CDS 101/110 Homework #5 Solution

Problem 1 (CDS 101, CDS 110): (10 points)

Let the state space system be the following form

$$\frac{dx}{dt} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x \tag{1}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

In this problem, the output y is the tilt angle ϕ, and the input u is the steering angle δ.

From (1),

$$\begin{aligned}
\dot{y} &= c_1\dot{x}_1 + c_2\dot{x}_2 \\
&= c_1(-a_1x_1 - a_2x_2 + u) + c_2x_1 \\
\ddot{y} &= c_1(-a_1\dot{x}_1 - a_2\dot{x}_2 + \dot{u}) + c_2\dot{x}_1 \\
&= -a_1c_1\dot{x}_1 - a_2c_1x_1 + c_1\dot{u} + c_2(-a_1x_1 - a_2x_2 + u) \\
&= -a_1(\dot{y} - c_2\dot{x}_2) - a_2c_1x_1 + c_1\dot{u} + c_2(-a_1x_1 - a_2x_2 + u) \\
&= -a_1(\dot{y} - c_2x_1) - a_2c_1x_1 + c_1\dot{u} + c_2(-a_1x_1 - a_2x_2 + u) \\
&= -a_1\dot{y} - a_2(c_1x_1 + c_2x_2) + c_1\dot{u} + c_2u \\
&= -a_1\dot{y} - a_2y + c_1\dot{u} + c_2u
\end{aligned}$$

Compare the given equation in the problem with the above equation

$$a_1 = 0 \qquad -a_2 = \frac{mgh}{J} \qquad c_1 = \frac{Dv_0}{Jb} \qquad c_2 = \frac{mv_0h}{Jb}$$

Then, the observability matrix is

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \frac{Dv_0}{Jb} & \frac{mv_0h}{Jb} \\ \frac{mv_0h}{Jb} & \frac{mgh}{J}\frac{Dv_0}{Jb} \end{bmatrix}$$

The system is observable if the observability matrix is full rank. W_o is not full rank when

$$det(W_o) = \left(\frac{Dv_0}{Jb}\right)^2 \frac{mgh}{J} - \left(\frac{mv_0h}{Jb}\right)^2 = 0$$
$$\left(\frac{Dv_0}{Jb}\right)^2 \frac{mgh}{J} = \left(\frac{mv_0h}{Jb}\right)^2$$
$$D^2g = mhJ$$

.

Hence, the system is observable except when $D^2g = mhJ$.

Problem 2 (CDS 110): (35 points)

The dynamics is given by

$$J_1\frac{d^2\phi_1}{dt^2}+c\left(\frac{d\phi_1}{dt}-\frac{d\phi_2}{dt}\right)+k\left(\phi_1-\phi_2\right)=k_I I$$
$$J_2\frac{d^2\phi_2}{dt^2}+c\left(\frac{d\phi_2}{dt}-\frac{d\phi_1}{dt}\right)+k\left(\phi_2-\phi_1\right)=T_d$$

Let $x_1=\phi_1$, $x_2=\phi_2$, $x_3=\omega_1/\omega_0$, and $x_4=\omega_2/\omega_0$ where $\omega_0=\sqrt{k(J_1+J_2)/(J_1J_2)}$. Then,

$$J_1\ddot{x}_1+c\left(\dot{x}_1-\dot{x}_2\right)+k\left(x_1-x_2\right)=k_I I$$
$$J_1\ddot{x}_2+c\left(\dot{x}_2-\dot{x}_1\right)+k\left(x_2-x_1\right)=T_d$$

So,

$$\begin{aligned}
\frac{dx_1}{dt}&=\dot{\phi}=\omega_1=\omega_0x_3\\
\frac{dx_2}{dt}&=\omega_0x_4\\
\frac{dx_3}{dt}&=\frac{\dot{\omega}_1}{\omega_0}=\frac{\ddot{x}_1}{\omega_0}\\
&=\frac{1}{\omega_0J_1}\left(c\left(\dot{x}_2-\dot{x}_1\right)+k\left(x_2-x_1\right)+k_I I\right)\\
&=-\frac{k}{\omega_0J_1}x_1+\frac{k}{\omega_0J_1}x_2-\frac{c}{J_1}x_3+\frac{c}{J_1}x_4+\frac{k_I}{\omega_0J_1}I\\
\frac{dx_4}{dt}&=\frac{k}{\omega_0J_2}x_1-\frac{k}{\omega_0J_2}x_2+\frac{c}{J_2}x_3-\frac{c}{J_2}x_4+\frac{1}{\omega_0J_2}T_d
\end{aligned}$$

In state space form,

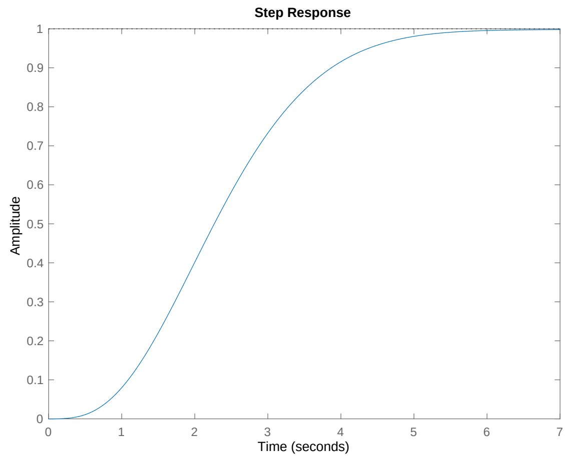
$$\dot{x}=\begin{bmatrix}0&0&\omega_0&0\\0&0&0&\omega_0\\-\frac{k}{\omega_0J_1}&\frac{k}{\omega_0J_1}&-\frac{c}{J_1}&\frac{c}{J_1}\\\frac{k}{\omega_0J_2}&-\frac{k}{\omega_0J_2}&\frac{c}{J_2}&-\frac{c}{J_2}\end{bmatrix}x+\begin{bmatrix}0\\0\\\frac{k_I}{\omega_0J_1}\\0\end{bmatrix}I+\begin{bmatrix}0\\0\\0\\\frac{1}{\omega_0J_2}\end{bmatrix}T_d$$

From Excercise 7.12, $J_1=10/9, J_2=10, c=0.1, k=1$, and $k_I=1$.

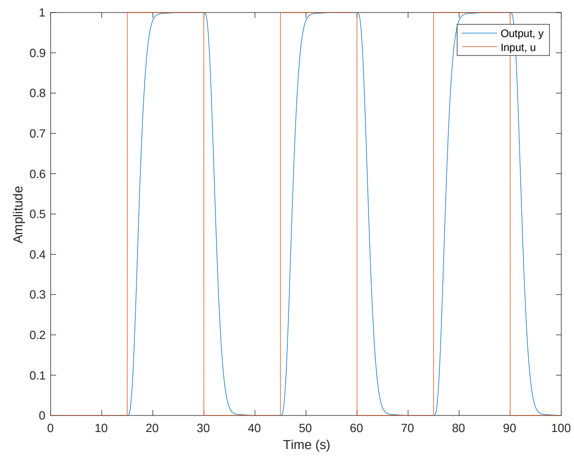
Let $u=-Kx+K_rx$. Then, $K=\begin{bmatrix}1.79&7.10&1.09&20.24\end{bmatrix}$ and $K_r=8.89$.

The observer gain is given by $L=\begin{bmatrix}646.9&9.9&297.9&38.0\end{bmatrix}^T$.

The simulation for two type of inputs are shown below:



(a) Step input



(b) Square input

The code is given below:

```
%Parameters
J1=10/9;J2=10;k=1;kd=0.1;ki=5;
w0=sqrt(k*(J1+J2)/J1/J2);a1=J2/(J1+J2);a2=J1/(J1+J2);
b1=kd/(w0^2*J1);b2=kd/(w0^2*J2);
g1=ki/(w0^2*J1);g2=1/(w0^2*J2);

%System matrices
A=[0 0 1 0;0 0 0 1;-a1 a1 -b1 b1;a2 -a2 b2 -b2]*w0;
B=[0;0;g1;0];
C=[0 1 0 0];
D=0;

% State feedback gain
P=[-1 -2 -1+1i -1-1i]; %Desired closed loop poles
K=place(A,B,P);

% Reference tracking gain
Kr = -1/(C/(A-B*K)*B);
Bref=Kr*B;

% Observer gain
P=[-4 -2 2*(-1+1i) 2*(-1-1i)];
L=place(A',C',P)';

% Closed loop system
LC = L*C;
BK = B*K;
ALCBK=A-LC-BK;
Atot=[A -BK; LC ALCBK];
Btot=[Bref; Bref];
Ctot=[0 1 0 0 0 0 0 0];
Dtot=0;
systot=ss(Atot,Btot,Ctot,Dtot);

%Simulate the system under step input
figure(1)
```

```
step(systot)

%Simulate the system under square input
[usquare,tsquare]=gensig('square',30,100,0.01);
[Ytot, Ttot, Xtot]=lsim(systot,usquare(:,1),tsquare);
graphtot=[Ytot usquare(:,1)];
figure(2)
plot(Ttot, graphtot);
ylabel('Amplitude')
xlabel('Time (s)')
legend('Output, y', 'Input, u')
```

Problem 3 (CDS 101, CDS 110): (5 points)

Transfer function is given by

$$G(s) = C(sI - A)^{-1}B$$

$$(sI - A)^{-1} = \frac{1}{s^2 - mgl/J_t}\begin{bmatrix} s & 1 \\ mgl/J_t & s \end{bmatrix}$$

$$G(s) = \frac{1}{J_t s^2 - mgl}$$

Problem 4 (CDS 101, CDS 110): (20 points)

(a) $G(s)$ can be simplified into

$$G(s) = k\frac{R_2Cs + 1}{C(kR_1 + R_2)s + 1}$$

Note that $s = j\omega$.

When $\omega \to 0$,

$$G(s) = k$$

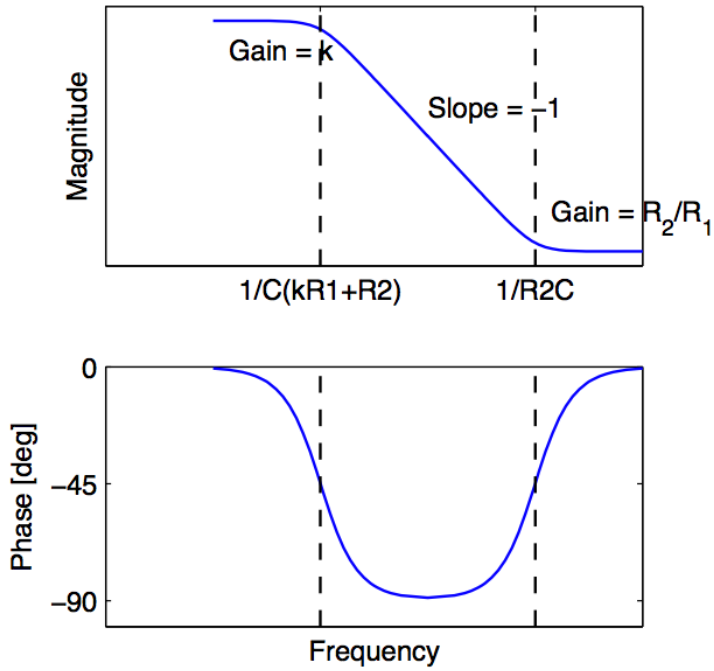
When $\omega \to \infty$,

$$G(s) = k\frac{R_2C}{C(kR_1 + R_2)}$$

Because $k \gg R_1 > R_2$,

$$G(s) = \frac{R_2}{R_1}$$

The Bode plot is given by



(b) Replace k in $G(s)$ with $\frac{k}{1+sT}$. Then,

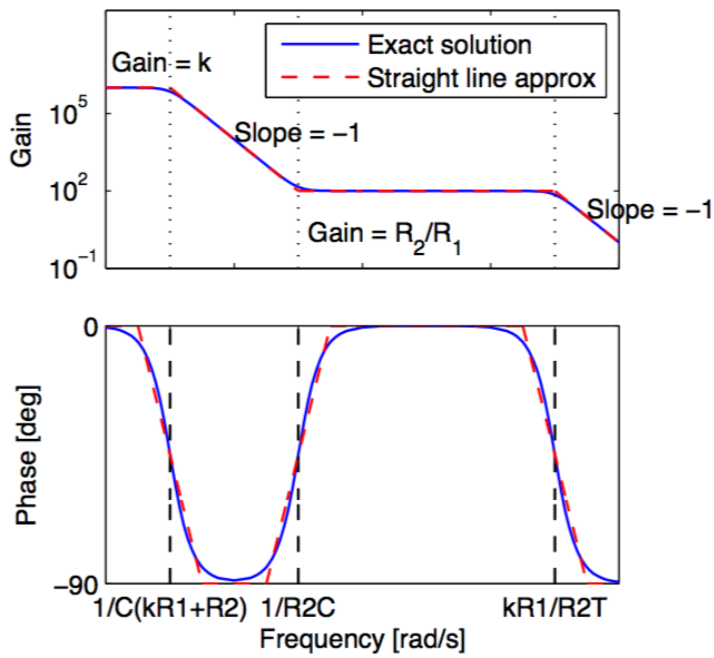
$$G(s) = k\frac{R_2Cs + 1}{CR_2Ts^2 + (kCR_1 + CR_2 + T)s + 1}$$

Plug in all the given values

$$G(s) = 10^8\frac{s+1}{s^2 + 1000101s + 100}$$

Then, the zero of $G(s)$ is -1, and the poles of $G(s)$ are -10^6 and -10^{-4}.

(c) The Bode plot is given by



Problem 5 (CDS 110): (15 points) Rewrite the dynamics in frequency domain

$$m_w x_w s^2 = -F + k_t(x_r - x_w) \implies F = k_t x_r - (k_t + m_w s^2)x_w$$

$$F = k(x_w - x_b) + c(x_w - x_b)s \implies x_w = \frac{F + (k+cs)x_b}{k+cs}$$

Combine both equation together.

$$F = k_t x_r - \frac{k_t + m_w s^2}{k+cs}F - (k_t + m_w s^2)x_b$$

$$F = \frac{k_t x_r - (k_t + m_w s^2)x_b}{1 + (k_t + m_w s^2)/(k+cs)}$$

Use $F = m_b a = m_b \ddot{x}_b = m_b s^2 x_b$.

$$x_b = \frac{a}{s^2}$$

We get

$$m_b a = \frac{k_t x_r - (k_t + m_w s^2)a/s^2}{1 + (k_t + m_w s^2)/(k+cs)}$$

Simplify

$$a = \frac{k_t}{m_b(1 + \frac{k_t + m_w s^2}{k+cs}) + \frac{k_t}{s^2} + m_w} x_r$$

So,

$$G_{ax_r}(s) = \frac{k_t}{m_b(1 + \frac{k_t + m_w s^2}{k + cs}) + \frac{k_t}{s^2} + m_w}$$

When $s = iw_t = i\sqrt{k_t/m_w}$, $k_t + m_w s^2 = 0 \implies G_{ax_r}(iw_t) = k_t/m_b$.