CDS 101/110: Lecture 5.1
Integral State Feedback

October 24, 2016

Goals:

- Brief discussion about integral feedback in state feedback

Reading:

- Åström and Murray, Feedback Systems-2e, Section 7.4
- For Wednesday: start reading FBS-2e, Sections 8.1-8.2.

State space controller design for linear systems

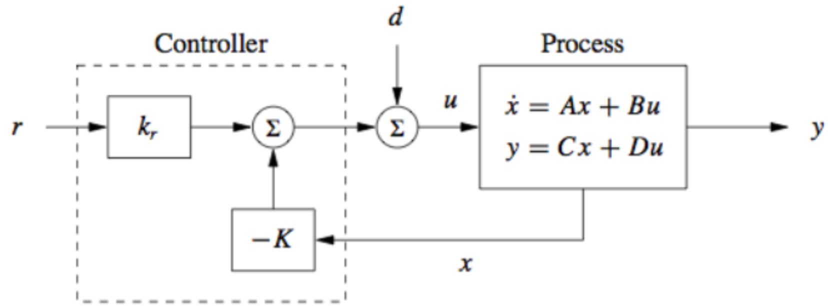
$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Goal: find a linear control law $u = -Kx + k_r r$ such that the closed loop system

$$\dot{x} = Ax + Bu = (A - BK)x + Bk_r r$$

is stable at equilibrium point x_e with $y_e = r$.



Theorem: If (A,B) is reachable, the eigenvalues of (A-BK) can be arbitrarily placed.

If the desired reference is constant at $r = 0$, then the system is a ***regulator***, and we need only choose K to stabilize. Else, we need to determine k_r.

- If we want *zero reference error* at *zero frequency,* for $r \neq 0$, then:
- $\dot{x} = (A - BK)x + Bk_r r, y = Cx \quad \rightarrow \quad (A - BK)x_e + Bk_r r = 0,\ y_e = Cx_e$
- $\therefore y_e = -C(A - BK)^{-1}Bk_r r$
- If we want $y_e = r$, then $\quad k_r = -\left[C(A - BK)^{-1}B\right]^{-1}$

Integral Feedback

Motivation:

- *Accurate* models (i.e., precise models for A,B,C matrices) are needed to ensure zero reference errors.
- An *Integral feedback* term can be used compensate for uncertainty in these terms.

Approach:

- Define a new state z such that $z = \int (y - r)\, dt$
- Require $z \to 0$, which in turn will ensure that $y \to r$ over time.
- Details:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Cx - r \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

$$\equiv \tilde{A}x + \tilde{B}_1 u + \tilde{B}_2 r$$

- If we find a feedback that stabilizes this system, then $\dot{z} \to 0$, and $y \to 0$
- Choose Feedback law: $u = -Kx - k_i z + k_r r = -K_i q + k_r r$
  - K_i is state feedback on the extended state $q = [x \quad z]^T$, which includes integral term

Integral Feedback

Analysis:

- Closed loop system looks like:
  - $\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A - BK & -Bk_i \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} Bk_r \\ -1 \end{bmatrix} r$
  - Where k_i is the “integral gain,” and gain k_r will be chosen as above:

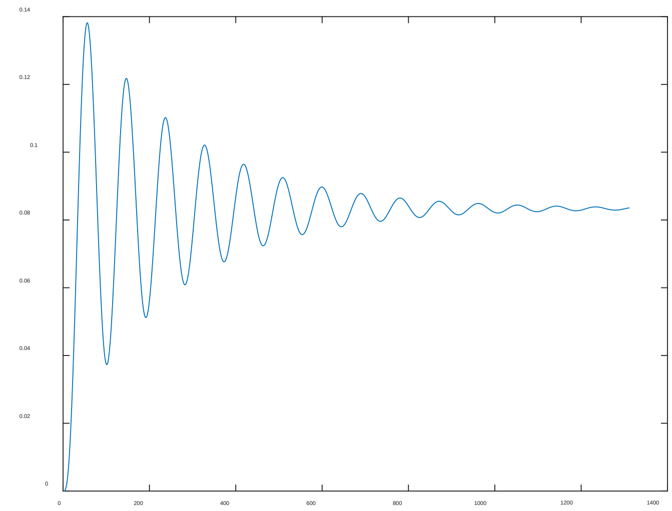
    $$k_r = -\left[C(A - BK)^{-1}B\right]^{-1}$$

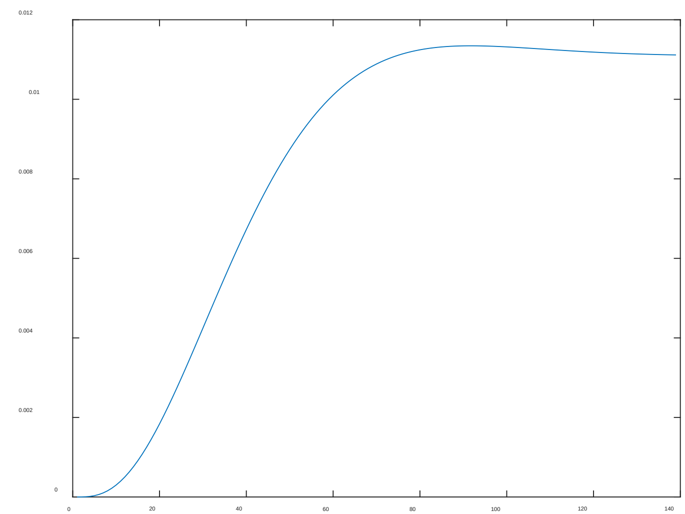
Integral Feedback Example

Uncontrolled System:

- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & -3 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

  $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad D = [0]$





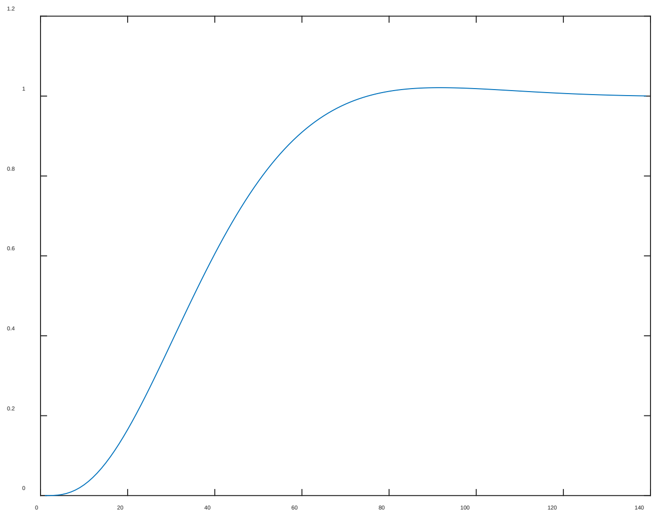
State Feedback:

- Let’s use state feedback ($u = -Kr$), and place the poles at $[-5, -3 \pm 3i]$. Using the MATLAB *place* function yields
  - $K = \begin{bmatrix} 78 & 43 & 8 \end{bmatrix}$
- Compare the unit step response of the open loop and closed loop systems. The steady state values is far from the desired value of 1

Integral Feedback Example

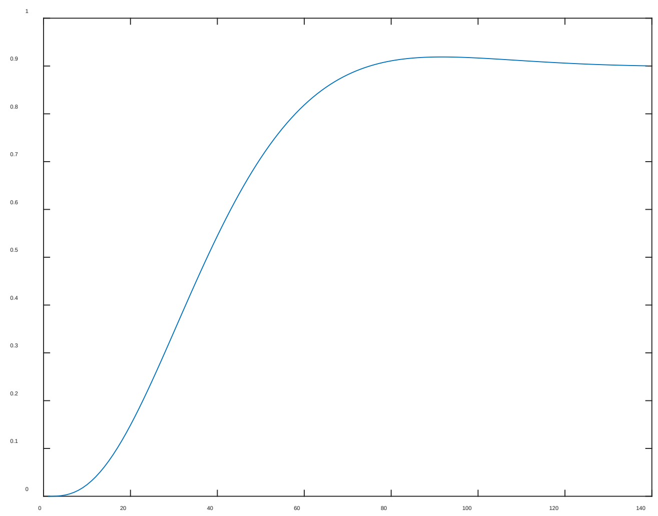
Reference Input :

- Now use $u = -Kr + k_r r$, with k_r chosen assuming perfect knowledge of (A,B,C,D)
  - $k_r = -\left[C(A - BK)^{-1}B\right]^{-1} = 90$
  - Note that step response now converges to unity gain



Model Mismatch :

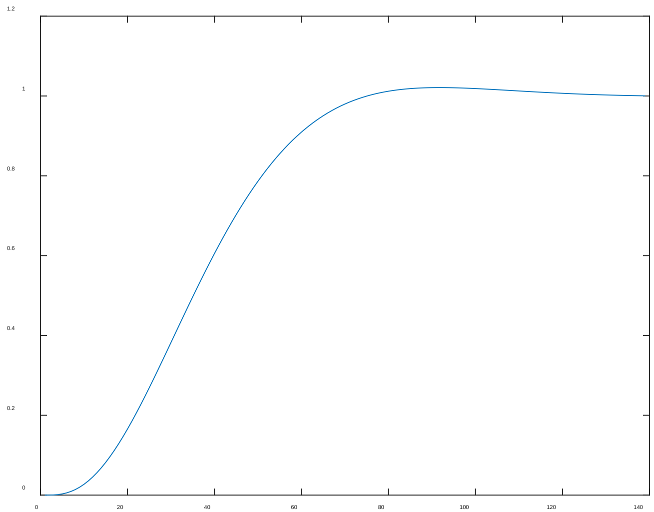
- Now perturb the matrices (A,B,C) by 10% (multiply them all by 0.9).
- Use feedback gain K and reference gain k_r which were computed for the unperturbed system matrices.
  - Step response now converges to about 10% steady state error.



Integral Feedback Example

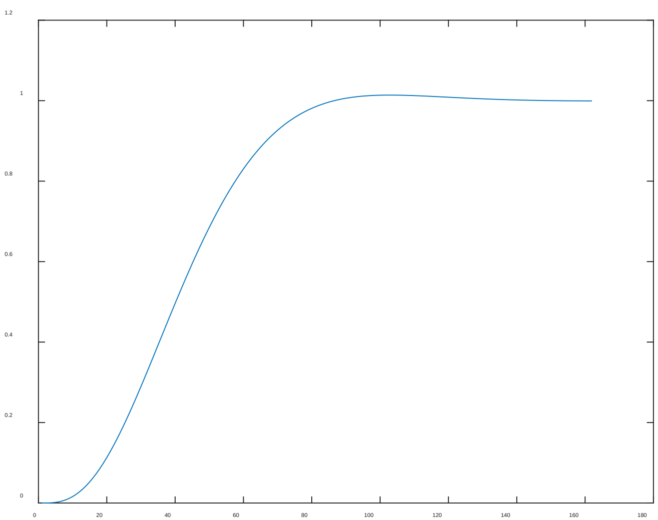
Add Integral State Feedback:

- Introduce new state $\dot{z} = y - r$;
  - Set the pole associated with integral feedback to -6.
  - Note that step response converges to unity gain for perfect model knowledge



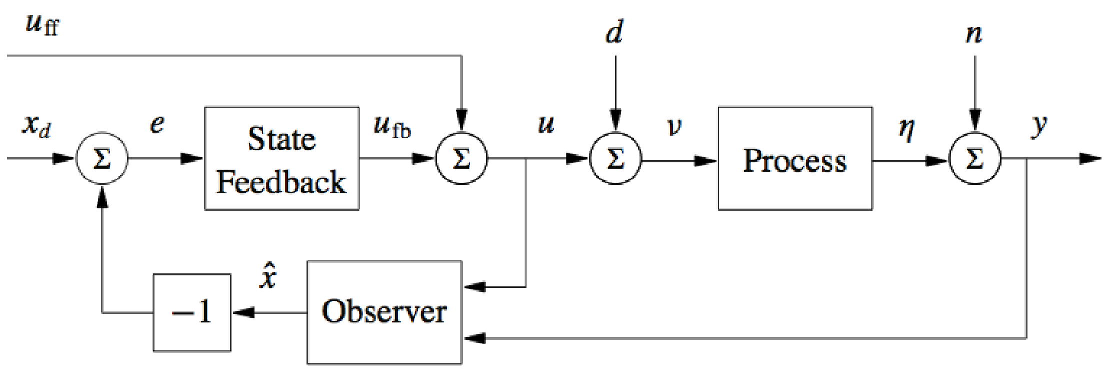
Integrate Feedback & Model Mismatch:

- Perturb the matrices (A,B,C) by 10% as above.
- Use feedback gain K_i and reference gain k_r of the unperturbed system matrices.
  - Step response converges to zero steady state error
  - But note (by looking at time scale difference in the plots) that response is slightly delayed.



Observers: First Look

Use *observer* to determine the current state if you can't measure it



- Estimator looks at inputs and outputs of plant and estimates the current state
- Can show that if a system is *observable* then you can construct and estimator
- Use the *estimated* state as the feedback

$$u = K\hat{x}$$

- Next week: basic theory of state estimation and observability
- CDS 112: *Kalman filtering* and theory of optimal observers