ME/CS 133(a): Solution to Homework #1
(Fall 2018/2019)

Solution to Problem 1:

Let the 2×1 vectors ${}^1\vec{v} = \begin{bmatrix} {}^1v_1 & {}^1v_2 \end{bmatrix}^T$ and ${}^2\vec{v} = \begin{bmatrix} {}^2v_1 & {}^2v_2 \end{bmatrix}^T$ have associated complex representations ${}^1\tilde{v} = {}^1v_1 + i\ {}^1v_2$ and ${}^2\tilde{v} = {}^2v_1 + i\ {}^2v_2$ respectively (where $i^2 = -1$). Recall that the goal of this problem is to show that the complex number formula:

$$ {}^1\tilde{v} = \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v}\ . \tag{1} $$

is equivalent to the planar coordinate transformation:

$$ {}^1\vec{v} = \vec{d}_{12} + R(\theta_{12})\ {}^2\vec{v}\ . \tag{2} $$

Let's evaluate the right hand side of expression (1) using the standard rules for multiplication of complex numbers[1]:

$$ \begin{aligned} \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v} &= (x + iy) + (\cos\theta_{12} + i\sin\theta_{12})({}^2v_1\ +\ i\ {}^2v_2) \\ &= (x +\ {}^2v_1\cos\theta_{12}\ -\ {}^2v_2\sin\theta_{12}) + i(y +\ {}^2v_1\sin\theta_{12}\ +\ {}^2v_2\cos\theta_{12}) \quad (3) \end{aligned} $$

where we have used Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$). Matching the real and complex portions of Equation (3) with the real and complex parts of ${}^1\tilde{v}$ in the left hand side of Equation (1), we see that

$$ \begin{aligned} {}^1v_1 &= x + {}^2v_1\cos\theta - {}^2v_2\sin\theta & (4) \\ {}^1v_2 &= y + {}^2v_1\sin\theta + {}^2v_2\cos\theta\ . & (5) \end{aligned} $$

These equations are equivalent to

$$ {}^1\vec{v} = \vec{d}_{12} + \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{bmatrix}\ {}^2\vec{v} \tag{6} $$

Solution to Problem 2: Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^B\vec{p}$ as seen by an observer in the fixed B frame. After displacement, the observer in the body fixed C frame also sees the pole in his/her coordinates at point ${}^B\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coodinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$$ {}^B\vec{p} = \vec{d}_{12} + R_{12}\ \ {}^B\vec{p}. $$

This equation can be solved to find the pole location:

$$ {}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} $$

[1] If $\tilde{a} = a_1 + ia_2$ and $\tilde{b} = b_1 + ib_2$, then $\tilde{a}\tilde{b} = (a_1b_2 - a_2b_2) + i(a_1b_2 + a_2b_1)$.

Of course, the matrix $(I - R_{12})$ must be invertible,which will alwas be true except when $R_{12} = I$. In this case, the motion is a pure translation, which is viewed as a rotation about the "pole at infinity."

B) In Frame B, the pole is located at: ${}^B\vec{p} \; = (I - R_{12})^{-1}\vec{d}_{12}$

C) In Frame C, the vector describing the pole has exactly the same value as seen by the observer in Frame B: ${}^C\vec{p} \; = (I - R_{12})^{-1}\vec{d}_{12}$

A) In Frame A, the expression for the pole vector is obtained by a simple coordinate transformation of the expression in Frame B: ${}^A\vec{p} \; = \vec{d}_{01} + R_{01} \; {}^B\vec{p} = \vec{d}_{01} + R_{01}(I - R_{12})^{-1}\vec{d}_{12}$

Problem 3: To find the pole of the displacement: $D_2 = (x, y, \theta) = (3.0, 2.0, 45.0^o)$, substitute into the above results:

$$\begin{aligned} {}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{pmatrix}\right]^{-1} \begin{bmatrix} 3.0 \\ 2.0 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 1 - \frac{\sqrt{2}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 3.0 \\ 2.0 \end{bmatrix} \\ &= \begin{bmatrix} ? \\ ? \end{bmatrix} \end{aligned} \tag{7}$$

You could report this result in Frame B, or transform the results to frame A.

$${}^A\vec{p} = \vec{d}_{01} + R_{01} \; {}^B\vec{p} = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix} + \begin{pmatrix} \cos(20^o) & -\sin(20^o) \\ \sin(20^o) & \cos(20^o) \end{pmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} ? \\ ? \end{bmatrix} \tag{9}$$

Problem 4: To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by D, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame B. The location of Frame B relative to Frame D is a pure translation of amount $-^1\vec{p}$, and therefore, $D_{DB} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame D, is obtained by a similarity transform $D_{DB}D_{12}D_{DB}^{-1}$:

$$D_{DB}D_{12}D_{DB}^{-1} = (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} \tag{10}$$

$$= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) \tag{11}$$

$$= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) \tag{12}$$

$$= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) \tag{13}$$

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{DB}D_{12}D_{DB}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame D, the displacement is a pure rotation by amount R_{12}.

Problem 5:

Part (a): There are many ways that one can prove that reflections preserve length. Here is one approach (see Figure 1).

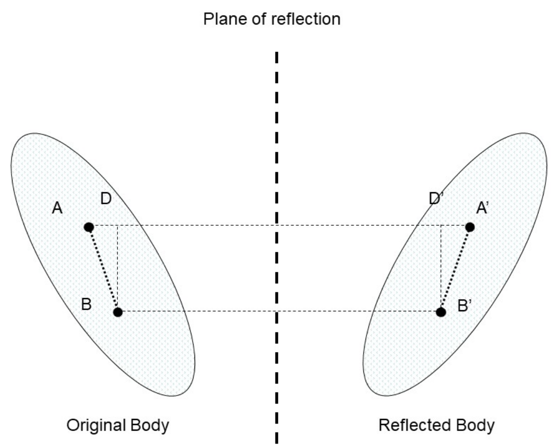

Figure 1: Geometry of Planar Rigid Body Reflection

Select any two non-identical points, A and B, in a rigid body. After reflection, those points become $A^{'}$ and $B^{'}$. Form the right triangle ABD, where the line BD is chosen to be perpendicular to the line $AA^{'}$. Similary, in the reflected body, form the right triangle $A^{'}B^{'}D^{'}$. Simple geometric arguments show that since the distance $|BD|$ and $|B^{'}D^{'}|$ are equal, and the distances $|AD|$ and $|A^{'}D^{'}|$ are equal, then $|AB| = |A^{'}B^{'}|$. Hence, the distance between A and B is preserved under reflection. Since A and B were chosen randomly, the result will hold for any non-identical pair of points in the body. Thus, distance is always preserved under reflection.

Part (b): Generally, physically meaningful planar displacements are not equivalent to a single reflection. To see this, define three points (A, B, C) in the body of Figure 1. Because the body is rigid, one can think of points (A, B, C) as forming a rigid triangle. Consider the triangle formed from the reflected points $(A^{'}, B^{'}, C^{'})$. Note that it is impossible physically translate (A, B, C) to $(A^{'}, B^{'}, C^{'})$. Finally, note that any rigid body planar displacement can generally be realized as the result of two sequential reflections.

Problem 6: (Problem 2.10(a,b,c) in MLS)

Part (b): If $\omega \in \mathbb{R}$, then let:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega J \qquad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that:

$$\hat{\omega}^2 = -\omega^2 I; \qquad \hat{\omega}^3 = \omega^3 J$$

Hence:

$$
\begin{aligned}
e^{\hat{\omega}\theta} &= I + \omega\theta J + \tfrac{(\omega\theta)^2}{2!}J^2 + \tfrac{(\omega\theta)^3}{3!}J^3 + \cdots \\
&= I + (\omega\theta)J - \tfrac{(\omega\theta)^2}{2!}I - \tfrac{(\omega\theta)^3}{3!}J + \cdots \\
&= (1 + \tfrac{(\omega\theta)^2}{2!} + \tfrac{(\omega\theta)^4}{4!} + \cdots)I + (\omega\theta - \tfrac{(\omega\theta)^3}{3!} + \cdots)J \\
&= \cos(\omega\theta)I + \sin(\omega\theta)J \\
\\
&= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix}
\end{aligned}
$$

We can think of the map $exp : so(2) \rightarrow SO(3)$ as embedding a planar rotation into $SO(3)$, where we have chosen the axis of rotation to be perpendicular to the x-y plane:

$$
\begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) & 0 \\ \sin(\omega\theta) & \cos(\omega\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Clearly this map can not be surjective, since there are many rotations in $SO(3)$ which can not be expressed as the exponential of an element in $so(2)$. If we assume that the product $\omega\theta$ is nonzero except at $\theta = 0$ (i.e., assume that $\omega \neq 0$ and $\theta \in [-\pi, \pi]$), then the map is not injective since $exp(\omega\theta) = exp(\omega\theta + 2\pi)$.

Part (c): This can be shown by brute-force calculation. If $R \in SO(2)$ then:

$$
\begin{aligned}
R\hat{\omega}R^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \omega \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \\
&= \omega \begin{bmatrix} 0 & -(\cos^2\theta + \sin^2\theta) \\ (\sin^2\theta + \cos^2\theta) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}
\end{aligned}
$$