Goals:
• Nyquist Example
• Introduce and review PID control.
• Show how to use “loop shaping” using PID to achieve a performance specification
• Discuss the use of integral feedback and anti-windup compensation

Reading:
• Åström and Murray, Feedback Systems 2-3, Sections 11.1 – 11.3
Nyquist Example (unstable system)

Consider \( L(s) = P(s)C(s) = \frac{k}{s(s-1)} \)

- Pole at the origin, and unstable pole at \( s = -1 \)
- \( \textbf{Q}: \) Does unity gain negative feedback stabilize this system?
- \( \textbf{Q}: \) Does closed loop stability depend upon gain, \( k \)?

Analysis of Closed Loop Poles

- \( G_{yr}(s) = \frac{k}{s^2 - s + k} \Rightarrow \) characteristic equation roots: \( s = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4k} \)
- Closed loop system is \textbf{always} unstable for any \( k \)

Nyquist Plot Analysis

- \textbf{Aside:} magnitude and phase (bode plot) of unstable pole:
  - Let \( H(s) = \frac{1}{s-a} \). \( H(i\omega) = \frac{1}{i\omega-a} \frac{-i\omega-a}{-i\omega-a} = \frac{-i\omega-a}{\omega^2+a^2} \)
  - Magnitude: \( |G(i\omega)| = \frac{\sqrt{\omega^2+a^2}}{\omega^2+a^2} = \frac{1}{\sqrt{\omega^2+a^2}} \)
  - Phase: \( \text{arg}(G(i\omega)) = \arctan \left( \frac{-\omega}{-a} \right) = \pm 180^\circ + \arctan \left( \frac{\omega}{a} \right) \)
Nyquist Example (unstable system)

Bode Plots of Open Loop \( L(s) = P(s)C(s) = \frac{k}{s(s-1)} \)

Nyquist Contour and Plot

- Must account for pole on the \( i\omega \) axis

\[
\begin{align*}
    & a) \ \omega = 0^+ \rightarrow +\infty \\
    & b) \ \omega = +\infty \rightarrow -\infty \\
    & c) \ \omega = -\infty \rightarrow \omega = 0^- \\
    & d) \ \omega = 0^- \rightarrow \omega = 0^+
\end{align*}
\]

\[
\omega = \varepsilon e^{i\phi} \text{ for } [-90^\circ, 90^\circ]
\]

\[
G(s = \varepsilon e^{i\phi}) \approx \frac{k}{-\varepsilon e^{i\phi}} = \frac{ke^{-i\phi}}{-\varepsilon}
\]
Nyquist Example (unstable system)

Nyquist Contour and Plot

\[ (d) \quad \omega = 0^- \rightarrow \omega = 0^+ \]
\[ \omega = \varepsilon e^{i\phi} \text{ for } [-90^\circ, 90^\circ] \]
\[ G(s = \varepsilon e^{i\phi}) \approx \frac{k}{-\varepsilon e^{i\phi}} \]
\[ = \frac{ke^{-i\phi}}{-\varepsilon} = \frac{k}{\varepsilon}(-\cos \phi + i \sin \phi) \]

Accounting:

- One open loop pole in RHP: \( P = 1 \)
- One clockwise encirclement of -1 point: \( N = 1 \)
- \( Z = N + P = 1 + 1 = 2 \) \( \Rightarrow \) two unstable poles in closed loop system

Homework: show that \( \frac{k_1(1+k_2s)}{s(s-1)} \) is stable for \( k_1 k_2 > 1 \)
Overview: PID control

Parametrized by:

- $k_p$, the “proportional gain”
- $k_i$, the “integral gain”
- $k_d$, the “derivative gain”

Intuition

- Proportional term: provides inputs that correct for “current” errors
- Integral term: insures steady state error goes to zero
- Derivative term: provides “anticipation” of upcoming changes (also provides “damping”)
- Controller specified in time domain, but can be analyzed in frequency domain

A bit of history on “three term control”

- First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
Utility of PID

- PID control is most common feedback structure in engineering systems
- For many systems, only need PI or PD (special case)
- Many tools for tuning PID loops and designing gains
Proportional Feedback

Simplest controller choice: $u = k_p e$

- Effect: lifts gain with no change in phase
- Good for plants with low phase up to desired bandwidth
- Bode: shift gain up by factor of $k_p$
- Step response: better steady state error, but with decreasing stability

Steady State error removed by feedforward: $u = k_p e + u_{ff}$
Specify bounds on the loop transfer function to guarantee desired performance.

\[ L(s) = P(s)C(s) \]

\[ H_{er} = \frac{1}{1 + L} \quad H_{yr} = \frac{L}{1 + L} \]

- Steady state error:
  \[ H_{er}(0) = 1/(1 + L(0)) \approx 1/L(0) \]
  \( \Rightarrow \) zero frequency ("DC") gain

- Bandwidth: assuming \( \sim 90^\circ \) phase margin
  \[ \frac{L}{1 + L}(j\omega_c) \approx \left| \frac{1}{1 + j} \right| = \frac{1}{\sqrt{2}} \]
  \( \Rightarrow \) sets crossover freq

- Tracking: \( X\% \) error up to frequency \( \omega_t \) \( \Rightarrow \) determines gain bound (1 + PC > 100/X)
Proportional + Integral Compensation

Use to eliminate steady state error
- Effect: lifts gain at low frequency
- Gives zero steady state error
- Handles modeling error
- Bode: infinite SS gain + phase lag
- Step response: zero steady state error, with smaller settling time, but more overshoot

\[
\omega_z = \frac{K_I}{K_p}
\]

Diagram showing the transfer functions and the output and input graphs with different values of \( k_i \).
Proportional + Integral + Derivative (PID)

C(s) = \( k_p + \frac{k_i}{s} + k_d s \)

\[ = k(1 + \frac{1}{T_i s} + T_d s) \]

\[ = \frac{kT_d}{T_i} \left( s + \frac{1}{T_i} \right) \left( s + \frac{1}{T_d} \right) \]

Derivative Action:

- \( u = k_p e + k_d \dot{e} = k_p \left( e + T_d \frac{de}{dt} \right) = k_p e_p \)
- \( e_p \) is 1st-order (linearized) prediction error at time \( t + T_d \)
- \( T_d \) is the derivative time constant
Implementing Derivative Action

Problems with derivatives
- High frequency noise amplified by derivative term
- Step inputs in reference can cause large inputs

Solution: modified PID control
- Use high frequency rolloff in derivative term
  - 1st-order filter gives finite gain at high frequency
  - use higher order filter if needed
- Don’t feed reference signal through derivative block
  - Useful when reference has unwanted high frequency content
  - Better solution: reference shaping via two DOF design (F(s) block)
- Many other variations (see text + refs)
Choosing PID gains ("tuning")

First order system: $P(s) = \frac{b}{s+a}$

• PI controller has char. poly: $s^2 + (a + bk_p)s + bk_i$
• Closed loop poles can be set arbitrarily:

Second Order System: $P(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0s+\omega_0^2}$

• PID controller allows closed loop poles to be set arbitrarily

Higher Order Systems:
• Use PID to controller a “reduced order” (simplified system)
• Use PID “knobs” to set performance for “dominant” modes

Zeigler-Nichols step response method
• Design PID gains based on step response
• Measure maximum slope + intercept
• Works OK for many plants (but underdamped)
• Maybe useful way to get a first cut controller, especially for higher order, or unknown order

Response to “bump test”
Ziegler-Nichols frequency response method

- Increase proportional gain (with zero derivative and integral gain) until system goes unstable → $k_c$
- Use critical gain and frequency as parameters
- Based on Nyquist plot

Variations

- Modified formulas (see text) give better response
- Relay feedback: provides automated way to obtain critical gain, frequency

\[
k_p = \frac{0.15\tau + 0.35T}{K\tau} \left(\frac{0.9T}{K\tau}\right), \quad k_i = \frac{0.46\tau + 0.02T}{K\tau^2} \left(\frac{0.3T}{K\tau^2}\right),
\]

\[
k_p = \frac{0.22k_c - 0.07}{K} \left(0.4k_c\right), \quad k_i = \frac{0.16k_c}{T_c} + \frac{0.62}{KT_c} \left(\frac{0.5k_c}{T_c}\right).
\]
PID Controllers are easy to implement.

\[ u = -\frac{Z_2}{Z_1} e = -\frac{R_2}{R_1} \frac{(1 + R_1 C_1 s)(1 + R_2 C_2 s)}{R_2 C_2 s} e \]

\[ k_p = \frac{R_1 C_1 + R_2 C_2}{R_1 C_2}, \quad T_i = R_1 C_1 + R_2 C_2, \quad T_d = \frac{R_1 R_2 C_1 C_2}{R_1 C_1 + R_2 C_2}. \]
Example: Cruise Control using PID - Specification

**Performance Specification**

- $\leq 1\%$ steady state error
  - Zero frequency gain $> 100$
- $\leq 10\%$ tracking error up to 10 rad/sec
  - Gain $> 10$ from 0-10 rad/sec
- $\geq 45^\circ$ phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

**Observations**

- Purely proportional gain won’t work: to get gain above desired level will not leave adequate phase margin
- Need to increase the phase from $\sim 0.5$ to 2 rad/sec and increase gain as well
Example: Cruise Control using PID - Design

Approach
- Use integral gain to make steady state error small (zero, in fact)
- Use derivative action to increase phase lead in the cross over region
- Use proportional gain to give desired bandwidth

Controller
- $Ti = 1/0.1; \ Td = 1/1; \ k = 2000$

$$C(s) = \frac{2000s^2 + 1.1s + 0.1}{s} = 2200 + \frac{200}{s} + 2000s$$

Closed loop system
- Very high steady state gain
- Adequate tracking @ 1 rad/sec
- $\sim 80^\circ$ phase margin
- Verify with Nyquist
Example: Cruise Control using PID - Verification

\[ P(s) = \frac{1}{m} \frac{r}{s + \frac{b}{m}} \times \frac{r}{s + a} \]

\[ C(s) = 2000 \frac{s^2 + 1.1s + 0.1}{s} \]

Observations
- **Very fast response** (probably too aggressive)
- Back off on Ti to get something more reasonable
Example: PID cruise control

\[ P(s) = \frac{1}{m} \frac{r}{s + \frac{b}{m}} = \frac{r}{s + \frac{b}{m}} \]

Ziegler-Nichols design for cruise controller
- Plot step response, extract \( \tau \) and \( a \), compute gains

Bode Diagrams

- \( P(s) \)
- \( C(s) \)
- \( L(s) \)

**Step Response**

\[ C(s) = K \left( 1 + \frac{1}{T_i s} + T_D s \right) \]

\[ K = 1.2/a \quad T_i = 2 \times \tau \quad T_d = \tau / 2 \]

- Result: sluggish 
  - increase loop gain 
  - more phase margin (shift zero)
Windup and Anti-Windup Compensation

Problem
- Limited magnitude input (saturation)
- Integrator “winds up” => overshoot

Solution
- Compare commanded input to actual
- Subtract off difference from integrator
Summary: Frequency Domain Design using PID

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking

\[ H_{ue}(s) = K_p + K_I \frac{1}{s} + K_DS \]

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID