CDS 101/110: Lecture 4.2
State Feedback

October 19, 2016

Goals:

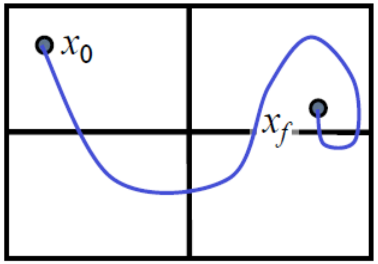
- Review state feedback controllers for linear/linearized systems
- Review reachability of a control system
- Introduce *Reachable Canonical Form*

Reading:

- Åström and Murray, Feedback Systems 2e, Ch 7

Reachability of Input/Output Systems

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = h(x) \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$



Defn An input/output system is *reachable* if for any $x_o, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u_{[0,T]} \in \mathbb{R}$ such that the solution of the dynamics starting from $x(0) = x_0$ and applying input $u(t)$ gives $x(T) = x_f$.

Note: the term "controllable" is also commonly used to describe this concept

Remarks:

- x_0, x_f need not be equilibria, and reachability is independent of output
- For LTI control systems,

$$\dot{x} = Ax + Bu, \qquad x\epsilon\mathbb{R}^n,\ A\epsilon\mathbb{R}^{nxn},\ \ B\epsilon\mathbb{R}^{nxr}, u\epsilon\mathbb{R}^r$$
$$y = Cx + Du, \qquad y\epsilon\mathbb{R}^m,\ A\epsilon\mathbb{R}^{mxn},\ \ B\epsilon\mathbb{R}^{mxr}$$

controllability can be assessed from the rank of:

$$W_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

I.e., if W_r has full rank (rank n), then the system is controllable.

- MATLAB: **ctrb**(A,B) constructs W_r. Test rank with **det**(.) or **rank**(.)

Example

Consider

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$

unstable poles at s=1, s=2

Can we use feedback to stabilize? $u = -Kx = -[k_1\ k_2]x$

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1\ k_2] = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix}$$

Characteristic equation of closed loop system:

$$\det(sI - A) = (s - 1 + k_1)(s - 2) = 0$$

Feedback can modify pole at s=1, but not pole at s=2.

Why? Choose invertible coordinate change z=Tx which diagonalizes A.

$$W_r = [B\ AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Clearly, the second “mode” is not controllable

State space controller design for linear systems

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

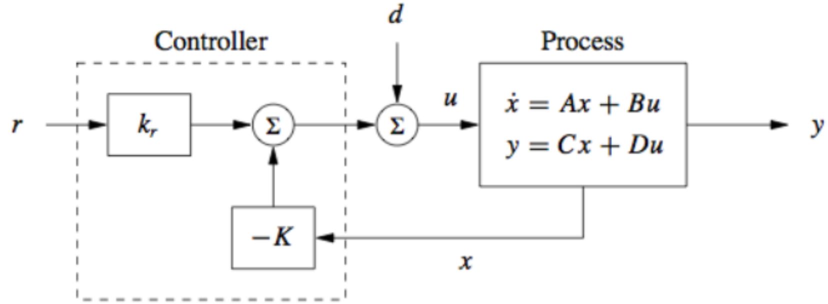
$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Goal: find a linear control law $u = -Kx + k_r r$ such that the closed loop system

$$\dot{x} = Ax + Bu = (A - BK)x + Bk_r r$$

is stable at equilibrium point x_e with $y_e = r$.

k_r only affects steady state gain



Theorem: If (A,B) is reachable, then the eigenvalues of (A-BK) can be set to any desired values.

Proof: (with a few limitations)

- Suppose A, B for single input/output system are in *reachable canonical form:*

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ & 1 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad C = [c_1 \quad c_2 \quad \cdots \quad c_n]$$

Proof: *(continued)*

- Can show that characteristic polynomial of A takes the form:

$$\lambda_A(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

- Let gain matrix, K, take the form:

$$K = [k_1 \quad k_2 \quad \cdots \quad k_{n-1} \quad k_n]$$

- Then

$$(A - BK) = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & \cdots & -a_{n-1} - k_{n-1} & -a_n - k_n \\ 1 & 0 & \cdots & 0 & 0 \\ & 1 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} k_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which yields a closed-loop characteristic polynomial of:

$$\lambda_{(A-BK)}(s) = s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_{n-1} + k_{n-1})s + (a_n + k_n) = 0$$

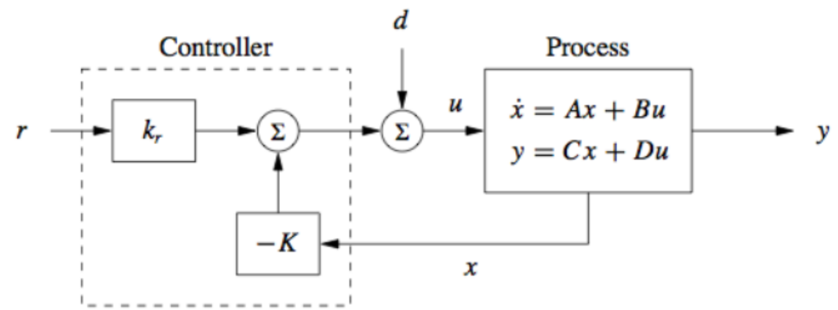
- If *desired* eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then desired characteristic polynomial is:

$$\lambda_d(s) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) = s^n + p_1 s^{n-1} + \cdots + p_n = 0$$

- Choose: $k_1 = p_1 - a_1,\ k_2 = p_2 - a_2,\ \ldots, k_n = p_n - a_n$
  - *Ackermann's method* (1972). MATLAB: *acker*. But *place* is better conditioned.

Reference Trajectory Tracking

- If the desired reference is constant at $r = 0$, then the system is a ***regulator***, and we need only choose K to stabilize. Else, we need to determine k_r.



- If we want *zero reference error* at *zero frequency,* for $r \neq 0$, then:

$$\dot{x} = (A - BK)x + Bk_r r, \; y = Cx \quad \rightarrow$$
$$(A - BK)x_e + Bk_r r = 0, \qquad y_e = Cx_e$$

- $\therefore y_e = -C(A - BK)^{-1}Bk_r r$
- If we want $y_e = r$, then

$$k_r = -\left[C(A - BK)^{-1}B\right]^{-1}$$

Check: Is x_e stable?

- Let $\mathrm{z} = \mathrm{x} - \mathrm{x_e}$. Then

$$\dot{z} = \dot{x} = (A - BK)x + \; Bk_r r$$
$$= (A - BK)(z + x_e) + Bk_r r = (A - BK)z + (A - BK)x_e + Bk_r r$$
$$= (A - BK)z + (A - BK)\left[-(A - BK)^{-1}Bk_r r\right] + Bk_r r = (A - BK)z$$

Converting to Reachable Canonical Form

Given: $\dot{x} = Ax + Bu, \quad y = Cx + Du,$ how do we convert to reachable canonical form?

- Introduce a coordinate transformation: $z = Tx$, with T invertible

$$\dot{z} = T\dot{x} = T(Ax + Bu) = \underbrace{TAT^{-1}}z + \underbrace{TB}_{\tilde{B}}u$$

- **Q:** Can we find T such that $\tilde{A}$ is in reachable canonical form?

$$\widetilde{W}_r = [\tilde{B} \quad \tilde{A}\tilde{B} \quad \cdots \quad \tilde{A}^{n-1}\tilde{B}] = \underbrace{\begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 & \cdots & * \\ 0 & 1 & -a_1 & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\text{Known from } \lambda_A(s) = s^n + a_1 s^{n-1} + \cdots + a_n}$$

- $\tilde{B} = TB, \tilde{A}\tilde{B} = TAT^{-1}TB = TAB$
- $\tilde{A}^2\tilde{B} = TA^2B$
- $\rightarrow \widetilde{W}_r = T[B \quad AB \quad \cdots \quad A^{n-1}B]$
  $= TW_r$
- $\rightarrow T = \widetilde{W}_r W_r^{-1}$
- **Note:** T exists if (A,B) reachable. Hence, any reachable system can be converted
  This proves that eigenvalue placement is possible for any reachable system

Implementation Details

How to choose closed loop Eigenvalues?

- Find coordinate transformation of A such that

Real eigenvalues

Complex conjugate eigenvalues

Jordan Block

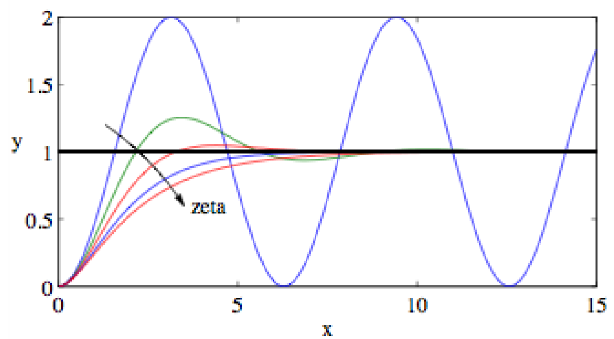
$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & \omega_3 & 0 & 0 \\ 0 & 0 & -\omega_3 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

- I.e., open loop system is a collection of first order systems, second order systems, and possibly Jordan block systems. Then design feedback (i.e., assignment of closed loop eigenvalues) for 1st and 2nd order systems (and possibly Jordan block)

For 2nd Order System

- Each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get a contribution of the form

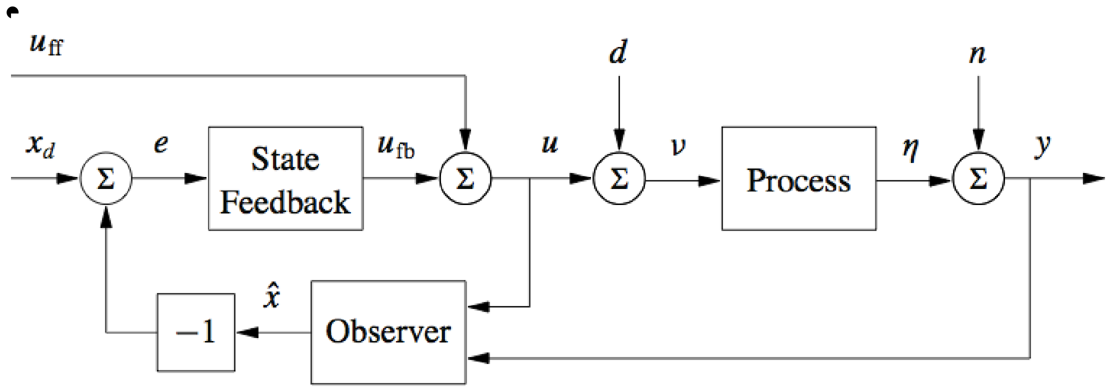
$$y_i(t) = e^{-\sigma_i t}\left(a\sin(\omega_i t) + b\cos(\omega_i t)\right)$$



- $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0, \quad \lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$

Implementation Details

Use *observer* to determine the current state if you can't measure it



- Estimator looks at inputs and outputs of plant and estimates the current state
- Can show that if a system is *observable* then you can construct and estimator
- Use the *estimated* state as the feedback

$$u = K\hat{x}$$

- Next week: basic theory of state estimation and observability
- CDS 112: *Kalman filtering* and theory of optimal observers

Example #2: Predator prey

(From FBS Section 4.7)

(growth rate)

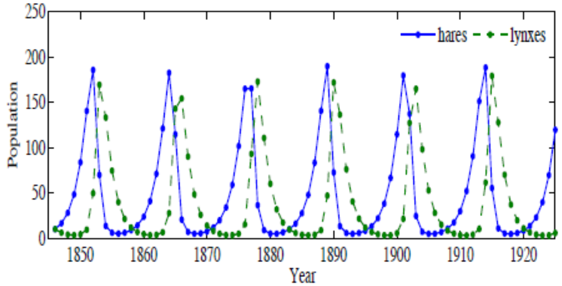
System dynamics

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k}\right) - \frac{aHL}{c+H}, \qquad H \geq 0,$$

$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0.$$

(prey consumption rate)

- Stable limit cycle with unstable equilibrium point at $H_e = 20.6$, $L_e = 29.5$
- Can we design the dynamics of the system by modulating the food supply ("u" in "$r+u$" term)

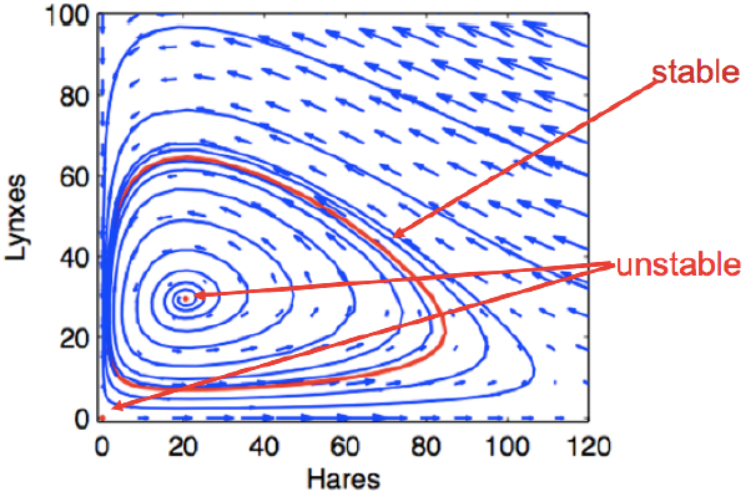


Q1: can we move from a given initial population of lynxes and rabbits to a specified one in time T by modulation of the food supply?

Q2: can we stabilize the lynx population around a desired equilibrium point (eg, L_d = ~30)?

- Try to keep lynx and hare population in check

Approach: try to stabilize using state feedback law



Example #2: Problem setup

Equilibrium point calculation

$$\frac{dH}{dt} = (r+u)H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL$$

- x_e = (20.6, 29.5), u_e = 0, L_e = 29.5

```
f = inline('predprey(0, x)', 'x');
xeq = fsolve(f, [20, 30])'; He = xeq(1); Le = xeq(2);

% Generate the linearization around the eq point
App = [
  -((a*c*k*Le + (c + He)^2*(2*He - k)*r)/((c + He)^2*
  (a*b*c*Le)/(c + He)^2, -d + (a*b*He)/(c + He)
];
Bpp = [He*(1 - He/k); 0];

% Check reachability
if (det(ctrb(App, Bpp)) ~= 0) disp "reachable"; end
```

Linearization

- Compute linearization around equilibrium point, x_e:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} \qquad \frac{dx}{dt} \approx A(x - x_e) + B(u - u_e) + \text{higher order terms}$$

- Redefine local variables: $z = x - x_e$, $v = u - u_e$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{acL_e}{(c+H_e)^2} - \frac{2H_e r}{k} + r & -\frac{aH_e}{c+H_e} \\ \frac{abcL_e}{(c+H_e)^2} & \frac{abH_e}{c+H_e} - d \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_e\left(1 - \frac{H_e}{k}\right) \\ 0 \end{bmatrix} v$$

- Reachable? YES, if a, $b \neq 0$ (check [B AB]) ⇒ can locally steer to any point

Example #2: Stabilization via eigenvalue assignment

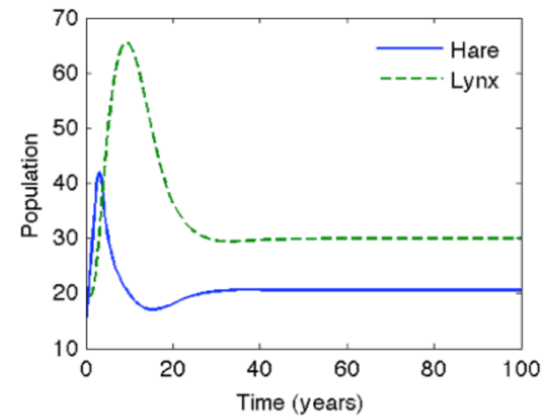
$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{acL_e}{(c+H_e)^2} - \frac{2H_e r}{k} + r & -\frac{aH_e}{c+H_e} \\ \frac{abcL_e}{(c+H_e)^2} & \frac{abH_e}{c+H_e} - d \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_e\left(1 - \frac{H_e}{k}\right) \\ 0 \end{bmatrix} v$$

Control design:

$$v = -Kz = -k_1(H - H_e) - k_2(L - L_e)$$
$$u = u_e + K(x - x_e)$$

Place poles at stable values

- Choose λ = -0.1, -0.2
- MATLAB: Kpp = place(App, Bpp, [-0.1; -0.2]);



Key principle: *design of dynamics*

- Use feedback to create a stable equilibrium point

More advanced: control to desired value $r = L_d$