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Koopman Operator Based Observer Synthesis for Control-Affine Nonlinear Systems

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Abstract— **We propose a new observer form based on Koopman operator theoretic framework for input-output nonlinear systems with control affine inputs. Based on this observer form, we describe an observer synthesis framework which exploits estimation techniques developed for Lipschitz systems and bilinear systems. We also formulate nonlinear observability rank condition in terms of the Koopman observer form, and numerically illustrate the benefits of the proposed framework.**

I. Introduction

Nonlinear state observer design has been an area of constant research for several decades and, despite important progress, many outstanding practical problems still remain unsolved [1], [2], [3]. Extended Kalman filter (EKF) continues to be one of most widely used practical nonlinear estimation approaches, but unfortunately has no general guarantees on convergence due to its reliance on linearization. Geometric observer techniques [1], [4], [5], [6] seek a coordinate transformation so that the state estimation error dynamics are linear in the new coordinates. Necessary and sufficient conditions for existence of such transformation have been established but in practice are extremely difficult to satisfy. There has also been work to propose observers for more specialized classes of nonlinear systems, such as Lipschitz nonlinear systems and bilinear systems. For instance, Lyapunov and Linear Matrix Inequalities based approaches have been developed for Lipschitz nonlinear systems [7], [8], [9], [10], [11], while Kalman type filters can be used for bilinear systems, see [12] and references therein. These observer design approaches are only applicable if the output is a linear function of the state, precluding its applicability for systems with nonlinear outputs. Some techniques have been proposed (see [13], [12] and references there in) which transforms output nonlinearities of a bilinear system into a linear form by expanding the state space. Along similar lines Carleman linearization approach [14], [15] provides a framework to transform a nonlinear system with control affine terms approximately into a higher dimensional bilinear form, however requires analyticity of the underlying system. Other approaches for exact/approximate bilinearization of arbitrary nonlinear systems has also been extensively studied [16], [12]; however these approaches suffer from similar limitations as the geometric observer techniques. The purpose of this paper is to explore a Koopman operator theoretic framework for observer synthesis which has the promise of alleviating some of the technical challenges stated above, and provide a rigourous yet practical approach for nonlinear estimation.

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Koopman operator is a linear but infinite-dimensional operator that governs the time evolution of observables or outputs defined on the state space of a dynamical system [17], [18]. Spectral properties of Koopman operator provide powerful means of analysis and decomposition of nonlinear dynamical systems. Recent theoretical and computational advances associated with Koopman operator are enabling new techniques for more effectively dealing with high dimensional complex nonlinear systems. The majority of the applications [19], [20], [17], [21], [22], [23] exploit Koopman framework for data driven nonlinear model reduction or spectral analysis. More recently [24], [25] formulated global stability property of fixed points and limit cycles in terms of specific eigenfunctions of the Koopman operator, and developed a numerical procedure to estimate the associated basin of attraction. Along similar lines, Perrron-Frobeniuos operator (which is adjoint of Koopman operator) based approaches have been used for nonlinear stability and observability analysis [26], [27]. In [28], authors developed a Koopman based linear observer design framework for discrete time nonlinear systems with no inputs. This framework uses specific Koopman eigenfunctions whose span contains the state and output function to construct an immersion which transforms the nonlinear system into a linear observer form for which one can employ standard Luenberger/Kalman based linear observers. The majority of the operator theoretic work, for example as referenced above has been restricted to an autonomous setting, i.e. systems with no inputs. An exception is a recent publication [29], which explores Koopman invariant subspaces to facilitate design of optimal control of nonlinear systems with full state feedback.

In this paper we demonstrate how Koopman framework can be used for immersion based observer design for input-output nonlinear systems. Motivated by the formulation in [28] for discrete time systems with no inputs, we develop a Koopman Observer Form (KOF) in continuous time setting which incorporates the input terms. We give conditions under which the KOF is finite dimensional. While the KOF looses the desirable linear structure (as with no inputs), we demonstrate that it can still facilitate the application of existing special observer design techniques for Lipschitz systems and bilinear systems in context of more general nonlinear systems. We demonstrate the benefits of the proposed framework numerically. We also formulate nonlinear observability rank conditions in terms of the KOF.

The paper is organized into seven sections. We start with mathematical preliminaries in Section II, followed by a review of Koopman operator theoretic framework in Section III. The KOF is introduced in Section IV, and observer design framework based on it is discussed in Section V. In Section VI we formulate nonlinear observability rank condition in terms of the KOF. Numerical example is presented in Section VIII, and paper is concluded in Section IX with directions for future research.

II. Mathematical Preliminaries

In this paper we consider input/output nonlinear systems with control affine terms

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{l} \mathbf{g}_i(\mathbf{x}) u_i, \quad (1)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}), \quad (2)$$

where, $\mathbf{x} \in \mathbf{X} \subset \mathbb{R}^d$ is state vector, $\mathbf{u} = (u_1, \cdots, u_l)^* \in \mathbf{U} \subset \mathbb{R}^l$ are the control inputs, $\mathbf{f} : \mathbf{X} \to \mathbb{R}^d$ is a vector field, $\mathbf{g}_i : \mathbf{X} \to \mathbb{R}^d, i = 1, \cdots, l$ are state dependent control coupling terms, and $\mathbf{h} : \mathbf{X} \to \mathbb{R}^m$ are the outputs. Throughout we will consider standard inner product $< \cdot, \cdot >$ on the Euclidean space. For any complex number $c \in \mathbb{C}$, we shall denote by $\mathrm{Re}(c)$, $\mathrm{Im}(c)$, $|c|$ and $\arg(c)$ as the real part, imaginary part, modulus and argument, respectively. The transpose of a vector or matrix will be represented by superscript a *.

Let $C^1(\mathbf{X})$ be the vector space of continuously differentiable scalar complex valued functions $r : \mathbf{X} \to \mathbb{C}$ on $\mathbf{X}$. The gradient of a function $r \in C^1(\mathbf{X})$ will be represented as a row vector $dr = (\frac{\partial r}{\partial x_1}, \cdots, \frac{\partial r}{\partial x_d})$, while Jacobian of any vector valued function $\mathbf{r} : \mathbf{X} \to \mathbb{C}^p$ with components $\mathbf{r} = (r_1, \cdots, r_p)^*, r_i \in C^1(\mathbf{X})$ will be denoted by,

$$\frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \begin{pmatrix} dr_1 \\ \vdots \\ dr_p \end{pmatrix} = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_d} \\ \vdots & \vdots & \vdots \\ \frac{\partial r_p}{\partial x_1} & \cdots & \frac{\partial r_p}{\partial x_d} \end{pmatrix}. \quad (3)$$

Let $\chi(\mathbf{X})$ be a set of all real valued vector fields $\mathbf{v} : \mathbf{X} \to \mathbb{R}^d$ on $\mathbf{X}$. Elements of $\chi(\mathbf{X})$ act as linear operators on $C^1(\mathbf{X})$ by Lie differentiation, s.t. for any $r \in C^1(\mathbf{X})$ and $\mathbf{v} \in \chi(\mathbf{X})$,

$$\mathcal{L}_{\mathbf{v}} r = < dr, \mathbf{v} > . \quad (4)$$

This definition extends naturally to vector valued functions, as follows

$$\mathcal{L}_{\mathbf{v}} \mathbf{r} = \begin{pmatrix} \mathcal{L}_{\mathbf{v}} r_1 \\ \vdots \\ \mathcal{L}_{\mathbf{v}} r_p \end{pmatrix} = \begin{pmatrix} < dr_1, \mathbf{v} > \\ \vdots \\ < dr_p, \mathbf{v} > \end{pmatrix}, \quad (5)$$

where, $\mathbf{r} = (r_1, \cdots, r_p)^*$. The fact that we work on subsets of Euclidean space is for simplicity, and the whole framework discussed in this paper can easily be extended to systems with phase spaces on manifolds.

III. Koopman Operator Theoretic Framework

In this section we review Koopman operator theory, details can be found in [17], [18]. Consider an autonomous dynamical system described by an ODE,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (6)$$

where, $\mathbf{x} \in \mathbf{X} \subset \mathbb{R}^d$ as above. Let $\Phi(t, \mathbf{x}_0)$ be the flow map i.e. solution of above ODE stating at the initial condition $\mathbf{x}_0$. Let $\mathcal{F}$ be a space of complex valued scalar functions $\psi : \mathbf{X} \to \mathbb{C}$, then Koopman (semi)group of operators $\mathcal{U}^t : \mathcal{F} \to \mathcal{F}$ associated with the flow Φ is defined by

$$(\mathcal{U}^t \psi)(\mathbf{x}) = \psi \circ \Phi(t, \mathbf{x}). \quad (7)$$

In this paper we assume $\mathcal{F} \subseteq C^1(\mathbf{X})$, see [25] and [30] for discussion on appropriate choices of $\mathcal{F}$. By definition, the Koopman operator is linear, and $\tilde{\psi}(t, \mathbf{x}) = \mathcal{U}^t \psi$ is the solution to the PDE

$$\frac{\partial \tilde{\psi}}{\partial t} = < d\tilde{\psi}, \mathbf{f} > = \mathcal{L}_{\mathbf{f}} \tilde{\psi}, \quad (8)$$

with boundary condition, $\tilde{\psi}(0, \mathbf{x}) = \psi(\mathbf{x})$.

An eigenfunction of the Koopman operator (or in short Koopman eigenfunction (KEF)) is an observable $\phi \in \mathcal{F}$ that satisfies:

$$\mathcal{U}^t \phi = e^{\lambda t} \phi, \quad (9)$$

where, $\lambda \in \mathbb{C}$ is referred to as the Koopman eigenvalue (KE) corresponding to KEF ϕ. It follows from (8) that the KEF satisfy the eigenvalue equation

$$\mathcal{L}_{\mathbf{f}} \phi = \lambda \phi. \quad (10)$$

In general the Koopman operator could possess continuous and residual parts of spectrum in addition to the point spectrum [30]. For our purposes only point spectrum of Koopman operator will suffice. In fact, in well-chosen $\mathcal{F}$ the continuous and residual parts of the Koopman spectra are empty for most types of attractors [25]. Note following:

- If ϕ_1 and ϕ_2 are KEFs with eigenvalues λ_1, λ_2 then $\phi_1^{k_1} \phi_2^{k_2}$ is also a KEF with eigenvalue $k_1 \lambda_1 + k_2 \lambda_2$ for any $k_1, k_2 \in \mathbb{R}$.
- KEF are smooth in the vicinity of the attractor, a property which contrasts Koopman operator with the Perron-Frobenius operator, i.e it's dual [25].

Let ϕ_i be an eigenfunction for the Koopman operator corresponding to the eigenvalue λ_i. Given a vector valued observable $\mathbf{r}(\mathbf{x})$, the Koopman mode (KM) $\mathbf{v}_i$, corresponding to ϕ_i is the vector of the coefficients of the projection of $\mathbf{r}(\mathbf{x})$ onto the span$\{\phi_i\}$ [17], [18]. $\mathbf{v}_i$ can be thought of as mapping from the observable space into a vector space $V \subset \mathbb{C}^p$; the map $\mathbf{r} \to \phi_i \mathbf{v}_i$ is then a vector-valued projection operator onto the subspace span $\{\phi_i\}$.

Note that KEs/KEFs (λ, ϕ) depend only on the dynamics (6), and the chosen function space $\mathcal{F}$, and not on any particular observable. On the other hand the KMs $\mathbf{v}$ are specific to a given observable. In this regard, we will refer to modes $\mathbf{v}^x$ for full state observable, i.e. $\mathbf{r}(\mathbf{x}) = \mathbf{x}$ as the *Koopman*

Modes (KMs), and modes $\mathbf{v}^{\mathbf{r}}$ for any other observable $\mathbf{r}$ as the *Output Koopman Modes* (OKMs). Finally, we will refer to the KEs, KEFs, KMs triplet i.e. $(\lambda_i, \phi_i, \mathbf{v}_i^x), i = 1, \cdots$ as the *Koopman tuple*.

IV. Koopman Observer Form

For the input-output system (1)-(2), consider the Koopman operator and its associated KEs/KEFs (λ, ϕ) corresponding to the flow Φ^0 induced by the unactuated part, i.e.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{11}$$

which is obtained by setting $\mathbf{u} \equiv \mathbf{0}$ in (1).

Condition I: Let $\mathcal{F}^n = \text{span}\{\phi_i\}_{i=1}^n$ be a *finite* subset of KEFs such that $\mathbf{x}, \mathbf{h}(\mathbf{x}) \in \mathcal{F}^n$ (where, $\mathbf{h}$ is the output in (2)), and so

$$\mathbf{x} = \sum_{i=1}^n \phi_i(\mathbf{x})\mathbf{v}_i^x, \quad \mathbf{h}(\mathbf{x}) = \sum_{i=1}^n \phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{h}}, \tag{12}$$

where, $\mathbf{v}_i^x \in \mathbb{C}^d, i = 1, \cdots, n$ are the KMs, and $\mathbf{v}_i^{\mathbf{h}} \in \mathbb{C}^m, i = 1, \cdots, n$ are the OKMs, as defined before.

Note that if λ is a complex KE with KEF ϕ, then the complex conjugate $\overline{\lambda}$ is also a KE with KEF $\overline{\phi}$, and

$$\mathcal{L}_{\mathbf{f}}\overline{\phi} = \overline{\lambda}\,\overline{\phi}. \tag{13}$$

For any real valued observable $\mathbf{h}$ it is straightforward to show that the OKMs also occur in conjugate pairs just as the KEs/KEFs.

Let $\lambda = |\lambda| e^{i \arg(\lambda)}$ be polar representation of a complex KE. From Eqns. (10) and (13) one can easily show

$$\begin{aligned} \mathcal{L}_{\mathbf{f}}\text{Re}(\phi) &= |\lambda|(\cos(\arg\lambda)\text{Re}(\phi) + \sin(\arg\lambda)\text{Im}(\phi)), \\ \mathcal{L}_{\mathbf{f}}\text{Im}(\phi) &= |\lambda|(\sin(\arg\lambda)\text{Re}(\phi) - \cos(\arg\lambda\text{Im}(\phi)). \end{aligned}$$

For compactness we express above relations in a vector form

$$\mathcal{L}_{\mathbf{f}}\begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{\mathbf{f}}\text{Re}(\phi) \\ -\mathcal{L}_{\mathbf{f}}\text{Im}(\phi) \end{pmatrix} = Q_\lambda \begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix}, \tag{14}$$

where,

$$Q_\lambda = |\lambda| \begin{pmatrix} \cos(\arg\lambda) & \sin(\arg\lambda) \\ -\sin(\arg\lambda) & \cos(\arg\lambda) \end{pmatrix}. \tag{15}$$

Since $\mathcal{L}Q_\lambda = Q_\lambda\mathcal{L}$, it also follows that

$$\begin{aligned} \mathcal{L}_{\mathbf{f}}^k \begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix} &= \mathcal{L}_{\mathbf{f}}^{k-1} Q_\lambda \begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix} \\ &= Q_\lambda \mathcal{L}_{\mathbf{f}}^{k-1} \begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix} \\ &= Q_\lambda^k \begin{pmatrix} \text{Re}(\phi) \\ -\text{Im}(\phi) \end{pmatrix}, \end{aligned} \tag{16}$$

for $k = 1, 2, \cdots$.

In what follows, we order KEFs $\{\phi_1, \phi_2, \cdots, \phi_n\}$ (and correspondingly KEs and KMs/OKMs) such that complex conjugate pairs appears adjacent to each other. Define $\mathcal{T}(\mathbf{x}) = (\hat{\phi}_1(\mathbf{x}), \hat{\phi}_2(\mathbf{x}), \cdots, \hat{\phi}_n(\mathbf{x}))^*$ as follows:

- $\hat{\phi}_i = \phi_i$ if i-th KEF is real, and
- $\hat{\phi}_i = 2\text{Re}(\phi_i)$ and $\hat{\phi}_{i+1} = -2\text{Im}(\phi_i)$, if i and $i+1$-th KEFs are complex conjugate pairs.

It is straightforward to show that KMD (12) can be expressed in terms of $\mathcal{T}(\mathbf{x})$ as

$$\mathbf{x} = C^x\mathcal{T}(\mathbf{x}), \quad \mathbf{h}(\mathbf{x}) = C^{\mathbf{h}}\mathcal{T}(\mathbf{x}), \tag{17}$$

where, $C^x \in \mathbb{R}^{d\times n}$ and $C^{\mathbf{h}} \in \mathbb{R}^{m\times n}$ are matrices formed from KMs and OKMs as follows. The i-th column of $C^{\mathbf{h}}$ is $\mathbf{v}_i^{\mathbf{h}}$ if i-th KEF is real, and $i, i+1$-th columns are $\text{Re}(\mathbf{v}_i^{\mathbf{h}})$ and $\text{Im}(\mathbf{v}_i^{\mathbf{h}})$, respectively if i and $i+1$-th KEFs are complex conjugate pairs. Similar procedure applies for constructing C^x.

Using the relation (14), one can write the Lie derivative of $\mathcal{T}(\mathbf{x})$ as:

$$\mathcal{L}_{\mathbf{f}}\mathcal{T}(\mathbf{x}) = \Lambda\mathcal{T}(\mathbf{x}), \tag{18}$$

where, Λ is a $n \times n$ real block diagonal matrix such that:

- Λ has a diagonal entry $\Lambda_{i,i} = \lambda_i$, if i-th KEF is real,
- Λ has a block diagonal entry $\begin{bmatrix} \Lambda_{i,i} & \Lambda_{i,i+1} \\ \Lambda_{i+1,i} & \Lambda_{i+1,i+1} \end{bmatrix} = Q_{\lambda_i}$ (see (15)), if i and $i+1$-th KEFs are complex conjugate pairs.

Consider a nonlinear change of coordinates defined by $\mathcal{T} : \mathbf{X} \to \mathbb{R}^n$,

$$\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix} = \mathcal{T}(\mathbf{x}(t)). \tag{19}$$

Following [28] we refer to this transformation as the *Koopman Canonical Transform* (KCT), and the coordinates $\mathbf{z}(t) \in \mathbb{R}^n$ as the *Koopman Canonical Coordinates* (KCC). From Eqn. (18) it follows that under change of coordinates (19),

$$\begin{aligned} \dot{\mathbf{z}} &= \frac{\partial\mathcal{T}(\mathbf{x})}{\partial\mathbf{x}}(\mathbf{f}(\mathbf{x}) + \sum_{i=1}^l \mathbf{g}_i(\mathbf{x})u_i), \\ &= \mathcal{L}_{\mathbf{f}}\mathcal{T}(\mathbf{x}) + \sum_{i=1}^l \mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x})u_i, \\ &= \Lambda\mathbf{z} + \sum_{i=1}^l \tilde{\mathbf{g}}_i(\mathbf{z})u_i, \end{aligned} \tag{20}$$

where,

$$\tilde{\mathbf{g}}_i(\mathbf{z}) = \mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x})|_{\mathbf{x}=C^x\mathbf{z}}. \tag{21}$$

Thus system (1)-(2) can be transformed into

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \sum_{i=1}^l \tilde{\mathbf{g}}_i(\mathbf{z})u_i, \tag{22}$$

$$\mathbf{y} = C^{\mathbf{h}}\mathbf{z}, \tag{23}$$

$$\mathbf{x} = C^{\mathbf{x}}\mathbf{z}. \tag{24}$$

Following [28], we will refer to the system (22)-(24) as the *Koopman Observer Form* (KOF). Note that the dimension of $\mathbf{z}$ in KOF will be greater than the dimension of state $\mathbf{x}$ of the original nonlinear system, i.e. $n \geq d$; and so the observer design based on KOF is an immersion based approach, see [3] and references therein. We further refer reader to [28] for comparison with geometric observer design approaches

[1], [4], [5], [6] which seek a diffeomorphism to obtain other types of observer forms to facilitate observer design.

Note that above derivation can be easily extended to more general nonlinear systems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (25)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}). \quad (26)$$

Writing the above system as

$$\dot{\mathbf{x}} = \mathbf{f}^0(\mathbf{x}) + \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}), \quad (27)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}), \quad (28)$$

where, $\mathbf{f}^0(\mathbf{x}) = \mathbf{f}(\mathbf{x}, 0)$ and $\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}) = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}, 0)$, and applying the KCT (based on $\mathbf{f}^0(\mathbf{x})$), one gets a KOF,

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \tilde{\mathbf{g}}(\mathbf{z}, \mathbf{u}), \quad (29)$$

$$\mathbf{y} = C^{\mathbf{h}}\mathbf{z}, \quad (30)$$

$$\mathbf{x} = C^{\mathbf{x}}\mathbf{z}. \quad (31)$$

where, $\tilde{\mathbf{g}}(\mathbf{z}, \mathbf{u}) = \frac{\partial \mathcal{T}(\mathbf{x})}{\partial \mathbf{x}} \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u})|_{\mathbf{x}=C^x\mathbf{z}}$.

We next consider three special cases of the KOF which will require working with the system (22)-(24). However, the Lipschitz KOF formulation (see Section IV-B) is applicable for the more general KOF (29)-(31) as well.

A. Linear KOF with No Inputs

Condition II: Assume that there are no input terms in (2), i.e. $\mathbf{g}_i \equiv 0, i = 1 \cdots, l$.

Under *Condition II* the KOF (22)-(24) reduces to linear time invariant system,

$$\dot{\mathbf{z}} = \Lambda\mathbf{z}, \quad (32)$$

$$\mathbf{y} = C^{\mathbf{h}}\mathbf{z}, \quad (33)$$

$$\mathbf{x} = C^{\mathbf{x}}\mathbf{z}, \quad (34)$$

which is continuous time analogue of discrete time KOF introduced in [28]. In this special case one can employ standard Luenberger/Kalman observer design approaches, see [28] for details.

B. Lipschitz KOF

Condition III: Assume that the nonlinear term

$$\Psi(\mathbf{z}, \mathbf{u}) = \sum_{i=1}^{l} \tilde{\mathbf{g}}_i(\mathbf{z}) u_i, \quad (35)$$

in (22) is locally Lipschitz in domain $\mathcal{D} \subset \mathbb{R}^n$ i.e.

$$||\Psi(\mathbf{z}_2, \mathbf{u}) - \Psi(\mathbf{z}_1, \mathbf{u})|| \leq \gamma ||\mathbf{z}_2 - \mathbf{z}_1||, \quad (36)$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}$, $\mathbf{u} \in \mathbf{U}$, and γ is the Lipschitz constant.

Under *Condition III* the KOF (22)-(24) becomes a Lipschitz system [7],

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \Psi(\mathbf{z}, \mathbf{u}), \quad (37)$$

$$\mathbf{y} = C^{\mathbf{h}}\mathbf{z}, \quad (38)$$

$$\mathbf{x} = C^{\mathbf{x}}\mathbf{z}, \quad (39)$$

where, $\Psi(\mathbf{z}, \mathbf{u})$ is locally Lipschitz. One can then employ observer design techniques for Lipschitz systems as discussed in Section V-A.

Note that any nonlinear system (25)-(26) can be expressed in the form (37) as long as $\mathbf{f}$ is continuously differentiable with respect to $\mathbf{x}$. Furthermore the resulting nonlinear term Ψ will satisfy (36) at least locally. However, that is not sufficient to apply observer design techniques for Lipschitz systems (see Section V-A) which also require that the output equation (2) be linear. The KCT on the other hand transforms the output equation into a linear form as well, making observer design techniques for Lipschitz systems applicable to a much broader class of nonlinear input-output systems.

C. Bilinear KOF

Condition IV: Assume that

$$\mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x}) = \mathbf{b}_i + \sum_{j=1}^{n} \mathbf{v}_j^{\mathbf{g}_i} \phi_i(\mathbf{x}), \quad (40)$$

where, $\mathbf{b}_i \in \mathbb{R}^n$ is a constant vector, and $\mathbf{v}_j^{\mathbf{g}_i}, j = 1 \cdots, n$ are the KMs for $\mathbf{g}_i, i = 1, \cdots, l$.

Under *Condition IV* the KOF (22)-(24) reduces to a bilinear control form

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \sum_{i=1}^{l} B^i \mathbf{z} u_i + B^0\mathbf{u}, \quad (41)$$

$$\mathbf{y} = C^{\mathbf{h}}\mathbf{z}, \quad (42)$$

$$\mathbf{x} = C^{\mathbf{x}}\mathbf{z}, \quad (43)$$

where, $B^0 = [\mathbf{b}_1, \cdots, \mathbf{b}_l]$ is a $n \times l$ matrix, and B^i is a $n \times n$ matrix constructed using Koopman modes $\{\mathbf{v}_j^{\mathbf{g}_i}\}$ using the procedure for constructing $C^{\mathbf{h}}/C^{\mathbf{x}}$ described in the previous section. Note that in case $\mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x}) - \mathbf{b}_i \notin \mathcal{F}^n$, one can expand the $\mathcal{F}^n$ to include additional KEFs such that $\mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x})$ lie in $\mathcal{F}^n$, and a closure is obtained. Observer design techniques for bilinear systems are discussed in Section V-B. Few additional remarks:

- Note that if $\mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x}) = \mathbf{b}_i \neq 0, i = 1, \cdots, l$, then $B^i \equiv 0$ and the bilinear KOF above reduces to a linear input-output KOF.
- The process of obtaining bilinear KOF (or linear KOF when no control inputs are present) is similar to the Carleman approach to linearization [14], [15]: both use an immersion to transform the nonlinear system to a bilinear (or linear) form. The key distinction is that in the bilinear KOF we use a specific set of KEFs (as prescribed by *Condition I*) to construct the transformation, while in the Carleman approach the transformation is constructed using tensor products of the state vector and relies on analyticity of the system (1)-(2). In future it will be worthwhile to investigate if the proposed Koopman based approach leads to more compact linear/bilinear KOF representation than the Carleman linearization for similar accuracy in approximation of the underlying nonlinear system, see Section VII for further discussion.

D. Computation of KCT

In order of compute the KOF for the system (1)-(2), one needs to determine the KCT (19) based on the autonomous part of (1) such that *Condition I* is satisfied. It is important to note that while there are infinitely many Koopman eigenvalues/eigenfunctions, we are only interested in those whose span contains $\mathbf{x}, \mathbf{h}(\mathbf{x})$ (where $\mathbf{h}(\mathbf{x})$ is the output function (2)).

A variety of techniques have been proposed in literature for computation of Koopman tuple i.e. Koopman eigenvalues, eigenfunctions and modes: harmonic averaging [31], [32], generalized Laplacian analysis [18], and Dynamic Mode Decomposition (DMD) and its variants (see [33] and references there in), and extended DMD [34], [35]. These approaches primarily differ in the function basis they use for obtaining a finite dimensional approximation of the Koopman operator, and use simulated traces from the underling dynamical system to compute a subset of spectra of the Koopman operator. We refer the reader to [28] for such an approach for computing KCT based on the extended DMD technique.

V. Observer Design using KOF

We next discuss observer design approaches for the nonlinear system (1)-(2) based on the KOF (22)-(24). Different design techniques are applicable under different assumptions (which result in different special cases of the KOF):

- Standard linear observer design under *Condition I* and *Condition II*,
- Observer design for Lipschitz systems under *Condition I* and *Condition III*,
- Observer design for bilinear systems under *Condition I* and *Condition IV*.

Below we review design techniques for Lipschitz systems and bilinear systems.

A. Observer Design for Lipschitz KOF

Observer design for Lipschitz systems of the form (37)-(38), i.e.

$$\begin{aligned} \dot{\mathbf{z}} &= A\mathbf{z} + \Psi(\mathbf{z}, \mathbf{u}), \\ \mathbf{y} &= C\mathbf{z}, \end{aligned} \quad (44)$$

have received much attention in the literature. For such systems one can seek an observer of the form:

$$\dot{\hat{\mathbf{z}}} = A\hat{\mathbf{z}} + \Psi(\hat{\mathbf{z}}, \mathbf{u}) + L(\mathbf{y} - C\hat{\mathbf{z}}), \quad (45)$$

and the goal is to find a gain L, such that the observer error $\mathbf{e}(t) = \mathbf{z} - \hat{\mathbf{z}}$ dynamics

$$\dot{\mathbf{e}} = (A - LC)\mathbf{e} + \Psi(\mathbf{z}, \mathbf{u}) - \Psi(\hat{\mathbf{z}}, \mathbf{u}), \quad (46)$$

is asymptotically stable. Since Thau's seminal paper [36] where he obtained a sufficient condition on L to ensure the asymptotic stability of the observer, a rich body of literature has emerged tackling the observer design problem [7], [8], [9], [10], [11]. The observer design technique proposed in [7] is based on quadratic Lyapunov function, and depends on the existence of a positive definite solution to an algebraic Ricatti equation. The procedure is iterative and involves following steps:

1. Choose $\epsilon > 0$ sufficiently small.
2. Check if the modified Ricatti equation
$$AP + PA^* + P(\gamma^2\mathcal{I} - \frac{1}{\epsilon}C^*C)P + (1+\epsilon)\mathcal{I} = 0, \quad (47)$$
has a positive definite solution P, where $\mathcal{I}$ is a $n \times n$ identity matrix.
3. If positive-definite P exists, then a choice
$$L = \frac{1}{2\epsilon}PC^*, \quad (48)$$
stabilizes the error dynamics. If not let $\epsilon \leftarrow \epsilon/2$ and repeat steps 2 and 3.

Raghavan's algorithm can sometime fail to converge even when the matrices (A, C) satisfy the usual observability assumptions and does not provide insights into what condition $A - LC$ should satisfy to ensure observer stability. In [8], necessary and sufficient conditions on L that ensure asymptotic stability of the observer were given, and a gradient based optimization method was proposed for designing the observer. The authors in [9] showed that the condition introduced in [8] is related to a modified H_∞ problem, and proposed a dynamical filter based observer design based on H_∞ optimization.

A major limitation of above approaches is that they work only for adequately small values of the Lipschitz constant [11]. When the Lipschitz constant is large or when the equivalent Lipschitz constant has to be chosen large due to the non-Lipschitz nature of the nonlinearity, above observer design techniques could fail to provide a solution. To address this limitation, several approaches have been proposed. For example, Raghavan [7] showed that the observer design approach discussed above might still be feasible using a state transformation. A technique based on Linear Matrix Inequalities (LMI) optimization has been proposed in [10] which leads to a robust H_∞ observer design. Using this approach the Lipschitz constant of the nonlinear system can be maximized so that the observer error dynamics is not only asymptotically stable but also the observer can tolerate some additive nonlinear uncertainty.

In [11], a less conservative approach to estimating the Lipschitz constant is proposed with the Lipschitz condition (36) expressed in a matrix form:

$$||\Psi(\mathbf{z}_2, \mathbf{u}) - \Psi(\mathbf{z}_1, \mathbf{u})|| \leq ||G(\mathbf{z}_2 - \mathbf{z}_1)||. \quad (49)$$

Note that the matrix G in this case could be a sparsely populated matrix. Hence, $||G(\mathbf{x}_1 - \mathbf{x}_2)|$ can be much smaller than the constant $\gamma||\mathbf{x}_1 - \mathbf{x}_2||$ used earlier in (35) for the same nonlinear function. Furthermore, it was shown in [11] that the error dynamics (46) is asymptotically stable if and only if an observer gain matrix L can be chosen such that

$$\begin{bmatrix} (A - LC)^*P + P(A - LC) + \epsilon G^*G & P \\ P & -\epsilon\mathcal{I} \end{bmatrix} < 0,$$

for some positive definite symmetric matrix P, and some real $\epsilon > 0$. The above inequality is nonconvex because it involves the product of P and L. By introducing $Y = PL$

above matrix inequality can be expressed as a LMI,

$$\begin{bmatrix} A^*P + PA - C^*Y^* - YC + \epsilon G^*G & P \\ P & -\epsilon\mathcal{I} \end{bmatrix} < 0, P > 0,$$

which can be solved using standard convex optimization techniques or the MATLAB LMI control toolbox. The gain can simply be computed as $L = P^{-1}Y$. The above LMI can also be replaced by an equivalent Riccati inequality in just one variable P

$$A^*P + PA + \epsilon G^*G + \frac{1}{\epsilon}PP - C^*L^*P - PLC = -\mu\mathcal{I} < 0, \quad (50)$$

for some $\beta \in \mathbb{R}$ and $\mu > 0$. This can be solved using Algebric Riccati Equation solver in MATLAB, and the gain can be computed via $L = \frac{\beta^2}{2}P^{-1}C^*$, see [11] for details.

B. Observer Design for Bilinear KOF

For bilinear systems of the form (41)-(42), i.e.

$$\begin{aligned} \dot{\mathbf{z}} &= A\mathbf{z} + \sum_{i=1}^{l} B^i \mathbf{z} u_i + B^0\mathbf{u}, \\ \mathbf{y} &= C\mathbf{z}, \end{aligned} \quad (51)$$

multiple formulations can be posed for observer design [12]. For instance, assume that $\mathbf{u}$ is known. For this case one can view (51) as a linear time varying system,

$$\begin{aligned} \dot{\mathbf{z}} &= A(t)\mathbf{z} + B^0\mathbf{u}, \\ \mathbf{y} &= C\mathbf{z}, \end{aligned} \quad (52)$$

where, $A(t) = A + \sum_{i=1}^{l} B^i u_i(t)$, and use well-known Kalman type methods of designing observers of time-varying linear system. For instance, if the system (52) is uniformly completely observable, and $A(t)$ is uniformly bounded in time, one can seek an observer of the form [3], [37],

$$\dot{\hat{\mathbf{z}}} = A(t)\hat{\mathbf{z}} + B^0\mathbf{u} + K(t)(\mathbf{y} - C\hat{\mathbf{z}}), \quad (53)$$

where, the gain $K(t) = M(t)C^*W^{-1}$ is computed based on solution of a matrix Riccati equation

$$\begin{aligned} \dot{M} &= A(t)M(t) + M(t)A^*(t) - M(t)C^*W^{-1}CM(t) \\ &+ V + \delta M(t), \end{aligned}$$

where, $M(0) = M_0 = M_0^* > 0$, $W = W^* > 0$, and with either $\delta \geq 2||A(t)||$ for all t, or $V = V^* > 0$. The rate of convergence can be tuned by either δ or V.

Design of an observer which will converge for all choices of $\mathbf{u}$ has also been considered, see [38], [12].

VI. Nonlinear Observability Criterion based on KOF

In this section we establish a nonlinear observability criterion for system (1)-(2) by exploiting the KOF. We will use standard notion of nonlinear observability [39], which we briefly recall first. A pair of points $\mathbf{x}_0$ and $\mathbf{x}_0'$ are *indistinguishable* (denoted $\mathbf{x}_0\mathbb{I}\mathbf{x}_0'$) if the system (1)-(2) with these two initial conditions realizes same input-output map for every admissible control input $\mathbf{u}(t), t \in [t_0, t_1]$. Note that indistinguishability $\mathbb{I}$ is an equivalence relation on $\mathbf{X}$. System (25)-(26) is said to be *nonlinearly observable at* $\mathbf{x}_0$ if $\mathbb{I}(\mathbf{x}_0) = \{\mathbf{x}_0\}$ and is *nonlinearly observable* if $\mathbb{I}(\mathbf{x}) = \{\mathbf{x}\}$ for every $\mathbf{x} \in \mathbf{X}$.

Let $\mathbf{Z}$ be the image of state space $\mathbf{X}$ under the KCT, i.e. $\mathbf{Z} = \mathcal{T}(\mathbf{X})$. Note that since $\mathbf{x} \equiv C^{\mathbf{x}}\mathbf{z}$ (by *Condition I*), the transformation $\mathcal{T}$ is an injective mapping onto its range $\mathbf{Z}$.

Theorem I: If *Condition I* and *Condition II* hold, and the pair $(\Lambda, C^{\mathbf{h}})$ is observable, i.e.

$$\text{rank}([C^{\mathbf{h}}, C^{\mathbf{h}}\Lambda, \cdots, C^{\mathbf{h}}\Lambda^{n-1}]^*) = n, \quad (54)$$

then the nonlinear system (1)-(2) is nonlinearly observable.

Proof: The proof is by contradiction. Assume (1)-(2) is not observable, and so there exists two distinct initial conditions $\mathbf{x}_0 \neq \mathbf{x}_0'$ such that they result in same output $\mathbf{y}(t)$ over any interval of time $[0, T]$. Let $\mathbf{z}_0 = \mathcal{T}(\mathbf{x}_0)$ and $\mathbf{z}_0' = \mathcal{T}(\mathbf{x}_0')$, then $\mathbf{z}_0 \neq \mathbf{z}_0'$ by injectivity of $\mathcal{T}$. By construction, the KOF will also produce same outputs when initialized at $\mathbf{z}_0$ or $\mathbf{z}_0'$. This is a contradiction, since under condition (54), the KOF (32)-(33) is observable, and so $\mathbf{z}_0 = \mathbf{z}_0'$.

Theorem II: If *Condition I* and *Condition IV* hold, and

$$\begin{aligned} &\text{rank}([C^{\mathbf{h}}, C^{\mathbf{h}}\Lambda, C^{\mathbf{h}}B^1, \cdots C^{\mathbf{h}}B^l, C^{\mathbf{h}}\Lambda^2, C^{\mathbf{h}}\Lambda B^1, \\ &\cdots, C^{\mathbf{h}}\Lambda B^l, C^{\mathbf{h}}B^1\Lambda, \cdots, C^{\mathbf{h}}(B^l)^{n-1}]) = n, \end{aligned} \quad (55)$$

then the nonlinear system (1)-(2) is nonlinearly observable for the space of control inputs $\mathbf{U}$ containing piecewise continuous input signals.

Proof: The condition (55) implies that there exists piecewise continuous input signal for which the bilinear KOF (41)-(43) is observable [40]. Then, following same arguments as in the proof of *Theorem I*, the proof of the *Theorem II* follows.

The above propositions provide sufficient conditions for nonlinear observability in terms of KOF. However, the observability conditions (54) and (55) required in above theorems, respectively are stringent. In fact, as long as the linear KOF (32)-(34), and the bilinear KOF (41)-(43) are observable (as discussed above) restricted to $\mathbf{Z}$ (i.e. any two states $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ are indistinguishable), *Theorem I* and *Theorem II* will hold true.

VII. Remarks on Assumptions for KOF

We discuss the implications of *Condition I*, which provides the key assumption for the proposed Koopman based observer design framework. Recall, *Condition I* requires that the $\mathbf{x}, \mathbf{h}(\mathbf{x})$ lie in a span of a finite dimensional subset of Koopman eigenfunctions associated with the autonomous part of the system (1). For $\mathbf{u} \equiv 0$, the KOF reduces to a linear form. Since in this case the KOF can only have a fixed point at the origin, an exact finite dimensional KOF may exist only for nonlinear systems with a single isolated fixed point. A similar assessment on existence of a finite-dimensional Koopman-invariant subspace was noted in [29] in context of full state feedback based control design using Koopman framework.

Furthermore, the Koopman decomposition (12) may require infinitely many terms, i.e.

$$\mathbf{x} = \lim_{N\to\infty} \sum_{i=1}^{N} \phi_i(\mathbf{x})\mathbf{v}_i^x, \mathbf{h}(\mathbf{x}) = \lim_{N\to\infty} \sum_{i=1}^{N} \phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{h}}. \quad (56)$$

Retaining only a finite number $n \gg 1$ of terms in above expansions, a truncated or approximate KOF can be obtained

$$\begin{aligned}\dot{\mathbf{z}} &= \Lambda\mathbf{z} + \sum_{i=1}^{l} \tilde{\mathbf{g}}_i(\mathbf{z})u_i,\\ \mathbf{y} &= C^{\mathbf{h}}\mathbf{z} + \Delta^{\mathbf{h}}(\mathbf{x}),\\ \mathbf{x} &= C^{\mathbf{x}}\mathbf{z} + \Delta^{\mathbf{x}}(\mathbf{x}),\end{aligned}$$

where, $\tilde{\mathbf{g}}_i(\mathbf{z}) = \mathcal{L}_{\mathbf{g}_i}\mathcal{T}(\mathbf{x})|_{\mathbf{x}=C^x\mathbf{z}+\Delta^{\mathbf{x}}(\mathbf{x})}$, and Δ denotes the truncation error. Note that while any continuously differentiable nonlinear system can always be written in above form with linear terms in state, and additive nonlinear terms (for instance by linearizing both dynamics (1) and the output equation (2)), the resulting nonlinear terms can only be bounded locally. On the other hand, at the expense of including sufficient number of terms n in the decomposition (56), the nonlinear terms $\Delta^{\mathbf{h}}(\mathbf{x}), \Delta^{\mathbf{x}}(\mathbf{x})$ can be made arbitrarily small, and thus the truncated KOF is expected to provide more accurate approximation over a larger portion of the state space. Further theoretical and numerical work is required for assessing the computational and estimation accuracy tradeoff resulting from the truncation of the KOF under different assumptions (i.e. *Condition* II, III and IV), and will be pursued in future work. Along similar lines as discussed in Section IV-C comparison with the Carleman linearization approach will also be worthwhile to investigate.

Additionally, we argue that the proposed framework is also useful for nonlinear systems with multiple fixed points or more general attractors. For instance, it was demonstrated in [28] that a Kalman filter designed based on an approximate KOF in basin of attraction (whose boundary is formed by an unstable limit cycle) of a fixed point (of a nonlinear system with no inputs) leads to a superior estimation performance compared to that based on the Extended Kalman Filter. In a more general setting, obtaining approximate KOFs in different basins of attraction and patching them for estimation is another worthwhile avenue of future research.

Finally, note that the requirement in *Condition I* for the full state observable $\mathbf{x}$ to lie in a finite dimensional span of Koopman eigenfunctions can be relaxed. For this case no modifications are required for KOF based observer design framework presented in this paper. However, the relationship $\mathbf{x} = C^{\mathbf{x}}\mathbf{z}$ can no longer be used to obtain the estimate of $\mathbf{x}$; rather, one will need to solve the system of nonlinear equations $\mathcal{T}(\mathbf{x}) - \mathbf{z} = 0$.

VIII. Numerical Demonstration

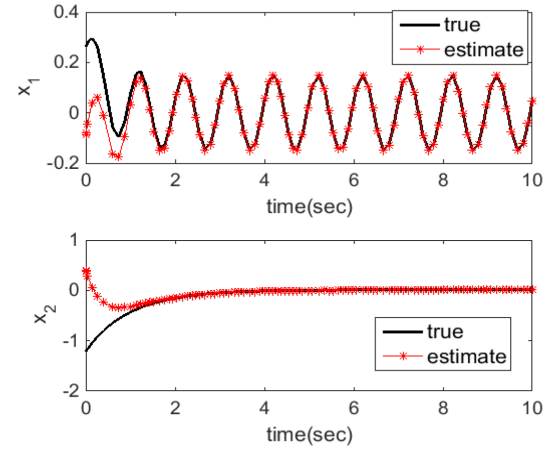
To illustrate KOF based observer design framework, we consider following system:

$$\dot{\mathbf{x}} = \begin{pmatrix} \rho x_1 \\ \mu(x_2 - x_1^2)\end{pmatrix} + \mathbf{g}(\mathbf{x})u, \quad (57)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) = x_1^2 + x_2, \quad (58)$$

where, $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}))^*$. It can be shown that for autonomous part of (57), i.e.

$$\dot{\mathbf{x}} = \begin{pmatrix} \rho x_1 \\ \mu(x_2 - x_1^2)\end{pmatrix}, \quad (59)$$



(a)

Fig. 1. Estimation results for system (57)-(58) using observer design based on the associated Lipschitz KOF.

ρ, μ are Koopman eigenvalues with eigenfunctions $\phi_\rho(\mathbf{x}) = x_1$, and $\phi_\mu(\mathbf{x}) = x_2 - \alpha x_1^2$, respectively, where $\alpha = \frac{\mu}{\mu - 2\rho}$. Also note that $2\rho, \rho + \mu$ etc. are Koopman eigenvalues with eigenfunctions $\phi_\rho^2, \phi_\rho\phi_\mu$ etc. Let $\phi_1 = \phi_\rho, \quad \phi_2 = \phi_\mu, \quad \phi_3 = \phi_\rho^2$, then it follows that $\mathbf{x} = \sum_{i=1}^{3}\phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{x}}$ where, $\mathbf{v}_1^{\mathbf{x}} = (1, 0)^*$, $\mathbf{v}_2^{\mathbf{x}} = (0, 1)^*$, and $\mathbf{v}_3^{\mathbf{x}} = (0, \alpha)^*$. Similarly, $\mathbf{h}(\mathbf{x}) = \sum_{i=1}^{3}\phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{h}}$, where $\mathbf{v}_1^{\mathbf{h}} = 0$, $\mathbf{v}_2^{\mathbf{h}} = 1$, and $\mathbf{v}_3^{\mathbf{h}} = 1 + \alpha$. Thus, using the KCT (19)

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_1 \end{pmatrix} = \mathcal{T}(\mathbf{x}) = \begin{pmatrix} \phi_\rho \\ \phi_\mu \\ \phi_{2\rho}\end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - \alpha x_1^2 \\ x_1^2 \end{pmatrix},$$

we get the KOF (22)-(24) with

$$\begin{aligned}\Lambda &= \mathrm{diag}(\rho, \mu, 2\rho), \quad C^{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \end{pmatrix},\\ C^{\mathbf{h}} &= \begin{pmatrix} 0 & 1 & 1+\alpha \end{pmatrix},\end{aligned}$$

and

$$\tilde{\mathbf{g}}(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ -2\alpha z_1 & 1 \\ 2z_1 & 0 \end{pmatrix}\begin{pmatrix} g_1(C^{\mathbf{h}}\mathbf{z}) \\ g_2(C^{\mathbf{h}}\mathbf{z})\end{pmatrix}. \quad (60)$$

For illustration purposes, we consider some specific choices of $\mathbf{g}(\mathbf{x})$. For example, with $\mathbf{g}(\mathbf{x}) = (1, 0)^*$,

$$\mathcal{L}_{\mathbf{g}}\mathcal{T}(\mathbf{x}) = \mathbf{b} + \sum_{j=1}^{n}\mathbf{v}_j^{\mathbf{g}}\phi_i(\mathbf{x}), \quad (61)$$

where, $\mathbf{b} = (1, 0, 0)^*$, and $\mathbf{v}_1^{\mathbf{g}} = (0, -2\alpha, 2)^*$, and $\mathbf{v}_2^{\mathbf{g}} = \mathbf{v}_3^{\mathbf{g}} = (0, 0, 0)^*$. So *Condition IV* is satisfied leading to a bilinear KOF, and one can employ observer design techniques discussed in Section V-B.

For $\mathbf{g}(\mathbf{x}) = (\cos(x_1), 0)^*$, one obtains a Lipschitz KOF with

$$\Psi(\mathbf{z}, u) = \cos(z_1)\begin{pmatrix} 1 \\ -2\alpha z_1 \\ 2z_1 \end{pmatrix}u, \quad (62)$$

where, we have assumed that the control inputs are bounded $|u| \leq u_m$. We use Raghavan's method discussed in Section V-A to design the observer gain L. Figure 1 shows the estimation results for periodically excited inputs $u(t) = \cos(2\pi t)$ so that $u_m = 1$.

IX. Conclusion

In this paper we introduced a Koopman Observer Form (KOF) and associated immersion based observer synthesis framework for general nonlinear input-output systems with control affine terms. By considering special cases of the KOF, we showed how existing observer design techniques for Lipschitz systems and bilinear systems can be used in context of estimation for more general nonlinear systems.

Further theoretical and numerical work is required for assessing the tradeoff between computational effort and estimation accuracy resulting from the truncation of the KOF. The use of KOF for control synthesis, and fault detection and isolation in dynamic systems is another avenue of future research which we are currently pursuing.

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References

[1] E. A. Misawa and J. K. Hedrick, "Nonlinear Observers: A State-of-the-Art Survey," *Journal of Dynamic Systems Measurement and Control*, vol. 111, no. 3, 1989.

[2] H. Nijmeijer and T. I. Fossen, *New Directions in Nonlinear Observer Design*. Springer, 1999, vol. 244.

[3] G. Besancon, *Nonlinear Observers and Applications*. Lecture Notes in Control and Information Sciences, Volume 363, 2007.

[4] H. Keller, "Nonlinear Observer Design by Tranformation into a Generalized Observer Canonical Form," *International Journal of Control*, vol. 46, no. 6, 1987.

[5] W. Respondek, *Introduction to Geometric Nonlinear Control; Linearization, Observability, Decoupling*. Trieste: Lectures given at the Summer School on Mathematical Control Theory, 2001.

[6] W. Kang, A. J. Krener, M. Xiao, and L. Xu, *A Survey of Observers for Nonlinear Dynamical Systems*. Data Assimilation for Atmospheric, Oceanic and Hydrologic Applications (Vol. II), 2013.

[7] S. Raghavan and J. K. Hedrick, "Observer design for a class of nonlinear systems," *International Journal of Control*, vol. 59, no. 2, pp. 515–528, 1994.

[8] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 3, pp. 397–401, 1998.

[9] A. M. Pertew, H. J. Marquez, and Q. Zhao, "H infinity observer design for Lipschitz nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 7, pp. 1211–1216, 2006.

[10] M. Abbaszadeh and H. J. Marquez, "A robust observer design method for continuous-time Lipschitz nonlinear systems," in *IEEE CDC*, 2006.

[11] G. Phanomchoeng and R. Rajamani, "Observer design for Lipschitz nonlinear systems using Riccati equations," in *American Control Conference*, 2010, pp. 6060–6065.

[12] D. Elliott, *Bilinear Control Systems: Matrices in Action*. Springer Publishing Company, 2009.

[13] W. J. Rugh, *Nonlinear System Theory: The Volterra/Wiener Approach*. Johns Hopkins University Press, 1981.

[14] T. Carleman, "Application de la théories des équations intégrales linéaires aux systémes d´équations différentielles non linéaires," *Acta. Math.*, vol. 59, pp. 63–87, 1932.

[15] K. Kowalski and W.-H. Steeb, *Nonlinear Dynamical Systems and Carleman Linearization*. World Scientific Publishing Co. Pte. Ltd., 1991.

[16] A. J. Krener, "Bilinear and nonlinear realizations of input output maps," *SIAM J. Control Optimization*, vol. 13, 1975.

[17] I. Mezic, "Analysis of Fluid Flows via Spectral Properties of the Koopman Operator," *Annual Review of Fluid Mechanics*, vol. 45, 2012.

[18] M. Budisic, R. M. Mohr, and I. Mezic, "Applied Koopmanism," *Chaos*, no. 22, 2012.

[19] C. Rowley, I. Mezic, S. Bagheri, P. Schlatter, and D. S. Henningson, "Spectral Analysis of Nonlinear Flows," *J. Fluid Mech*, vol. 641, 2009.

[20] K. K. Chen, J. H. Tu, and C. W. Rowley, "Variants of dynamic mode decomposition: connections between Koopman and Fourier analyses," *Journal of Nonlinear Science*, vol. 6, no. 22, 2012.

[21] B. Eisenhower, T. Maile, M. Fischer, and I. Mezic, "Decomposing building system data for model validation and analysis using the Koopman operator," in *Fourth National Conference of IBPSA-USA*, 2010.

[22] Y. Susuki and I. Mezic, "Nonlinear Koopman Modes and Power System Stability Assessment Without Models ," *IEEE Transactions on Power Systems*, vol. 29, no. 2, 2014.

[23] M. O. Williams, C. W. Rowley, I. Mezić, and I. G. Kevrekidis, "Data fusion via intrinsic dynamic variables: An application of data-driven koopman spectral analysis," *EPL (Europhysics Letters)*, vol. 109, no. 4, p. 40007, 2015.

[24] A. Mauroy and I. Mezic, "A spectral operator-theoretic framework for global stability," in *IEEE CDC*, 2013.

[25] ——, "Global stability analysis using the eigenfunctions of the Koopman operator," *arXiv preprint arXiv:1408.1379*, 2015.

[26] U. Vaidya, "Observability gramian for nonlinear systems," in *IEEE CDC*, 2007.

[27] U. Vaidya and P. G. Mehta, "Lyapunov Measure for Almost Everywhere Stability," *IEEE Trans. Automat. Contr.*, vol. 53, no. 1, pp. 307–323, 2008.

[28] A. Surana and A. Banaszuk, "Linear observer synthesis for nonlinear systems using Koopman Operator theoretic framework," in *10th IFAC Symposium on Nonlinear Control Systems*, 2016.

[29] S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz, "Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control," *PloS one*, vol. 11, no. 2, p. e0150171, 2016.

[30] R. Mohr and I. Mezic, "Construction of eigenfunctions for scalar-type operators via Laplace averages with connections to the Koopman operator," *arXiv preprint arXiv:1403.6559*, 2014.

[31] I. Mezic, "Spectral properties of dynamical systems, model reduction and decompositions," *Nonlinear Dynamics*, vol. 41, no. 1-3, 2005.

[32] I. Mezic and A. Banaszuk, "Comparison of systems with complex behavior," *Physica D*, vol. 197, pp. 101–133, 2004.

[33] J. H. Tu, C. W. Rowley, D. M. Luchtenburg, S. L. Brunton, and J. N. Kutz, "On Dynamic Mode Decomposition: Theory and Applications," *J. of Comp. Dynamics*, vol. 2, no. 1, 2014.

[34] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley, "A Data–Driven Approximation of the Koopman Operator: Extending Dynamic Mode Decomposition," *Journal of Nonlinear Science*, vol. 25, no. 6, pp. 1307–1346, 2015.

[35] M. O. Williams, C. W. Rowley, and I. G. Kevrekidis, "A Kernel-Based Approach to Data-Driven Koopman Spectral Analysis," *arXiv preprint arXiv:1411.2260*, 2014.

[36] F. Thau, "Observing the state of non-linear dynamic systems," *International Journal of Control*, vol. 17, no. 3, pp. 471–479, 1973.

[37] J. Gauthier and A. Kupka, *Deterministic observation theory and applications*. Cambridge Univ Press, 1997.

[38] D. Williamson, "Observation of Bilinear Systems with Application to Biological Control," *Automatica*, vol. 13, no. 3, 1977.

[39] R. Hermann and A. J. Krener, "Nonlinear Controllability and Observability," *IEEE Transactions on Automatic Control*, vol. 22, no. 5, 1977.

[40] O. M. Grasselli and A. Isidori, "Deterministic state reconstruction and reachability of bilinear processes," in *IEEE Joint Automatic Control Conference*, 1977.