koenig, Konolige, etc.

**Feature-Based:**
Collection of landmark locations and correlated uncertainty

Smith/Self/Cheeseman, Durrant–Whyte, Leonard, Nebot, Christensen, etc.

**Topological:**
Collection of nodes and their interconnections

Kuipers/Byun, Chong/Kleeman, Dudek, Choset, Howard, Mataric, etc.
# Three Major Map Models

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Specialized estimation techniques & literatures
Gmapping

**Occupancy Grid:** “map” is a grid of “cells”: \{x_{i,j}^m\}

- \(x_{i,j}^m = 0\) if cell (i,j) is empty; \(x_{i,j}^m = 1\) if cell (i,j) is occupied

- \(p \left(x_{k+1}^r, \{x_{i,j}^m\}_{k+1} \mid x_{1:k}^r, \{x_{i,j}^m\}_k, y_{1:k+1}\right)\) (estimate cell occupancy probability)

Gmapping:
- Uses a *Rao-Blackwellized* particle filter for estimator
- Actually computes \(p \left(x_{1:T}^r, \{x_{i,j}^m\} \mid x_{1:k}^r, x_k^m, y_{1:k+1}\right)\)
Axioms of Set-Based Probability

Probability Space:
- Let $\Omega$ be a set of experimental outcomes (e.g., roll of dice)
  $$\Omega = \{A_1, A_2, \ldots, A_N\}$$
  - the $A_i$ are “elementary events” and subsets of $\Omega$ are termed “events”
  - Empty set $\emptyset$ is the “impossible event”
  - $S=\{\Omega\}$ is the “certain event”
- A probability space $(\Omega, F, P)$
  - $F$ = set of subsets of $\Omega$, or “events”, $P$ assigns probabilities to events

Probability of an Event—the Key Axioms:
- Assign to each $A_i$ a number, $P(A_i)$, termed the “probability” of event $A_i$
- $P(A_i)$ must satisfy these axioms
  1. $P(A_i) \geq 0$
  2. $P(S) = 1$
  3. If events $A, B \in \Omega$ are “mutually exclusive,” or disjoint, elements or events $(A \cap B = \emptyset)$, then
     $$P(A \cup B) = P(A) + P(B)$$
Axioms of Set-Based Probability

As a result of these three axioms and basic set operations (e.g., DeMorgan’s laws, such as $A \cup B = \overline{A \cap B}$)

- $P(\emptyset) = 0$
- $P(A) = 1 - P(\overline{A}) \Rightarrow P(A) + P(\overline{A}) = 1$, where $\overline{A}$ is complement of $A$
- If $A_1, A_2, \ldots, A_N$ mutually disjoint
  \[ P(A_1 \cup A_1 \cup \cdots \cup A_N) = P(A_1) + P(A_1) + \cdots + P(A_N) \]

For $\Omega$ an infinite, but countable, set we add the “Axiom of infinite additivity”

3(b). If $A_1, A_2, \ldots$ are mutually exclusive,

\[ P(A_1 \cup A_1 \cup \cdots) = P(A_1) + P(A_1) + \cdots \]

We assume that all countable sets of events satisfy Axioms 1, 2, 3, 3(b)

But we need to model uncountable sets…
Continuous Random Variables (CRVs)

Let $\Omega = \mathbb{R}$ (an uncountable set of events)

- **Problem:** it is not possible to assign probabilities to subsets of $\mathbb{R}$ which satisfy the above Axioms

- **Solution:**
  - let “events” be intervals of $\mathbb{R}$: $A = \{x_{\ell} \leq x \leq x_u\}$, and their countable unions and intersections.
  - Assign probabilities to these events

$$P(x_{\ell} \leq x \leq x_u) = \text{Probability that } x \text{ takes values in } [x_{\ell}, x_u]$$

- $x$ is a “continuous random variable (CRV).

Some basic properties of CRVs
- If $x$ is a CRV in $[L, U]$, then $P(L \leq x \leq U) = 1$
- If $y$ in $[L, U]$, then $P(L \leq y \leq x) = 1 - P(y \leq x \leq U)$
Probability Density Function (pdf)

E.g.

- **Uniform Probability pdf:**
  \[ p(x) = \frac{1}{b-a} \]

- **Gaussian (Normal) pdf:**
  \[ p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \]
  
  \( \mu \) = “mean” of pdf
  
  \( \sigma \) = “standard deviation”

Most of our Estimation theory will be built on the Gaussian distribution
Expectation

Expectation: (key for estimation)

- Let $x$ be a CRV with distribution $p(x)$. The expected value (or mean) of $x$ is

\[ E[x] = \int_{-\infty}^{\infty} xp(x)dx \quad E[g(x)] = \int_{-\infty}^{\infty} g(x)p(x)dx \]

Mean Square:

\[ E[x^2] = \int_{-\infty}^{\infty} x^2p(x)dx \]

Variance:

\[ \sigma^2 = E[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2p(x)dx \quad \mu(x) = E[x] \]
A stochastic system whose state is characterized by a time evolving CRV, $x(t)$, $t \in [0,T]$.

- At each $t$, $x(t)$ is a CRV.
- $x(t)$ is the “state” of the random process, which can be characterized by

$$P[x_l \leq x(t) \leq x_u] = \int_{x_l}^{x_u} p(x,t)dx$$

Random Processes can also be characterized by:

- Joint probability function

$$P[x_{1l} \leq x(t_1) \leq x_{1u}; x_{2l} \leq x(t_2) \leq x_{2u}] = \int_{x_{1l}}^{x_{1u}} \int_{x_{2l}}^{x_{2u}} p(x_1,x_2,t_1,t_2) dx_1 dx_2$$

- A random process $x(t)$ is **Stationary** if $p(x,t+\tau)=p(x,t)$ for all $\tau$

- Correlation Function

$$E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} x_1 x_2 p(x_1,x_2,t_1,t_2) dx_1 dx_2 \equiv \rho(t_1,t_2)$$
Joint and Conditional Probability

• \( P(X = x \text{ and } Y = y) = P(x, y) \)

• If \( X \) and \( Y \) are independent then
  \[ P(x, y) = P(x) \cdot P(y) \]

• \( P(x \mid y) \) is the probability of \( x \) given \( y \)
  \[ P(x \mid y) = \frac{P(x, y)}{P(y)} \]
  \[ P(x, y) = P(x \mid y) \cdot P(y) \]

• If \( X \) and \( Y \) are independent then
  \[ P(x \mid y) = P(x) \]

Conditional independence

\[ P(x, y \mid z) = P(x \mid z) \cdot P(y \mid z) \]

Equivalent to

• \( P(x \mid z) = P(x \mid z, y) \)
• \( P(y \mid z) = P(y \mid z, x) \)
Law of Total Probability, Marginals

**Discrete case**

\[
\sum_x P(x) = 1
\]

\[
P(x) = \sum_y P(x, y)
\]

\[
P(x) = \sum_y P(x \mid y)P(y)
\]

\[
P(x \mid y) = \sum_z p(x \mid y, z)p(z \mid y)
\]

**Continuous case**

\[
\int p(x) \, dx = 1
\]

\[
p(x) = \int p(x, y) \, dy
\]

\[
p(x) = \int p(x \mid y)p(y) \, dy
\]

\[
p(x \mid y) = \int p(x \mid y, z)p(z \mid y) \, dz
\]
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ \Rightarrow \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]

Normalization

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \eta P(y \mid x)P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_{x} P(y \mid x)P(x)} \]

Bayes Rule with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z)P(x \mid z)}{P(y \mid z)} \]
Simple Example

- Suppose robot measures $z$
- What is $P(\text{open}|z)$?

- $P(\text{open}|z)$ is diagnostic.
- $P(z|\text{open})$ is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:
  - Causal knowledge can come from a frequentist approach
  - Causal knowledge can come from a model.

$$P(\text{open}|z) = \frac{P(z|\text{open})P(\text{open})}{P(z)}$$