ME/CS 133(a): Solution to Homework #3

Problem 1: (Problem 6(a,b,d) in Chapter 2 of MLS).

Part (a): Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{p})$ be unit quaternions—i.e., $QQ^* = PP^* = 1$. The set of unit quaternions is a group if you can show that: (i) multiplication is associative; (ii) the product of group elements yields a group element; (iii) the set contains an identity element; (iv) every group element has an inverse element, and the inverse is in the group.

(i) Since quaternions themselves are a group, multiplication between them is associative. So, for any quaternions R, S, and T, $R(ST) = (RS)T$. Then, in particular, for unit quaternions this property must hold.

Alternatively, one show also associativity directly, by showing that for 3 unit quaternions Q, P, R, we have $P(QR) = (PQ)R$; note that this holds for quaternions of any magnitude, as we do not have to use that they are unit quaternions in the proof. This method involves applying the quaternion multiplication formula, $QP = (q_0p_0 - \vec{q}\cdot\vec{p}, q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})$, twice in evaluating each side, and then using the vector identities $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ and $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$. Full mathematical details are not shown here, but via these identities, one can show that $P(QR) = (PQ)R = (r_0p_0q_0 - r_0(\vec{q}\cdot\vec{p}) - q_0(\vec{r}\cdot\vec{p}) - p_0(\vec{r}\cdot\vec{q}) - \vec{r}\cdot(\vec{q}\times\vec{p}), r_0(q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p}) + (q_0p_0 - \vec{q}\cdot\vec{p})\vec{r} + \vec{r}\times(q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p}))$.

(ii) The product of unit quaternions, QP, is a unit quaternion if and only if $QP(QP)^* = 1$. We will show that $(QP)^* = P^*Q^*$. From here, it follows that $QP(QP)^* = QPP^*Q^* = QQ^* = 1$, using associativity of quaternion multiplication and that Q, P are both unit quaternions.

Now, we just need to prove that $(QP)^* = P^*Q^*$. We demonstrate this via quaternion multiplication and using that for any quaternion Q, $Q^* = (q_0, -\vec{q})$:

$$
\begin{aligned}
QP &= (q_0p_0 - \vec{q}\cdot\vec{p}, q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p}) \\
(QP)^* &= (q_0p_0 - \vec{q}\cdot\vec{p}, -q_0\vec{p} - p_0\vec{q} - \vec{q}\times\vec{p}) \\
P^*Q^* &= (p_0, -\vec{p})\cdot(q_0, -\vec{q}) = (q_0p_0 - (-\vec{p})\cdot(\vec{q}), q_0(-\vec{p}) + p_0(-\vec{q}) + (-\vec{q}\times -\vec{p})) \\
&= (q_0p_0 - \vec{q}\cdot\vec{p}, -q_0\vec{p} - p_0\vec{q} - \vec{q}\times\vec{p}) \\
&= (QP)^*
\end{aligned}
$$

(iii) The identity quaternion is: $e = (1, 0, 0, 0) = 1 + 0\cdot i + 0\cdot j + 0\cdot k$. This is a unit quaternion, as $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

(iv) The inverse of any unit quaternion Q is Q^*, which is also a unit quaternion, since $Q^*(Q^*)^* = Q^*Q = (QQ^*)^* = 1^* = 1$.

Part (b): If a unit quaternion, Q, has real part q_0 and vector part $\vec{q}$, and $\vec{x} = [x_1\ x_2\ x_3]^T$ is represented as a pure quaternion $X = (0, x_1, x_2, x_3) = 0 + \vec{x}$, then:

$$XQ^{-1} = (\vec{x} \cdot \vec{q}, q_0\vec{x} - (\vec{x} \times \vec{q})),$$

where $\vec{x} \cdot \vec{q}$ is the real part and $q_0\vec{x} - (\vec{x} \times \vec{q})$ is the vector part.

Similarly, the product QXQ^{-1} is:

$$QXQ^{-1} = (q_0(\vec{x} \cdot \vec{q}) - \vec{q} \cdot (q_0\vec{x} - \vec{x} \times \vec{q}), q_0(q_0\vec{x} - \vec{x} \times \vec{q}) + (\vec{x} \cdot \vec{q})\vec{q} + \vec{q} \times (q_0\vec{x} - \vec{x} \times \vec{q}))$$

The real part of QXQ^{-1} is:

$$(\vec{x} \cdot \vec{q})q_0 - \vec{q} \cdot [q_0\vec{x} - (\vec{x} \times \vec{q})] = q_0(\vec{x} \cdot \vec{q}) - q_0(\vec{x} \cdot \vec{q}) + \vec{q} \cdot (\vec{x} \times \vec{q}) = 0,$$

where the equality comes from the identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$.

Thus QXQ^{-1} is a pure quaternion when X is.

The vector part of QXQ^{-1} is:

$$\begin{aligned} q_0(q_0\vec{x} - \vec{x} \times \vec{q}) &+ (\vec{x} \cdot \vec{q})\vec{q} + \vec{q} \times (q_0\vec{x} - \vec{x} \times \vec{q}) \\ &= {q_0}^2\vec{x} - q_0(\vec{x} \times \vec{q}) + (\vec{x} \cdot \vec{q})\vec{q} + q_0(\vec{q} \times \vec{x}) - \vec{q} \times (\vec{x} \times \vec{q}) \\ &= q_0^2\vec{x} - 2q_0(\vec{x} \times \vec{q}) + (\vec{x} \cdot \vec{q})\vec{q} - [(\vec{q} \cdot \vec{q})\vec{x} - (\vec{x} \cdot \vec{q})\vec{q}] \\ &= [q_0^2 - (\vec{q} \cdot \vec{q})]\vec{x} + 2[(\vec{x} \cdot \vec{q})\vec{q} + q_0(\vec{q} \times \vec{x})] \end{aligned}$$

where we have used the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Next, we need to verify that the vector part of QXQ^* describes the point to which $\vec{x}$ is rotated under the rotation associated with Q.

Firstly, we will show that the vector part of QXQ^* can be rewritten as $\vec{x} + 2[\vec{q} \times (\vec{q} \times \vec{x}) + q_0(\vec{q} \times \vec{x})]$. That is, we would like to show:

$$\vec{x} + 2[\vec{q} \times (\vec{q} \times \vec{x}) + q_0(\vec{q} \times \vec{x})] = [q_0^2 - (\vec{q} \cdot \vec{q})]\vec{x} + 2[(\vec{x} \cdot \vec{q})\vec{q} + q_0(\vec{q} \times \vec{x})] \tag{1}$$

We cancel $2q_0(\vec{q} \times \vec{x})$ from both sides and use the fact that $q_0^2 = 1 - \vec{q} \cdot \vec{q}$, which comes from the fact that Q is a unit quaternion. This gives us that Equation (1) is equivalent to the following:

$$\vec{x} + 2[\vec{q} \times (\vec{q} \times \vec{x})] = [1 - 2(\vec{q} \cdot \vec{q})]\vec{x} + 2[(\vec{x} \cdot \vec{q})\vec{q}]$$

Canceling $\vec{x}$ from both sides and dividing by 2, we find that this is equivalent to:

$$\vec{q} \times (\vec{q} \times \vec{x}) = -(\vec{q} \cdot \vec{q})\vec{x} + (\vec{x} \cdot \vec{q})\vec{q}$$

From the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$, we know that this is a true statement for any vectors $\vec{x}$ and $\vec{q}$, thus proving the desired statement.

Now, we would like to show that the vector expression $\vec{v} := \vec{x} + 2[\vec{q} \times (\vec{q} \times \vec{x}) + q_0(\vec{q} \times \vec{x})]$ is equal to $R\vec{x}$, where R is the rotation matrix associated with the quaternion Q. We will transform first from quaternion coordinates to the equivalent angle-axis coordinates $(\theta, \vec{\omega})$, and then from angle-axis coordinates to the corresponding rotation matrix R.

First, recall from page 34 of MLS that $Q = (q_0, \vec{q}) = \big(\cos(\frac{\theta}{2}), \vec{\omega} \sin \frac{\theta}{2}\big)$. This gives us that $\theta = \pm 2 \arccos(q_0)$, and that $\vec{\omega} = \frac{\vec{q}}{\sin(\frac{\theta}{2})}$ if $\theta \neq 0$ and $\vec{\omega} = 0$ otherwise. Since both possible $(\theta, \vec{\omega})$ pairs give the same equivalent rotation R, without loss of generality, we can pick $\theta = 2 \arccos(q_0)$.

We can ignore the case in which $\theta = 0$, since then $\vec{\omega} = \vec{q} = 0$, and it is clear that $R = I$, $R\vec{x} = \vec{x}$, and the vector part of QXQ^* is $\vec{x}$.

Next, we transform from angle-axis coordinates $(\theta, \vec{\omega})$ to $R = e^{\theta\hat{\omega}}$. We can do this using Rodrigues' formula, $R = e^{\theta\hat{\omega}} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$. Note that this form of Rodrigues' formula is valid when $\vec{\omega}$ is a unit vector; here we can confirm that this is indeed the case:

$$||\vec{\omega}||^2 = \frac{||\vec{q}||^2}{\sin^2(\frac{\theta}{2})} = \frac{1 - q_0^2}{\sin^2(\frac{\theta}{2})} = \frac{1 - \cos^2(\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})} = \frac{\sin^2(\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})} = 1$$

At this point, we have developed expressions to calculate θ and $\vec{\omega}$ from Q, and to calculate R from θ and $\vec{\omega}$. Showing that $\vec{v} = R\vec{x}$ by hand is doable, but involves tedious algebra. The attached Matlab code demonstrates the usage of Matlab's symbolic toolbox to show that the two expressions are equivalent (using Mathematica would also work well here).

```
% Code for ME/CS 133 Homework 3 problem 1b.

clear; clc; close all;

syms q0 q1 q2 q3 theta x1 x2 x3     % Declare symbolic variables

% Define vectors x and q, where q is the vector portion of the
 quaterion:
x = [x1; x2; x3];
q = [q1; q2; q3];

% Define theta and w_hat in term of elements of quaternion Q:
theta = @(q0)(2*acos(q0));

w_hat = @(q1, q2, q3, theta)([0, -q3, q2; q3, 0, -q1; -q2, q1,
 0] ./ ...
    sin(theta/2));

% Use Rodriguez' formula to transform from theta and w_hat to a
 rotation
% matrix R. Note that we can use the form of the equation for w being
 a
% unit vector, since we have shown that this is the case.
R = @(w_hat, theta)(eye(3) + w_hat .* sin(theta) + ...
    w_hat^2 .* (1 - cos(theta)));

% Rotate point x by rotation R:
fprintf('x under rotation R:\n');

% Evaluate R in terms of elements of Q.
w_hat_eval = w_hat(q1, q2, q3, theta(q0));
R_eval = R(w_hat_eval, theta(q0));

% Simplify and print out the answer:
R_eval = simplify(R_eval * x);
collect(R_eval, [x1, x2, x3])

% Vector expression for vector portion of QXQ*:
v = x + 2 .* (cross(q, cross(q, x)) + q0 .* cross(q, x));

fprintf('Vector portion of QXQ*:\n');
collect(v, [x1, x2, x3])
```

```
x under rotation R:

ans =

 (- 2*q2^2 - 2*q3^2 + 1)*x1 + (2*q1*q2 - 2*q0*q3)*x2 + (2*q0*q2 +
 2*q1*q3)*x3
 (2*q0*q3 + 2*q1*q2)*x1 + (- 2*q1^2 - 2*q3^2 + 1)*x2 + (2*q2*q3 -
 2*q0*q1)*x3
 (2*q1*q3 - 2*q0*q2)*x1 + (2*q0*q1 + 2*q2*q3)*x2 + (- 2*q1^2 - 2*q2^2
 + 1)*x3
```

Vector portion of QXQ:*

```
ans =

 (- 2*q2^2 - 2*q3^2 + 1)*x1 + (2*q1*q2 - 2*q0*q3)*x2 + (2*q0*q2 +
 2*q1*q3)*x3
 (2*q0*q3 + 2*q1*q2)*x1 + (- 2*q1^2 - 2*q3^2 + 1)*x2 + (2*q2*q3 -
 2*q0*q1)*x3
 (2*q1*q3 - 2*q0*q2)*x1 + (2*q0*q1 + 2*q2*q3)*x2 + (- 2*q1^2 - 2*q2^2
 + 1)*x3
```

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Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix A_1A_2 requires 3 multiplications and 2 additions. Hence, the product A_1A_2 requires a total of 27 multiplications and 18 additions.

(ii) Let Q_1 and Q_2 be quaternions, with respective real and vector parts q_{10}, q_{20} and $\vec{q}_1$, $\vec{q}_2$. The real part of the quaternion product, $q_{10}q_{20} - \vec{q}_1 \cdot \vec{q}_2$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The vector part, $\vec{q}_3 = q_{10}\vec{q}_2 + q_{20}\vec{q}_1 + \vec{q}_1 \times \vec{q}_2$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions, since the evaluation of each element of the resultant vector requires 3 multiplications and 2 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_0^2 + \vec{q} \cdot \vec{q}$, it is possible to show that the vector part of QXQ^{-1} in part (b) above can be rearranged (shown in part b) to the form:

$$\vec{x} + 2[\vec{q} \times (\vec{q} \times \vec{x}) + q_0(\vec{q} \times \vec{x})]$$

Since $\vec{q} \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Problem 2: We can use the "particle counting" argument that was used in class during the discussion of planar kinematics. Let's solve the problem for rigid bodies moving in an n-dimensional space. Then we can specialize to the case $n = 3$ (3-dimensional Euclidean space).

The particles that make up a rigid body in an n-dimensional Euclidean space each have n degrees-of-freedom (DOF) when they are not constrained to be in a rigid body. The key thing to recognize is that the number of constraints needed to join the particles to make a rigid body. For particles in an n-dimensional space, the total number of DOF for N particles, which are not constrained to be a rigid body, is nN. The first particle, P_1, has no constraints on its motion. Particle P_2 has one constraint on its location to be joined to the rigid body, etc. Partial P_n has $(n-1)$ constraints. Particles $P_{n+1}, \ldots, P_N$ have n constraints. So, the total DOF has of the rigid body is the sum of the DOF of all particles without constraints, minus the number of constraints:

$$Nn \;-\; [(N-n)n + \sum_{i=1}^{n}(n-i)] \;=\; n^2 - \sum_{i=1}^{n}(n-i) = n^2 \;-\frac{1}{2}(n^2-n) = \frac{1}{2}(n^2+n)$$

For $n = 3$, we get the answer:

$$\frac{1}{2}(3^2 + 3) = 6.$$

Problem # 3: (Problem 11(a,b,d) in Chapter 2 of the MLS text)

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

To show that the exponential of the twist is in $SE(2)$, we must show that $e^{\phi\hat{\xi}}$ takes the homogeneous form of a displacement, i.e.

$$e^{\phi\hat{\xi}} = \begin{bmatrix} R & p \\ \vec{0} & 1 \end{bmatrix},$$

where R is a rotation matrix and p is any vector in $\mathbb{R}^2$.

First, let's consider the case of $\xi = (v, \omega)$ with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus, $\hat{\xi}^n = 0$ for $n \geq 2$, and

$$e^{\phi\hat{\xi}} = I + \phi\hat{\xi} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$ by the appropriate scaling of ϕ. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let us define a new twist, $\hat{\xi}'$:

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}^{'}$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}^{'}} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}^{'}} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): In the pure translation case, we know from part (a) that $w = 0$ and the twist matrix takes the form

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}.$$

.

Then, $(\hat{\xi})^\vee = [v_x v_y 0]^T$.

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as a coordinate transformation of a twist $\xi^{'} = (0, 0, \omega)$—which is pure rotation—by a transformation h, which is pure translation by $\vec{q}$. $\xi^{'}$ is a pure rotation about the origin, and then we translate using the displacement matrix h:

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix}$$

Then, $\hat{\xi} = h\hat{\xi}^{'}h^{-1}$. Note that $e^{\hat{\xi}\phi} = e^{h\hat{\xi}^{'}h^{-1}\phi} = he^{\hat{\xi}^{'}\phi}h^{-1}$. Thus, the twist matrix $\hat{\xi}^{'}$ corresponds to a rotation in a coordinate frame with origin at $\vec{q}$, while multiplication by h and h^{-1} corresponds to observing this rotation at the global origin.

$$\xi = \mathrm{Ad}_h \xi^{'} = (h\hat{\xi}^{'}h^{-1})^\vee, \tag{2}$$

where $\hat{\xi}^{'} = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}$. Expanding Eq. (2) gives:

$$\xi = (h\hat{\xi}^{'}h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} \omega q_y \\ -\omega q_x \\ \omega \end{bmatrix} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix},$$

where the last step assumes that $\omega = 1$.

Part (d): Let

$$g = \begin{bmatrix} A & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $A \in SO(2)$ and $\vec{p} \in \mathbb{R}^2$. Then direct calculation shows that $\dot{g}g^{-1}$ and $g^{-1}\dot{g}$ are twists:

$$\hat{V}^s = \dot{g}g^{-1} = \begin{bmatrix} \dot{A} & \dot{\vec{p}} \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} A^T & -A^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \dot{A}A^T & \dot{\vec{p}} - \dot{A}A^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix}$$

To finish showing that $\hat{V}^s$ is a twist, we must show that $\dot{A}A^T$ is skew-symmetric. This can be shown as in lecture:

$$AA^T = I \rightarrow \dot{A}^T A + A^T \dot{A} = 0 \rightarrow A^T \dot{A} = -(\dot{A}^T A) = -(A^T \dot{A})^T$$

Similarly,

$$\hat{V}^b = g^{-1}\dot{g} = \begin{bmatrix} A^T & -A^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \dot{A} & \dot{\vec{p}} \\ \vec{0}^T & 0 \end{bmatrix} = \begin{bmatrix} A^T\dot{A} & A^T\dot{\vec{p}} \\ \vec{0}^T & 0 \end{bmatrix},$$

Similarly, we can show that $A^T\dot{A}$ is skew-symmetric by differentiating both sides of $A^T A = I$.

The spatial and body velocities have definitions analogous to those for 3-dimensional rigid bodies.

Problem 4:

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. To show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

form a basis for $SU(2)$, let A, B, C, and D be real numbers. Then, the matrix formed by the product of A, B, C,and D with these matrices is:

$$A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + B\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + C\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + D\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} A+iB & C+iD \\ C-iD & A-iB \end{bmatrix}.$$

This is a matrix in $SU(2)$ for any choice of A, B, C, and D where $A^2 + B^2 + C^2 + D^2 = 1$. Any matrix in $SU(2)$ can be written as a linear combination of these 4 matrices in this way, and thus they span $SU(2)$. The 4 matrices are also linearly independent (none can be written as a linear combination of the others), and thus they form a basis for $SU(2)$.

Thus these four basis matrices for $SU(2)$ are in 1-to-1 correspondence with the 1, i, j, and k basis elements for the quaternions. Thus, the scalar elements A, B, C, and D are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a unit quaternion be represented by $q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The correspondence is then:

$$\lambda_1 = A = Re(z) = \frac{z + z^*}{2} \tag{3}$$

$$\lambda_2 = B = Im(z) = \frac{i(z^* - z)}{2} \tag{4}$$

$$\lambda_3 = C = Re(w) = \frac{w + w^*}{2} \tag{5}$$

$$\lambda_4 = D = Im(w) = \frac{i(w^* - w)}{2} \tag{6}$$

Part (b): The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\cos\frac{\phi}{2}, \omega_x \sin\frac{\phi}{2}, \omega_y \sin\frac{\phi}{2}, \omega_z \sin\frac{\phi}{2})$, where ϕ is the rotation about an axis represented by a unit vector $\vec{\omega} = [\omega_x\ \omega_y\ \omega_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes.

Consider a rotation about the x-axis of amount γ. Since $\vec{w} = [1,\ 0,\ 0]^T$, this is represented by the quaternion in which $\lambda_1 = \cos\frac{\gamma}{2}$, $\lambda_2 = \sin\frac{\gamma}{2}$, $\lambda_3 = \lambda_4 = 0$:

$$M_x = \begin{bmatrix} (\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2}) & 0 \\ 0 & (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}.$$

Similarly, a rotation of amount ϕ about the y-axis has $\lambda_1 = \cos\frac{\phi}{2}$, $\lambda_2 = 0$,$\lambda_3 = \sin\frac{\phi}{2}$, $\lambda_4 = 0$. This can be represented as:

$$M_y = \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

Finally, a rotation of amount ψ about the z-axis has $\lambda_1 = \cos\frac{\psi}{2}$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = \sin\frac{\psi}{2}$, and can be represented as:

$$M_z = \begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

The product of these matrices, $M_z M_y M_x$, yields the result.