Kinematics of Motion Capture based on Quaternions

This set of notes derives a technique to estimate the displacement of a rigid body using *markers* placed on the body, and a camera system to track the marker positions.

1 Least Squares Solution

Assume that a rigid body lies in position #1. A body-fixed reference frame aligns with a fixed observing reference frame in this position. Then body then displaces by a translation $\vec{d}_{12}$ and a rotation $R_{12} \in SO(3)$ to a position #2. Assume that at least three noncolinear *marker* points can be identified in the body at the first position: $(P_1, P_2, \ldots, P_N)$. After displacements, the three points are located at $(Q_1, Q_2, \ldots, Q_N)$. Clearly,

$$Q_i \;=\; \vec{d}_{12} + R_{12} P_i \qquad i = 1, 2, 3 \;.$$

In a motion capture context, the points are associated with body-fixed *markers*, whose positions are readily measured with a camera system. However, we must expect some error in the measurement of the marker locations, which we will model as zero mean noise. Our goal is to estimate $\vec{d}_{12}$ and R_{12} from these measurements. We will use a *least squares* approach to finding the displacement estimates from the noisy measurements.

Let an *error*, e_i, in the i^{th} point coordinate be defined as follows:

$$e_i \;=\; Q_i - \vec{d}_{12} - R_{12} P_i \;.$$

That is, if the location of points P_i and Q_i where measured by the camera system with no errors, and if we knew $\vec{d}_{12}$ and R_{12} exactly, then the error e_i would be zero. Since measurement errors must be expected, the best estimate of the displacement is found by minimizing the following error:

$$E \;=\; \sum_{i=1}^{N} ||e_i||^2 \;=\; \sum_{i=1}^{N} ||Q_i - \vec{d}_{12} - R_{12} P_i||^2 \;.$$

That is, the best estimate of $\vec{d}_{12}$ and R_{12} are the ones whic minimize this error function. To simplify the evaluation of this expression, let us introduce the following *centroids* of the body-fixed points:

$$\bar{P} \;=\; \frac{1}{N} \sum_{i=1}^{N} P_i \qquad\qquad \bar{Q} \;=\; \frac{1}{N} \sum_{i=1}^{N} Q_i \tag{1}$$

and express the marker point coordinates with respect to these centroids:

$$P_i^{'} = P_i - \bar{P}_i \qquad Q_i^{'} = Q_i - \bar{Q}_i \;.$$

The error term e_i can be expressed in these adjusted coordinates as follows:

$$e_i \;=\; Q_i - \vec{d}_{12} - R_{12} P_i \;=\; Q_i^{'} + \bar{Q} - \vec{d}_{12} - R_{12}(P_i^{'} + \bar{P}_i) \;=\; Q_i^{'} - R_{12} P_i^{'} - z$$

where $z = -\vec{d_{12}} + \bar{Q} - R_{12}\bar{P}$. In these adjusted coordinates, the total error takes the form:

$$E \; = \; \sum_{i=1}^{N} \; ||Q_i^{'} - R_{12}P_i^{'} - z||^2 \; = \; \sum_{i=1}^{N} \; ||Q_i^{'} - R_{12}P_i^{'}||^2 \; - 2z \cdot (Q_i^{'} - R_{12}P_i^{'}) + z^2 \; . \tag{2}$$

Note that the third term, z^2, can only be minimized if $z = 0$, which implies that:

$$\vec{d_{12}} \; = \; \bar{Q} \; - \; R_{12}\bar{P} \; . \tag{3}$$

That is, once R_{12} is known, $\vec{d_{12}}$ can be found from Equation (3), and the expression only depends upon the centroids of the marker points.

The second term of Equation (2) vanishes since $\sum_{i=1}^{N} Q_i^{'} \; = \; \sum_{i=1}^{N} P_i^{'} = 0$ by the definition of centroid.

$$2z \cdot \sum_{i=1}^{N} \; (Q_i^{'} - R_{12}P_i^{'}) \; = \; 2z \cdot \big[\sum_{i=1}^{N} Q_i^{'} - R_{12} \sum_{i=1}^{N} \; P_i^{'} \big] = 0 \; .$$

Thus, R_{12} is found by minimizing the first term of Equation (2)

$$R_{12} \; = \; \arg\min \sum_{i=1}^{N} \; ||Q_i^{'} \; - \; R_{12}P_i^{'}||^2 \; . \tag{4}$$

Note that because rotation matrices preserve the lengths of vectors, each term $Q_i^{'} - R_{12}P_i^{'}$ is minimized by aligning vector $R_{12}P_i^{'}$ with vector $Q_i^{'}$ as closely as possible. Hence, Equation (4) is equivalent to:

$$R_{12} = \arg\max \; \sum_{i=1}^{N} Q_i^{'} \cdot (R_{12}P_i^{'}) \; . \tag{5}$$

As will be shown below, it is easiest to solve this optimization problem by converting it to use a quaternion representation of the rotation R_{12}.

2 Quaternion Review

Recall that a quarterion, q, takes the form

$$q = q_0 + q_x i + q_y j + q_z k$$

where basis elements i, j, and k obey the rules:

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= -ji = k \\ ik &= -ki = -j \\ jk &= -kj = i \; . \end{aligned}$$

The quaternion can also be simply represented as a 4-tuple, $q = (q_0, q_x, q_y, q_z)$, with the basis elements implicit. When the context is clear we can interpret the 4-tuple as a 4×1 vector.

If two quarternions, q and r, take the form:

$$q = q_0 + q_x i + q_y j + q_z k \qquad r = r_0 + r_x i + r_y j + r_z k$$

then the product of the two quarternions takes the form:

$$\begin{aligned} r \cdot q &= (r_0 q_0 - r_x q_x - r_y q_y - r_z q_z) + (r_0 q_x + r_x q_0 + r_y q_z - r_z q_y) i \\ &\quad + (r_o q_y - r_x q_z + r_y q_0 + r_z q_x) j + (r_0 q_z + r_x q_y - r_y q_x + r_z q_0) k \end{aligned}$$

Note that this product can also be represented in the following way

$$rq = \begin{bmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & -r_z & r_y \\ r_y & r_z & r_0 & -r_x \\ r_z & -r_y & r_x & r_0 \end{bmatrix} q \triangleq \mathcal{R} q \tag{6}$$

where quarternion q is treated as a 4×1 vector. In a similar way

$$qr = \begin{bmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & r_z & -r_y \\ r_y & -r_z & r_0 & r_x \\ r_z & r_y & -r_x & r_0 \end{bmatrix} q \triangleq \bar{\mathcal{R}} q \tag{7}$$

Also note that $r^* q = \mathcal{R}^T q$ and $q r^* = \bar{\mathcal{R}}^T q$, where r^* denotes the *conjugate* of r: $r^* = (r_0, -r_x, -r_y, -r_z)$.

Finally, let $\odot$ denote a dot product operator between two quaternions. That is, if we interpret quaternion r as a 4×1 vector $r = \begin{bmatrix} r_0 & r_x & r_y & r_z \end{bmatrix}$ and quaternion q as the 4×1 vector $q = \begin{bmatrix} q_0 & q_x & q_y & q_z \end{bmatrix}$, then

$$r \odot q = \begin{bmatrix} r_0 \\ r_x \\ r_y \\ r_z \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{bmatrix} = r_0 q_0 + r_x q_x + r_y q_y + r_z q_z \ .$$

3 Estimating Displacements using Quaternions

Let q_{12} be the unit quaternion which represents the same rotation as $R_{12} \in SO(3)$. Let $p_i^{'}$ be the *pure* or *vector* quaternion that represents the vector $P_i^{'}$. That is,

$$P_i^{'} = \begin{bmatrix} P_{i,x}^{'} \\ P_{i,y}^{'} \\ P_{i,z}^{'} \end{bmatrix} \qquad \Rightarrow \qquad p_i^{'} = (0, P_{i,x}^{'}, P_{i,y}^{'}, P_{i,z}^{'}) \ .$$

Similarly, let $q_i^{'} = (0, Q_{i,x}^{'}, Q_{i,y}^{'}, Q_{i,x}^{'})$ be the vector quaternion that represents the vector $Q_i^{'}$. Recall that the product of the rotation matrix $R_{12} \in SO(3)$ and the vector $P_i^{'} \in \mathbb{R}^3$, $R_{12}P_i^{'}$, can be represented in terms of quaternions as:

$$q_{12}p_i^{'}q_{12}^{*} \ .$$

Hence, the least squares estimate of R_{12} in Equation (5) can be expressed as

$$q_{12} = \arg\max \sum_{i=1}^{N} q_i^{'} \odot (q_{12}p_i^{'}q_{12}^{*}) \ . \tag{8}$$

To solve Equation (8), note that (using Equation (7)) $q_{12}p_i^{'}q_{12}^{*}$ can be expressed as $\bar{\mathcal{Q}}_{12}^{T}(q_{12}p_i^{'})$. Hence,

$$\begin{aligned}
q_{12} &= \arg\max \sum_{i=1}^{N} q_i^{'} \odot (q_{12}p_i^{'}q_{12}^{*}) \ = \arg\max \sum_{i=1}^{N} q_i^{'} \odot (\bar{\mathcal{Q}}_{12}^{T} q_{12}p_i^{'}) \\
&= \arg\max \sum_{i=1}^{N} (\bar{\mathcal{Q}}_{12}q_i^{'}) \odot (q_{12}p_i^{'}) = \arg\max \sum_{i=1}^{N} (q_i^{'}q_{12}) \odot (q_{12}p_i^{'}) \\
&= \arg\max \sum_{i=1}^{N} (\mathcal{Q}_i^{'}q_{12}) \odot (\bar{\mathcal{P}}_i^{'}q_{12}) = \arg\max \ q_{12}^{T}\big[\sum_{i=1}^{N} (\mathcal{Q}_i^{'})^{T}\bar{\mathcal{P}}_i^{'}\big]q_{12} \\
&\triangleq q_{12}^{T}\big[\sum_{i=1}^{n} N_i\big]q_{12} \triangleq q_{12}^{T}Nq_{12}
\end{aligned}$$

where matrices $\bar{\mathcal{Q}}_i^{'}$ and $\mathcal{P}_i$ are patterned after Equations (6) and (7):

$$\bar{\mathcal{P}}_i^{'} = \begin{bmatrix} 0 & -P_{i,x}^{'} & -P_{i,y}^{'} & P_{i,z}^{'} \\ P_{i,x}^{'} & 0 & P_{i,z}^{'} & -P_{i,y}^{'} \\ P_{i,y}^{'} & -P_{i,z}^{'} & 0 & P_{i,x}^{'} \\ P_{i,z}^{'} & P_{i,y}^{'} & -P_{i,x}^{'} & 0 \end{bmatrix} \qquad \mathcal{Q}_i^{'} = \begin{bmatrix} 0 & -Q_{i,x}^{'} & -Q_{i,y}^{'} & -Q_{i,z}^{'} \\ Q_{i,x}^{'} & 0 & -Q_{i,z}^{'} & Q_{i,y}^{'} \\ Q_{i,y}^{'} & Q_{i,z}^{'} & 0 & -Q_{i,x}^{'} \\ Q_{i,z}^{'} & -Q_{i,y}^{'} & Q_{i,x}^{'} & 0 \end{bmatrix} \tag{9}$$

and

$$N = \begin{bmatrix} (S_{xx}+S_{yy}+S_{zz}) & S_{yz}-S_{zy} & S_{zx}-S_{xz} & S_{xy}-Syx \\ S_{yz}-S_{yz} & (S_{xx}-S_{yy}-S_{zz}) & S_{xy}+S_{yx} & S_{zx}+S_{xz} \\ S_{zx}-S_{xz} & S_{xy}+S_{yx} & (-S_{xx}+S_{yy}-S_{zz}) & S_{yz}+S_{zy} \\ S_{xy}-S_{yx} & S_{zx}+S_{xz} & S_{yz}+S_{zy} & (-S_{xx}-S_{yy}+S_{zz}) \end{bmatrix} \tag{10}$$

where the 3×3 matrix S has the form:

$$S = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} = \sum_{i=1}^{N} Q_i^{'}(P_i^{'})^{T} \ . \tag{11}$$

Note that $q_{12}^{T}Nq_{12}$ will be maximized with respect to q_{12} when the 4×1 vector q_{12} aligns with the eigenvector associated with the maximum eigenvalue of N.

4 Summary

Let $(P_1, P_2, \ldots, P_N)$ denote the positions of a set of markers attached to a rigid body in the first position (before a displacement). After the rigid body displaces to a second position, the marker locations are described by positions $(Q_1, Q_2, \ldots, Q_N)$. The goal is to estimate the rigid body displacement $(\vec{d}_{12}, R_{12})$, where $\vec{d}_{12}$ is the translation of the rigid body between the two positions, and R_{12} denotes the relative orientation of the body in the second position with respect to the first position.

Here is a brief summary of an approach that uses the derivations above:

- Compute the centroids of the points in the first and second positions from Equation (1): $\bar{P}$ and $\bar{Q}$.
- Compute the coordinates of the points with respect to the centroids: $P_i^{'} = P_i - \bar{P}$, $Q_i^{'} = Q_i - \bar{Q}$, for $i = 1, \ldots, N$.
- Compute the S-matrix in Equation (11)
- Compute the N-matrix, Equation (10)
- Find the eigenvector of N associated with the largest eigenvalue of N. Normalize the eigenvector to ensure that it is a unit quaternion.
- Find the equivalent rotation matrix R_{12} to the unit quaternion found in the last step.
- Find the displacement, $\vec{d}_{12}$, using R_{12} found in the last step and Equation (3).