ME/CS 133(a): The Classical Matrix Groups

The notes provide a brief review of *matrix groups*, with a particular focus on the "classical" matrix groups. The primary goal is to motivate the language and symbols used to represent rotations ($\mathbb{SO}(2)$ and $\mathbb{SO}(3)$) and spatial displacements ($\mathbb{SE}(2)$ and $\mathbb{SE}(3)$).

1 Groups

Definition 1 *: A group, G, is a mathematical structure with the following characteristics and properties:*

i. the group consists of a set of elements $\{g_j\}$ which can be indexed. The indices j may form a finite, countably infinite, or continous (uncountably infinite) set.

ii. An associative binary group operation, denoted by $'*'$, termed the group product*. The product of two group elements is also a group element:*

$$\forall\ g_i, g_j\ \in\ G \qquad g_i\ *\ g_j\ =\ g_k, \quad \text{where } g_k\ \in\ G.$$

The associativity of the group operation implies that $(g_i\ *\ g_j)\ *\ g_k\ =\ g_i\ *\ (g_j\ *\ g_k)$.

iii. A unique group identify element, e, with the property that: $e\ *\ g_j\ =\ g_j$ for all $g_j\ \in G$.

iv. For every $g_j\ \in\ G$, there must exist an inverse element, g_j^{-1}, such that

$$g_j^{-1}\ *\ g_j\ =\ e.$$

Note that the above definition introduces the identity e as a *left identity* (i.e., the identity multiplies a group element on the left). Similarly, the inverse of group element g_i was defined as a *left inverse*, where the inverse element multiplies the group element on the left. The group definition can be used to show that e is also a right identity (i.e., $e * g\ =\ g * e\ =\ g$) and g^{-1} is a right inverse of g ($g^{-1}\ *\ g\ =\ g\ *\ g^{-1}\ =\ e$).

Proof: (that the left inverse g^{-1} is also a right inverse)

$$g^{-1} = e * g^{-1} = (g^{-1} * g) * g^{-1} = g^{-1} * (g * g^{-1}) \qquad (1)$$

Next note that by the definition of the left inverse introduced above

$$e = (g^{-1})^{-1} * g^{-1}\ .$$

Substitute from Equation (1) the expression for g^{-1}, and then simplify:

$$e = (g^{-1})^{-1} * g^{-1} * (g * g^{-1}) = ((g^{-1})^{-1} * g^{-1}) * (g * g^{-1}) = e * (g * g^{-1}) = g * g^{-1}$$

where the last equality arises from the left identity definition of e. Since we have shown that $e = g * g^{-1}$, then g^{-1} must also be a *right inverse* of g.

Proof: (that the left identity e is also a right identity)

$$g = e * g = (g * g^{-1}) * g = g * (g^{-1} * g) = g * e$$

where the second equality used the just-proved relationship that $e = g * g^{-1}$. Hence, e is also a right identify.

Simple examples of groups include the integers, $\mathbb{Z}$, with addition as the group operation, and the real numbers mod zero, $\mathbb{R} - \{0\}$, with multiplication as the group operation.

1.1 The General Linear Group, $GL(N)$

The set of all $N \times N$ invertible matrices with the group operation of matrix multiplication forms the *General Linear Group* of dimension N. This group is denoted by the symbol $GL(N)$, or $GL(N, \mathbb{K})$ where $\mathbb{K}$ is a field, such as $\mathbb{R}$, $\mathbb{C}$, etc. Generally, we will only consider the cases where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, which are respectively denoted by $GL(N, \mathbb{R})$ and $GL(N, \mathbb{C})$. By default, the notation $GL(N)$ refers to real matrices; i.e., $GL(N) = GL(N, \mathbb{R})$.

The identity element of $GL(N)$is the identify matrix, and the inverse elements are clearly just the matrix inverses. If matrix A is invertible (implying that $det(A) \neq 0$), then matrix A^{-1} is invertible as well. Note that the product of invertible matrices is necessarily invertible. This can be shown as follows. If matrices A and B are invertible (i.e. $A, B \in GL(N)$), then $det(A) \neq 0$ and $det(B) \neq 0$. Hence, $det(AB) = det(A)\ det(B) \neq 0$. Similarly, $det[(AB)^{-1}] = det[A^{-1}]\ det[B^{-1}] = (1/det(A))\ (1/det(B)) \neq 0$. Thus, a matrix which is formed from the product of two invertible matrices is invertible and in $GL(N)$.

2 Subgroups

A subgroup, H, of G (denoted $H \subseteq G$) is a subset of G which is itself a group under the group operation of G. Note that this subgroup must contain the identity element.

The General Linear Group has several important subgroups, which as a family make up the *Classical Matrix Subgroups.*

2.1 The Classical Matrix Subgroups

The Special Linear Group, $\mathbb{SL}(N)$, consists of all members of $GL(N)$ whose determinant has a value of +1. To see that this set of matrices forms a group, note that if $A, B \in SL(N)$, then to show that $A * B \in SL(N)$, note that $det(AB) = det(A) \cdot det(B) = 1 \cdot 1 = 1$. Also, for any $A \in SL(N)$, $det(A^{-1}) = [det(A)]^{-1} = [1]^{-1} = 1$, so that every inverse is a member of $\mathbb{SL}(N)$.

The Orthogonal Group, $\mathbb{O}(N)$, consists of all real $N \times N$ matrices with the property that:

$$A^T\ A\ =\ I \qquad \text{for all } A\ \in\ \mathbb{O}(N)$$

(Note that this relationship and the group properties also implies that for any $A\ \in\ \mathbb{O}(N)$, $A\ A^T\ =\ I$ as well). As described in class, the group $\mathbb{O}(N)$ can represent spherical displacements in N-dimensional Euclidean space. To check that $\mathbb{O}(N)$ forms a group, note that:

- The product of two orthogonal matrices is an orthogonal matrix. Let $A, B\ \in\ \mathbb{O}(N)$. Then: $(AB)^T(AB)\ =\ B^T\ A^T\ A\ B\ =\ B^T\ B\ =\ I$, and thus the product AB is orthogonal.
- Recall that the inverse of an orthogonal matrix is the same as its transpose: $A^T\ =\ A^{-1}$ for all $A\ \in\ \mathbb{O}(N)$. Thus, since $A^T A\ =\ I$ for othogonal matrices, it is also true that the inverse of A, A^{-1}, is an orthogonal matrix: $[A^{-1}]^T\ A^{-1}\ =\ [A^T]^T\ A^T\ =\ A\ A^T\ =\ I.$

The Special Orthogonal Group, $\mathbb{SO}(N)$, consists of all orthogonal matrices whose determinants have value $+1$. To show that these matrices form a group, we can immediately apply the results from the analyses of $\mathbb{O}(N)$ and $\mathbb{SL}(N)$ above to further show that the product of matrices in $\mathbb{SO}(N)$ has determinant $+1$, and that the inverses of all matrices in $\mathbb{SO}(N)$ have determinant $+1$.

The Unitary Group, $\mathbb{U}(N)$, consists of orthogonal matrices with complex matrix entries: $\mathbb{U}(N)\ =\ \mathbb{O}(N, \mathbb{C})$. Note that in this case of complex valued matrices, the matrix transpose operation is replaced by the Hermitian operation (transpose and complex conjugation): $A^*\ A\ =\ I$ for all $A\ \in\ \mathbb{U}(N)$, where A^* is the transposed complex conjugate of A.

The Special Unitary Group, $\mathbb{SU}(N)$, consists of those unitary matrices with determinant having value $+1$.

The Special Euclidean Group, $\mathbb{SE}(N)$, consists of all rigid body transformations of N-dimensional Euclidean space which preserve the length of vectors (i.e., distances between points). Matrices in $\mathbb{SE}(2)$ describe planar rigid body displacements, while matrices in $\mathbb{SE}(3)$ describe spatial rigid body displacements. Matrices g in $\mathbb{SE}(N)$ take the form:

$$g\ =\ \begin{bmatrix} R & \vec{d} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $R\ \in\ \mathbb{SO}(N)$, $\vec{d}\ \in\ \mathbb{R}^N$, and the vector $\vec{0}$ is an N-vector whose elements are identically zero. If $\vec{p}_1$ and $\vec{p}_2$ are two vectors in $\mathbb{R}^n$, and $\vec{p}_1, h$ and $\vec{p}_2, h$ are their homogeneous coordinates, then $g(\vec{p}_{2,h} - \vec{p}_{1,h})$ is a homogeneous vector equivalent to $R(\vec{p}_2\ -\ \vec{p}_1)$, and $||R(\vec{p}_2 - \vec{p}_1|| = ||(\vec{p}_2 - \vec{p}_1||$

2.2 Some Simple Examples

- $GL(1)\ =\ \mathbb{R} - \{0\}$.

- $GL(1, \mathbb{C}) \;=\; \mathbb{C} - \{0\}$.
- $\mathbb{O}(1) \;=\; \{1, -1\}$.
- $\mathbb{SO}(1) \;=\; \{1\}$.
- $\mathbb{SU}(1) \;=\; \{e^{i\theta}\}$, for all $\theta \;\in \mathbb{R}$.
- $\mathbb{SO}(2) \;=\; 2 \times 2$ matrices of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note, the groups $\mathbb{SO}(2)$ and $SU(1)$ are *isomorphic* because there is a one-to-one correspondence between every element in the two groups.