CDS 110: Solution to Homework #1

Problem 1 (CDS 101, CDS 110): (5 points)

In this problem, you were to Search the web for an article in the popular press about a feedback and control system.

Your grade on this simple problem depended upon your completeness in including the key components:

(a): Describe the feedback system using the terminology given in the article.

(b): describe the underlying process or system being controlled,

(c): Describe the sensor, actuator and computational element.

Problem 2 (CDS 101, CDS 110): (20 points)

You were given a simplified model of a cruise control for a car

$$m\dot{v} = -av + u + d$$

where m is the mass of the car, v is the car's forward velocity, a is a "drag coefficient" which models the effects of air drag, u is the control input (a lumped model which accounts for the effect of the motor's output on the traction force at the wheel-to-ground contact), and d is a disturbance.

Part (a): You were to consider an *open loop* control strategy defined by

$$u = \hat{a}v_{ref}$$

where $\hat{a}$ is an *estimate* of the drag coefficient and v_{ref} is the desired reference speed. Note that the estimate $\hat{a}$ may not be accurate. Compute the steady state response for this strategy as a function of $\beta = a/\hat{a}$. You should plot v_{ss}/v_{ref} as a function of β, where v_{ss} is your steady state solution.

Solution: In steady state, $\dot{v} = 0$, so that $v_{ss} = u/a$, where u is the control input and v_{ss} is the steady state velocity. This result assumes that $d = 0$, which was part of the problem statement. Since $u = \hat{a}v_{ref}$ in this simple control, $v_{ss} = (\hat{a}/a)v_{ref} = (1/\beta)v_{ref}$. Or, $v_{ss}/v_{ref} = (1/\beta)$. Hence, v_{ss} will actually equal v_{ref} in the obvious case where the estimate, $\hat{a}$, of the parameter a is exactly correct, or when $\beta = 1$..

Part (b): This part considers a *proportional closed loop* control strategy:

$$u = -k_p(v - v_{ref})$$

where v_{ref} is the desired reference speed and k_p is the *proportional gain*. Compute the steady state response for this strategy when $kp = 10\hat{a}$, and again plot the result as a function of β.

Solution: Since $\dot{v} = 0$ in steady state, and we ignore the disturbane, we have $0 = -av_{ss} - k_p(v_{ss} - v_{ref})$. Solving for v_{ss} yields:

$$v_{ss} = (\frac{k_p}{a + k_p})v_{ref}.$$

Hence, as k_p becomes larger, the steady state velocity more closely approaches the reference velocity. If $k_p = 10\hat{a}$, then $(v_{ss}/v_{ref}) = (10\hat{a}/(a + 10\hat{a})) = \left(\frac{10}{10+\beta}\right)$. Hence, with this feedback structure, there will typically be a steady state error for most reasonable values of β, but the error is not too sensitive to the value of β.

Part (c): Next consider a *proportional-integal* (PI) feedback control strategy.

$$u = -k_p(v - v_{ref}) - k_i \int_0^t (v - v_{ref})dt.$$

Again, compute the steady state response, and compare the result to your answers found in Parts (a) and (b). *Hint:* in one way of solving this problem, introduce $q = \int_0^t (v - v_{ref})dt$, and then $\dot{q} = (v - v_{ref})$

Solution: With this feedback, the system equations are:

$$\dot{v} = -av - k_p(v - v_{ref}) - k_i \int_0^t (v - v_{ref})dt. \tag{1}$$

There are a variety of arguments to use that all lead to the same conclusion: $v_{ss} = v_{ref}$. In one argument, take the time derivative of Equation (1), and set all of the derivative terms to zero. In another argument, set $\dot{v} = 0$ in (1), and solve (using the identity $\dot{q} + v_{ref} = v$):

$$0 = -a(\dot{q} + v_{ref}) - k_p\dot{q} - k_i q = -(a + k_p)\dot{q} - av_{ref} - k_i q.$$

For the right hand side of this equation to be zero, we require $\dot{q} = (v_{ss} - v_{ref}) = 0$ and $q = -av_{ref}/k_i$. In other words, the integral term estimates the factor av_{ref}!

Problem 3 (CDS 110): (15 points) Consider the differential equations:

$$\frac{dy}{dt} + 3y = 4u, \tag{2}$$

$$\frac{d^2}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u \tag{3}$$

where u is the system input, and y is the system output.

Part (a): Solve each of these equations.

Solution: For the first order equation (2), let us first find the homogeneous solution, $y_h(t)$:

$$\frac{dy}{dt} = -3t; \quad \Rightarrow \quad y_h(t) = e^{-3t}y(0).$$

To find the particular solution, recall from the class notes that a general first order o.d.e. and its solution take the form

$$\dot{x} + p(t)x = g(t) \quad \Rightarrow \quad x(t) = \frac{1}{\mu}(t)\int \mu(\tau)g(\tau)d\tau$$

where $\mu(t) = e^{\int p(t)dt}$. For this particular problem, $p(t) = 3$ and $g(t) = 4u(t)$, and hence $\mu(t) = e^{-3t}$, and

$$y_p(t) = e^{-3t}\int e^{3\tau}4u(\tau)d\tau = 4\int e^{-3(t-\tau)}u(\tau)d\tau.$$

Putting the homogeneous and inhomogeneous solutions together yields:

$$y(t) = e^{-3t}y(0) + 4\int_0^t e^{-3(t-\tau)}u(\tau)d\tau.$$

For the second order equation (3), let us assume an exponential solution to the homogeneous part of the equation: e^{st}. Assuming this solution, the characteristic equation of the homogeneous o.d.e is $s^2 + s2s + 1 = 0$, which has two repeated roots at $s = -1$. Hence, the homogeneous solution is:

$$y_h(t) = C_1e^{-t} + C_2te^{-t}.$$

For the particular solution, one has to assume a structural form for the input function in order to have any hope of finding an analytical solution. Here are two examples, which are related to parts (b) and (c) of this problem.

First assume that $u = 1$, which has particular solution $y_p(t) = 1$. In this case, the solution is $y = 1 + C_1e^{-t} + C_2te^{-t}$.

As a second particular solution example, let us assume a solution of the form $y_p(t) = G(s)e^{st}$. Substituting this solution into (3), and noting that $\dot{y}_p(t) = se^{st}G(s)$ and $\ddot{y}_p(t) = s^2e^{st}G(s)$, one finds:

$$G(s) = \frac{2s+1}{s^2 + s2 + 1}.$$

Part (b): Find the response of each system to: (1) a unit step input, $u(t) = 1$; (2) an exponential signal $u(t) = e^{st}$.

Solution: For o.d.e. (2), the output for a unit step input is:

$$y(t) = e^{-3t}y(0) + 4\int_0^t e^{-3(t-\tau)}d\tau = e^{-3t}y(0) + 4e^{-3t}\int_0^t e^{3\tau}d\tau \tag{4}$$

$$= e^{-3t}y(0) + 4e^{-3t}\frac{e^{3\tau}}{3}\Big|_0^t = e^{-3t}y(0) + 4e^{-3t}(e^{3t}-1)/3 \tag{5}$$

$$= e^{-3t}y(0) + \frac{4}{3}(1-e^{-3t}). \tag{6}$$

For the exponential input $u = e^{st}$,

$$y(t) = e^{-3t}y(0) + 4\int_o^t e^{-3(t-\tau)}e^{s\tau}d\tau = e^{-3t}y(0) + 4e^{-3t}\int_0^t e^{(3+s)\tau}d\tau \tag{7}$$

$$= e^{-3t}y(0) + 4e^{-3t}\frac{e^{(3+s)\tau}}{(3+s)}\Big|_0^t = e^{-3t}y(0) + \frac{4}{s+3}(e^{st}-e^{-3t}) \tag{8}$$

For o.d.e. (3), we derived the step response in the last part of this problem.

Part (c): Derive the transfor functions for each of these systems.

Solution: For o.d.e. (2), assume a solution of the form $y_p(t) = G(s)e^{st}$ for input $u(t) = e^{st}$. Substituting these terms into the o.d.e. yields:

$$sG(s)e^{st} + 3G(s)e^{st} = 4e^{st}$$

from which we can infer that

$$G(s) = \frac{4}{s+3} \ .$$

The transfer function for o.d.e. (3) was derived above.