CDS 101/110: Lecture 7.2
Loop Analysis of Feedback Systems

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Goals:
• Why Nyquist Diagrams?
• Gain margin and phase margin
• Examples!

Reading:
• Åström and Murray, Feedback Systems, Chapter 10, Sections 10.1-10.4,
What can you do with a Nyquist Analysis?

Set Up (somewhat artificial):

- **Given:** $P(s)$
  - (any unstable roots known)
- **Given:** $C(s)$
  - (any unstable roots known)
- **Q:** can negative output feedback stabilize the system (stable $G_{yr}(s)$)?

Possible Solutions:

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)}$$

- Compute and check poles of $G_{yr}$
- Find another way to determine existence of unstable poles without computing roots of $d_p(s)d_c(s) + n_p(s)n_c(s)$

The Nyquist plot logic

- Poles of $G_{yr}(s)$ are zeros of
  $$1 + P(s)C(s) = \frac{d_p(s)d_c(s) + n_p(s)n_c(s)}{d_p(s)d_c(s)}$$
- If $G_{yr}(s)$ is unstable, then it has at least one pole in RHP
- An unstable pole of $G_{yr}(s)$ implies and unstable (RHP) zero of $1 + P(s)C(s)$
- Nyquist plot and Nyquist Criterion allow us to determine if $1 + PC$ has RHP zeros without polynomial solving.
Argument Principle
(underlying Nyquist Criterion)

As \( s \) moves clockwise around \( \Gamma \), \( L(s) \) must rotate around the origin by \( 2\pi \) for each pole inside the contour, and by \( -2\pi \) for each zero inside the contour

\[
\alpha(s) = \sum_{i=1}^{m} \psi_i(s) - \sum_{j=1}^{n} \phi_j(s)
\]

As \( s \) moves clockwise around \( \Gamma \), \( L(s) \) must rotate around the origin by \( 2\pi \) for each pole inside the contour, and by \( -2\pi \) for each zero inside the contour

\[
f(s) = \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{r_1(s)e^{i\psi_1(s)} r_2(s)e^{i\psi_2(s)} \cdots r_m(s)e^{i\psi_m(s)}}{l_1(s)e^{i\phi_1(s)} l_2(s)e^{i\phi_2(s)} \cdots l_n(s)e^{i\phi_n(s)}}
\]

\[
= M(s)e^{i\alpha}
\]

\[
P \# \text{RHP poles of open loop } L(s) \quad (\text{from } P(s), C(s)\text{poles})
\]

\[
N \# \text{clockwise encirclements of -1} \quad (\text{from Nyquist plot})
\]

\[
Z \# \text{RHP zeros of } 1 + L(s)
\]

Then \( Z_{RHP} = N + P \)
Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not how stable

Gain margin
- How much we can modify the loop gain and still have the system be stable
- Determined by the location where the loop transfer function crosses 180° phase

Phase margin
- How much “phase delay” can be added while system remains stable
- Determined by the phase at which the loop transfer function has unity gain

Bode plot interpretation
- Look for gain = 1, 180° phase crossings
- MATLAB: margin(sys)
Nyquist Plot Example #1

\[ C(s) = k \frac{s + a}{s + b} \]

\[ P(s) = \frac{1}{(s+1)^3} \quad C(s) = k \frac{s+a}{s+b} \quad 1 < a < b \]

Goal #1: Is closed loop system stable?
Goal #2: Does stability vary with gain?

\[ L(s) = \frac{k(s + a)}{(s + 1)^3(s + b)} \]

\[ k = 5, a = 2, b = 3 \]
**Nyquist Plot Example #1**

\[ P(s) = \frac{1}{(s+1)^3} \]

\[ C(s) = k \frac{s + a}{s + b} \]

\[ 1 < a < b \]

Nyquist:
- \( P = 0 \)
- \( N = +2 \)
- \( Z_{RHP} = 2 \)

2 Encirclements of -1 point

Unstable!
Nyquist Plot Example #1
(alternative analysis without Nyquist)

\[ L(s) = \frac{n_P(s)n_C(s)}{d_P(s)d_C(s)} = k \frac{n_P(s)n_C'(s)}{d_P(s)d_C'(s)} = k \frac{1}{(s + 1)^3} \frac{s + a}{s + b} \]

\[ G_{yr}(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = k \frac{kP(s)C'(s)}{1 + kP(s)C'(s)} = \frac{k n_P(s)n_C'(s)}{d_P(s)d_C' + k n_P(s)n_C'(s)} \]

The **Root Locus** studies how the poles of \( G_{yr}(s) \) vary with \( k \)

- When \( k \to 0 \), the poles of \( G_{yr}(s) \) approach the poles of \( L(s) \)
- As \( k \to \infty \), the poles of \( G_{yr}(s) \) approach the zeros of \( L(s) \) (or infinity)

For this problem, as \( k \) increases, two of the poles of \( G_{yr}(s) \) become unstable.
Example: Proportional + Integral* speed controller

\[ C(s) = K_p + \frac{K_i}{s + 0.01} \]

\[ P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a} \]

Remarks
- \( N = 0, P = 0 \Rightarrow Z = 0 \) (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don’t have to compute closed loop response
Example: cruise control

Effect of additional sensor dynamics
- New speedometer has pole at $s = 10$ (very fast); problems develop in the field
- What’s the problem? A: insufficient phase margin in original design (not robust)

$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$
Preview: control design

Approach: Increase phase margin
- Increase phase margin by reducing gain ⇒ can accommodate new sensor dynamics
- Tradeoff: lower gain at low frequencies ⇒ less bandwidth, larger steady state error

\[ P(s) = \frac{1}{m} \times \frac{r}{s + b/m} \times \frac{s + a}{s + c/m} \]

\[ C(s) = \alpha \left( K_p + \frac{K_i}{s + 0.01} \right) \]

\[ G(s) = \frac{10}{s + 10} \]