CDS 101/110: Lecture 3.3
Some Details on Linear Systems

Goals for Today:

- Review Convolution Integral (again!): board notes
- Sampled Data Sytems
- Cart-Pendulum problem

Reading:

- Åström and Murray, FBS-2e, Ch. 6.4 (assuming you read 6.1-6.3)

Linear Discrete Time Control Systems

Basic Form of Linear Discrete-Time control system. $t_{k+1} - t_k = dt$

- $x(t = t_{k+1}) = A_k\, x(t = t_k) + B_k u(t = t_k);$
  $y(t = t_k) = C_k x(t = t_k) + D\_k\, u(t = t_k)$
  } General Form
- $x_{k+1} = A_k x_k + B_k u_k;$ $y_k = C_k x_k + D_k u_k$
- $x[k+1] = Ax[k] + Bu[k];$ $y[k] = Cx[k] + Du[k]$
  } Some Common Notation

Sampled Data System (using Convolution Integral)

- $x(t + dt) = e^{Adt} x(t) + \int_t^{t+dt} e^{A(t+dt-\tau)}\, Bu(\tau) = \Phi x(t) + \Gamma u(t)$
  - $\Phi = \mathrm{e}^{Adt}$
  - $\Gamma = \left(\int_0^{dt} e^{As}\, ds\right) B = A^{-1}\left(e^{Adt} - I\right) B$
  } LTI

Must make Assumption about $u(\tau)$

Assumes zero-order hold

If A invertible

Linear Discrete Time Control Systems

Inverse between sampled/continuous time representations (LTI)

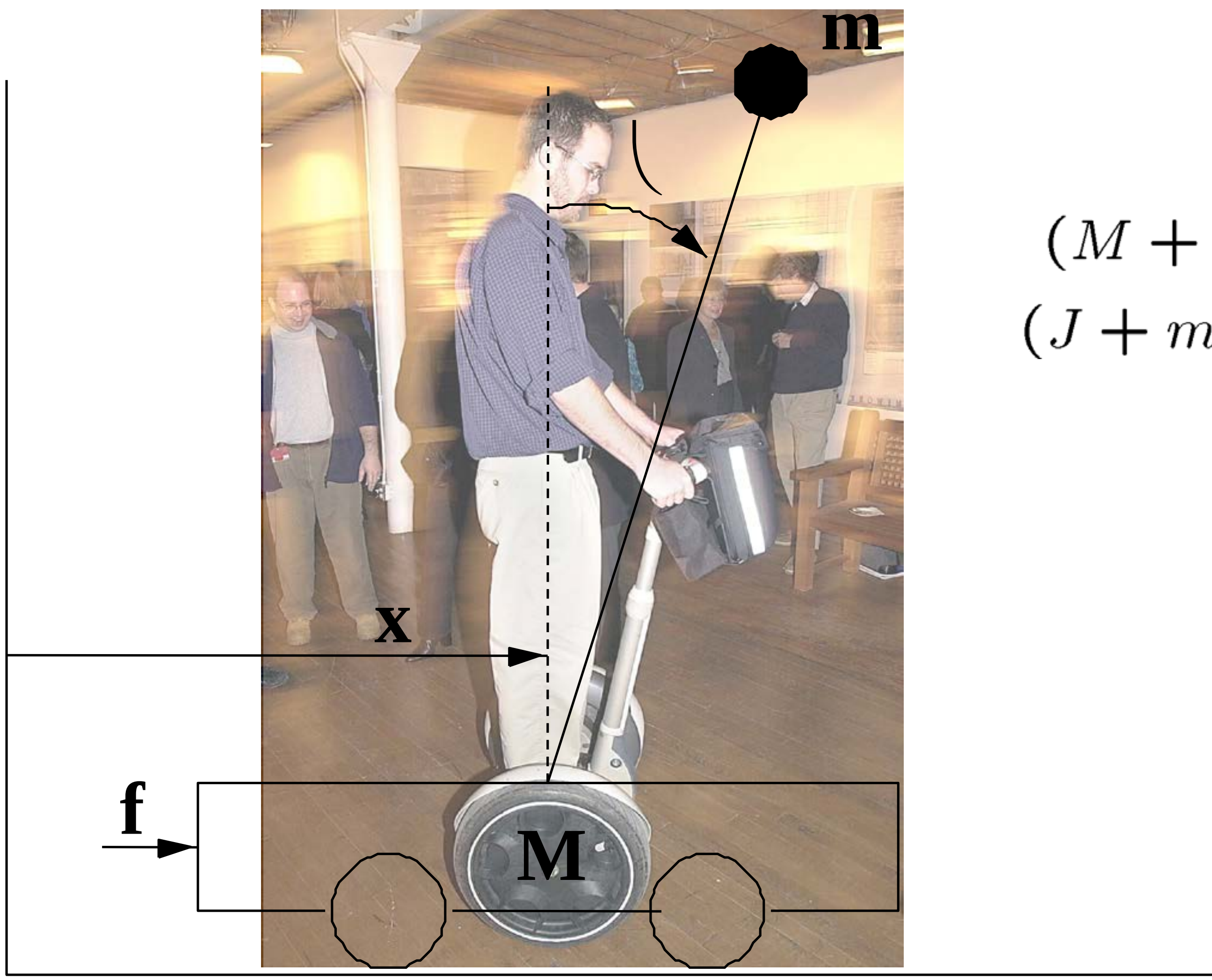
$$- A = \frac{1}{dt}\log(\Phi)\;; \qquad B = \left(\int_0^{dt} e^{As}\, ds\right)^{-1}\Gamma$$

Stability: $x_{k+1} = Ax_k$

- $x_1 = Ax_0; \quad x_2 = Ax_1 = A^2 x_0; \quad \cdots\;; \quad x_n = A^n x_0$
- $\lim_{n\to\infty} A^n x_o = 0$ if $\rho(A) < 1$, where

  $\rho(A) = max\{|\lambda|: \lambda\ is\ an\ eigenvalue\ of\ A\}$

Example: Inverted Pendulum on a Cart

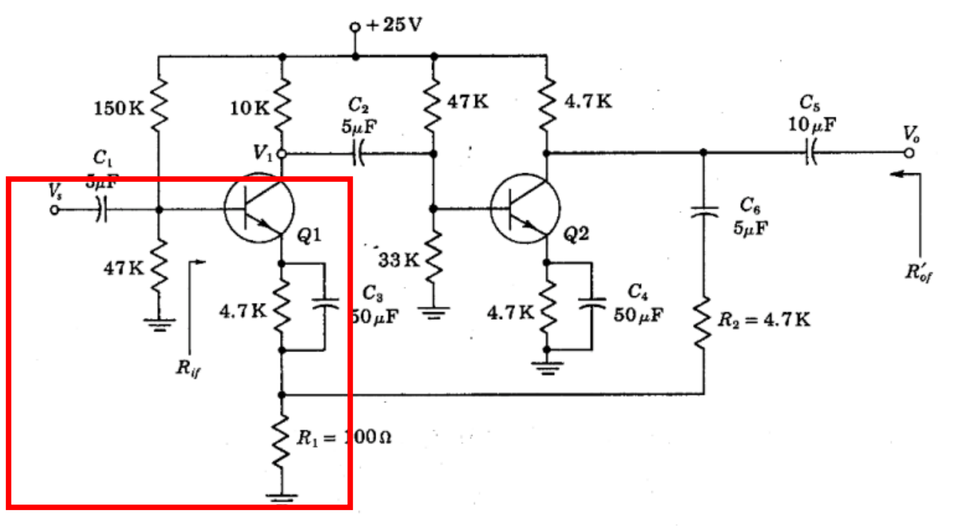


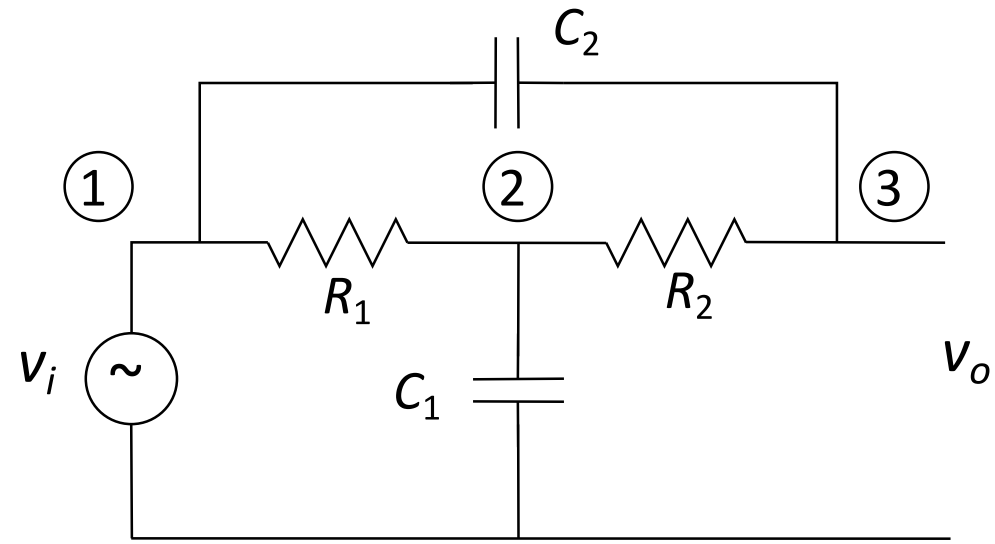
$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = 0$

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example: Electrical Circuit





"Bridged Tee Circuit"

- Derivation based on Kirchoff's laws for electrical circuits (Ph 2)
  - Sum of currents at nodes = 0:

$$C_1 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2 - v_3}{R_2} \qquad C_2 \frac{d(v_3 - v_1)}{dt} = -\frac{v_3 - v_2}{R_2}$$

  - Rewrite in terms of new states: $v_{c1} = v_2$, $v_{c2} = v_3 - v_1$

$$\frac{d}{dt}\begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\ V_{c2} \end{bmatrix} v_i$$

$$v_O = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$