CDS 101/110: Lecture 5.1
Integral State Feedback

October 24, 2016

Goals:
• Brief discussion about integral feedback in state feedback

Reading:
• Åström and Murray, Feedback Systems-2e, Section 7.4
• For Wednesday: start reading FBS-2e, Sections 8.1-8.2.
State space controller design for linear systems

\[
\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, \ x(0) \text{ given} \\
y = Cx \quad u \in \mathbb{R}, \ y \in \mathbb{R}
\]

**Goal:** find a linear control law \( u = -Kx + k_r r \) such that the closed loop system

\[
\dot{x} = Ax + Bu = (A - BK)x + Bk_r r
\]

is stable at equilibrium point \( x_e \) with \( y_e = r \).

**Theorem:** If \((A,B)\) is reachable, the eigenvalues of \((A-BK)\) can be arbitrarily placed.

If the desired reference is constant at \( r = 0 \), then the system is a **regulator**, and we need only choose \( K \) to stabilize. Else, we need to determine \( k_r \).

- If we want zero reference error at zero frequency, for \( r \neq 0 \), then:
  - \[
  \dot{x} = (A - BK)x + Bk_r r, \ y = Cx \quad \rightarrow \quad (A - BK)x_e + Bk_r r = 0, \ y_e = Cx_e
  \]
  - \[
  \therefore \ y_e = -C(A - BK)^{-1}Bk_r r
  \]
- If we want \( y_e = r \), then \[
  k_r = -[C(A - BK)^{-1}B]^{-1}
  \]
**Integral Feedback**

**Motivation:**

- *Accurate* models (i.e., precise models for A,B,C matrices) are needed to ensure zero reference errors.
- An *Integral feedback* term can be used to compensate for uncertainty in these terms.

**Approach:**

- Define a new state $z$ such that $z = \int (y - r) \, dt$
- Require $z \to 0$, which in turn will ensure that $y \to r$ over time.
- Details:

$$
\frac{d}{dt} [x] = \begin{bmatrix} Ax + Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Cx - r \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} [x] + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r
$$

$$
\equiv \ddot{A}x + \dddot{B}_1u + \dddot{B}_2r
$$

- If we find a feedback that stabilizes this system, then $\dot{z} \to 0$, and $y \to 0$
- Choose Feedback law: $u = -Kx - k_i z + k_r r = -K_i q + k_r r$
  
  - $K_i$ is state feedback on the extended state $q = [x ~ z]^T$, which includes integral term
Integral Feedback

Analysis:

- Closed loop system looks like:

  \[
  \frac{d}{dt} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A - BK & -B k_i \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \cdot k_r & -1 \end{bmatrix} r
  \]

  - Where \( k_i \) is the “integral gain,” and gain \( k_r \) will be chosen as above:

    \[
    k_r = -[C (A - BK)^{-1} B]^{-1}
    \]
Integral Feedback Example

Uncontrolled System:

- \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & -3 \end{bmatrix} \)
- \( B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)
- \( C = [1 \ 0 \ 0] \)
- \( D = [0] \)

State Feedback:

- Let's use state feedback \((u = -Kr)\), and place the poles at \([-5, -3 \pm 3i]\). Using the MATLAB \textit{place} function yields
  - \( K = [78 \ 43 \ 8] \)
- Compare the unit step response of the open loop and closed loop systems. The steady state values is far from the desired value of 1.
Integral Feedback Example

Reference Input:

- Now use $u = -K r + k_r r$, with $k_r$ chosen assuming perfect knowledge of (A,B,C,D)
  
  \[- k_r = -[C(A - BK)^{-1}B]^{-1} = 90\]
  
  - Note that step response now converges to unity gain

Model Mismatch:

- Now perturb the matrices (A,B,C) by 10% (multiply them all by 0.9).

- Use feedback gain $K$ and reference gain $k_r$ which were computed for the unperturbed system matrices.
  
  - Step response now converges to about 10% steady state error.
Add Integral State Feedback:

- Introduce new state \( \dot{z} = y - r; \)
  - Set the pole associated with integral feedback to -6.
  - Note that step response converges to unity gain for perfect model knowledge.

Integrate Feedback & Model Mismatch:

- Perturb the matrices (A,B,C) by 10% as above.
- Use feedback gain \( K_i \) and reference gain \( k_r \) of the unperturbed system matrices.
  - Step response converges to zero steady state error.
  - But note (by looking at time scale difference in the plots) that response is slightly delayed.
Observers: First Look

Use *observer* to determine the current state if you can’t measure it.

- Estimator looks at inputs and outputs of plant and estimates the current state.
- Can show that if a system is *observable* then you can construct and estimator.
- Use the *estimated* state as the feedback.

\[ u = K\hat{x} \]

- Next week: basic theory of state estimation and observability.
- CDS 112: *Kalman filtering* and theory of optimal observers.