CDS 101/110: Lecture 9.1
Frequency DomainLoop Shaping

November 23, 2016

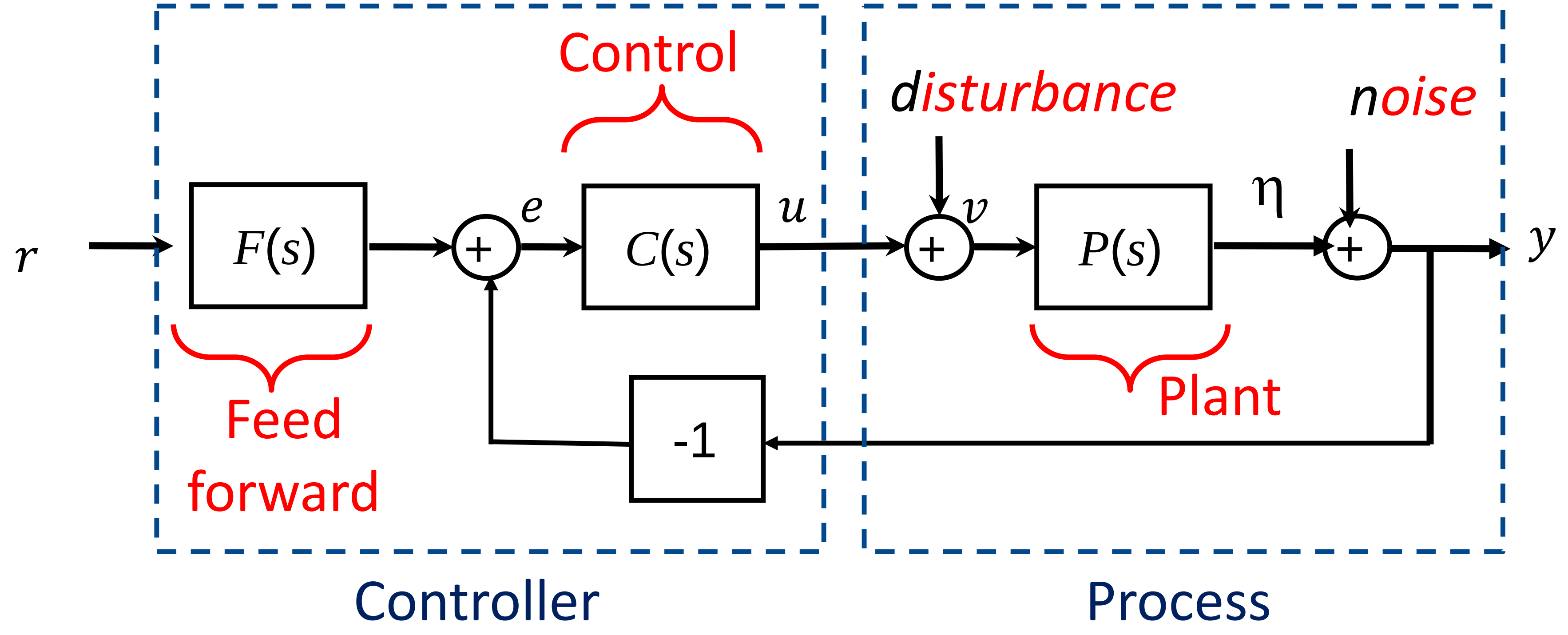
Goals:

- Review Basic Loop Shaping Concepts
- Work through example(s)

Reading:

- Åström and Murray, Feedback Systems 2-e, Section 12.1, 12.2-12.4, 12.6
- I.e., we are not going to cover Section 12.2 (feedforward design) and 12.5 (root locus).
- Section 12.6 will be mainly discussed next week.

General Loop Transfer Functions



r = reference input
e = error
u = control
v = control + disturbance
η = true output (***what we want to control***!)
y = measured output

System "outputs"

System "inputs"

$$\begin{pmatrix} y\\ \eta\\ v\\ u\\ e\end{pmatrix} = \begin{pmatrix} \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC}\\ \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{-PC}{1+PC}\\ \frac{CF}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC}\\ \frac{CF}{1+PC} & \frac{-PC}{1+PC} & \frac{-C}{1+PC}\\ \boxed{\frac{F}{1+PC}} & \frac{-P}{1+PC} & \frac{-1}{1+PC}\end{pmatrix}\begin{pmatrix} r\\ d\\ n\end{pmatrix}$$

"Gang of Seven"

"Gang of Six"

TF $= \frac{PCF}{1+PC}$ | T $= \frac{PC}{1+PC}$ | PS $= \frac{P}{1+PC}$

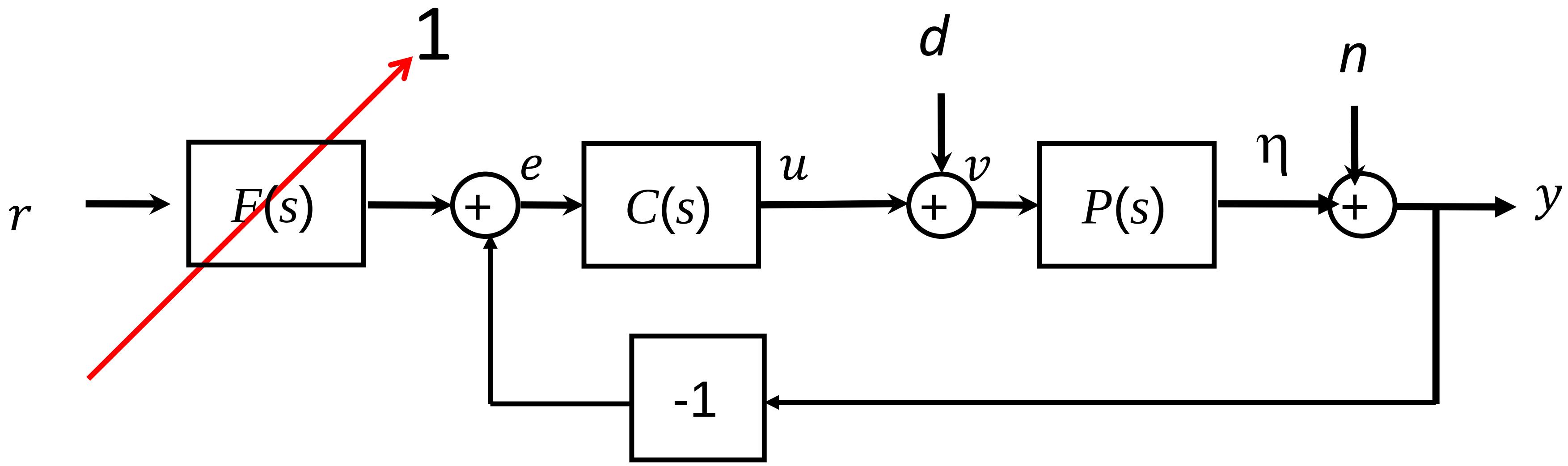
CFS $= \frac{CF}{1+PC}$ | CS $= \frac{C}{1+PC}$ | S $= \frac{1}{1+PC}$

Response of (y, u) to r

Response of u to (d,n)

Response of y to (d,n)

Key Loop Transfer Functions



***F*(s) = 1: Four unique transfer functions define performance ("Gang of Four")**

Sensitivity: Function

$$G_{er} = S(s) = \frac{1}{1+L(s)}$$

Complementary Sensitivity Function:

$$G_{yr} = T(s) = \frac{L(s)}{1+L(s)}$$

Load Sensitivity Function:

$$G_{yd} = PS(s) = \frac{P(s)}{1+L(s)}$$

Noise Sensitivity Function:

$$G_{yn} = CS\ (s) = \frac{C(s)}{1+L(s)}$$

$$\boxed{L(s) = P(s)C(s)}$$

"Gang of Four"

(the "sensitivity" functions)

Characterize most performance criteria of interest

Rough Loop Shaping Design Process

***A Process*: sequence of (nonunique) steps**

1. Start with plant and performance specifications

2. If plant not stable, first stabilize it (e.g., PID)

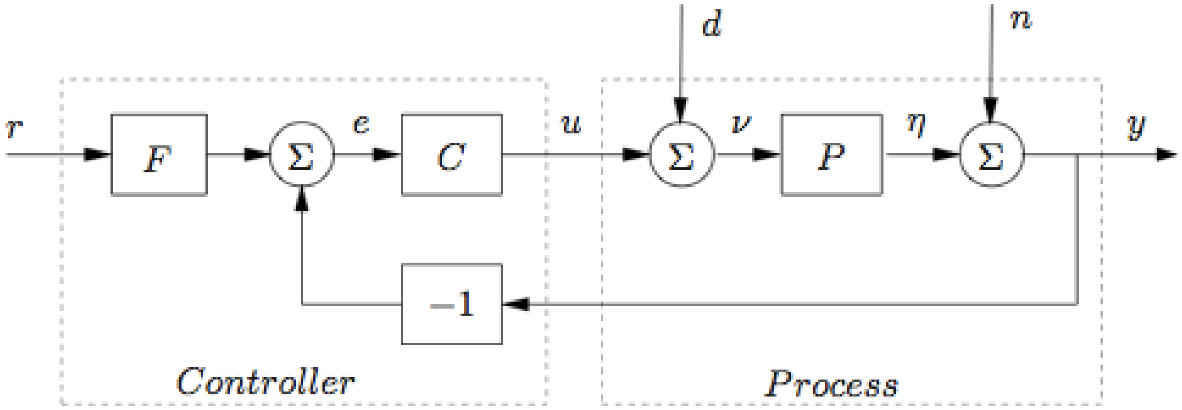
3. Adjust/increase simple gains

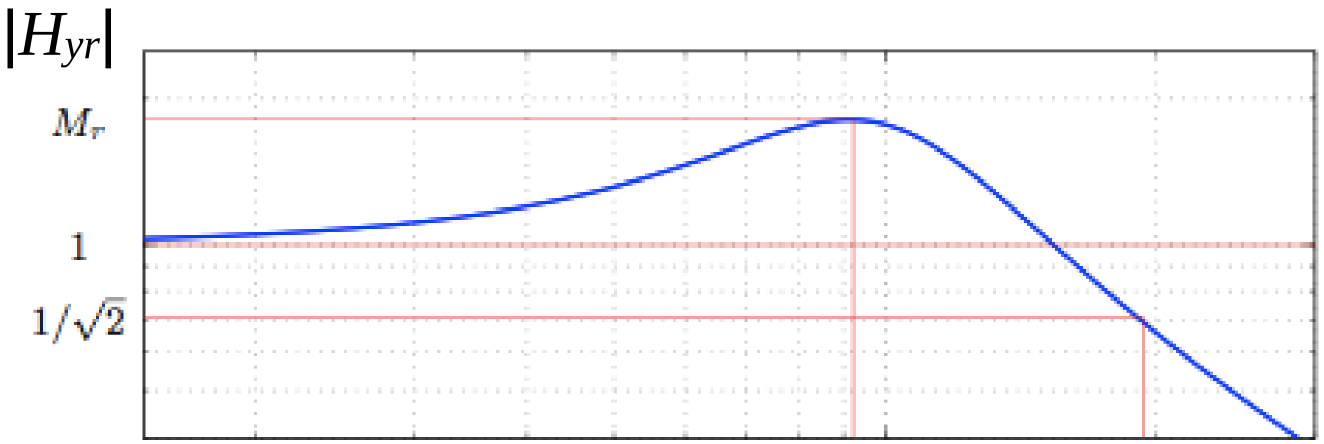
- Increase proportional gain for tracking error
- Introduce integral term for steady-state error
- Will derivative term improve overshoot?

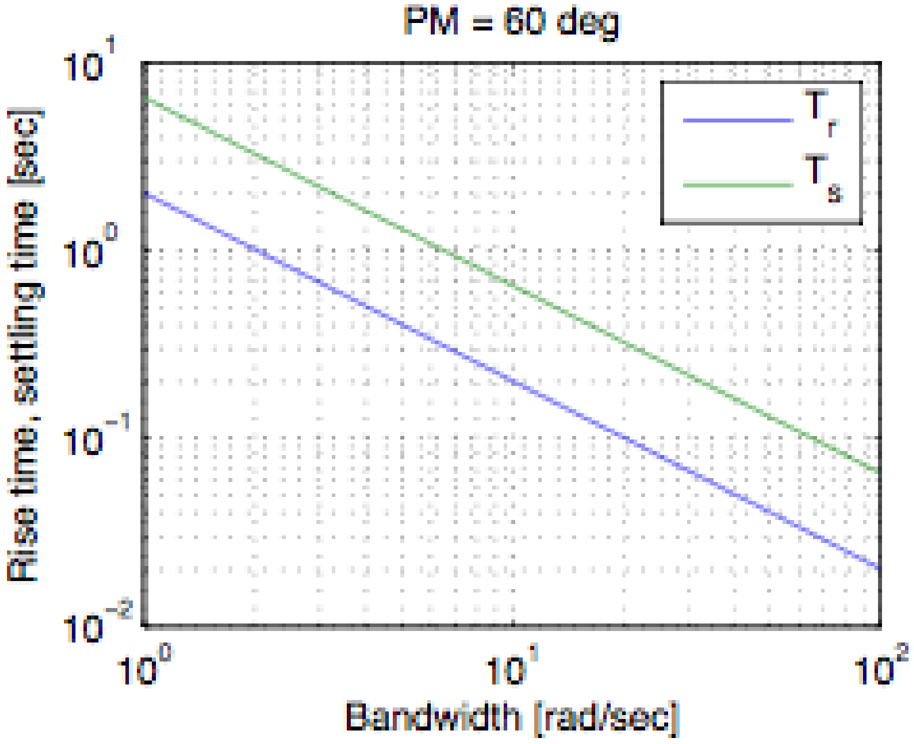
4. Analyze/adjust for stability and/or phase margin

- Adjust gains for margin
- Introduce *Lead* or *Lag Compensators* to adjust phase margin at crossover and other critical frequencies
- Consider PID if you haven’t already

Summary of Specifications







Key Idea: convert ***closed loop** specifications* on

$$G_{yr}(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{L\,(s)}{1 + L(s)}$$

to equivalent specifications on ***loop*** system $L(s)$

- Time domain spec.s can often be converted to frequency domain spec.s

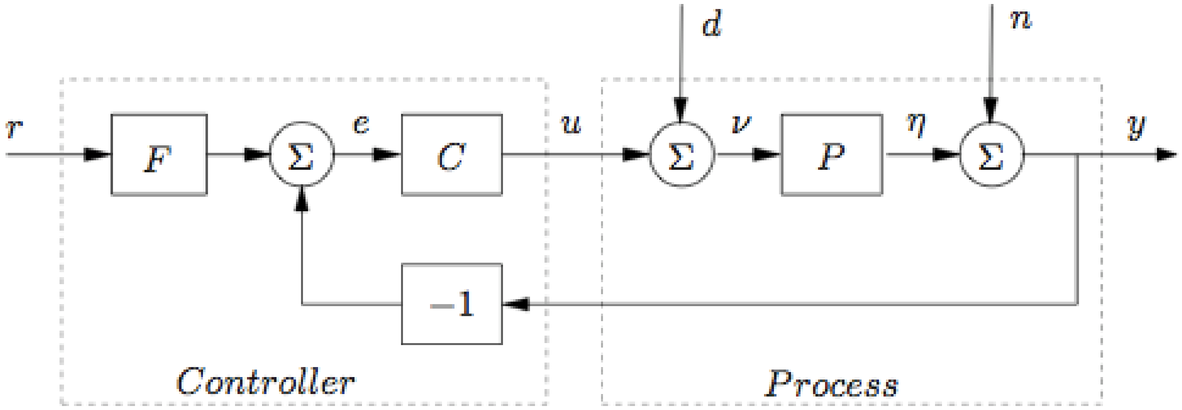
Steady-state tracking error $< X\%$ $\Rightarrow$ $|L(0)| > 1/X$

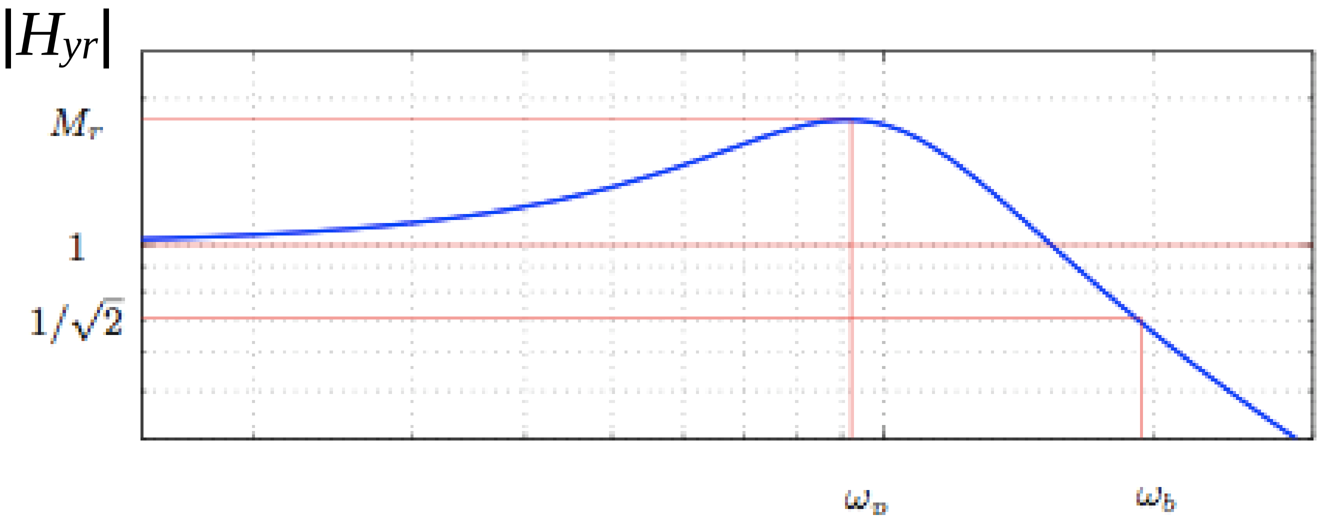
Tracking error $< Y\%$ up to frequency f_t Hz $\Rightarrow$ $|L(i\omega)| > 1/Y$ for $\omega < 2\pi f_t$

Bandwidth of $\boldsymbol{\omega_b}$ rad/sec $\Rightarrow$ $|L(i\omega_b)| = 1$

- Usually needed for rise/settling time spec.

Summary of Specifications

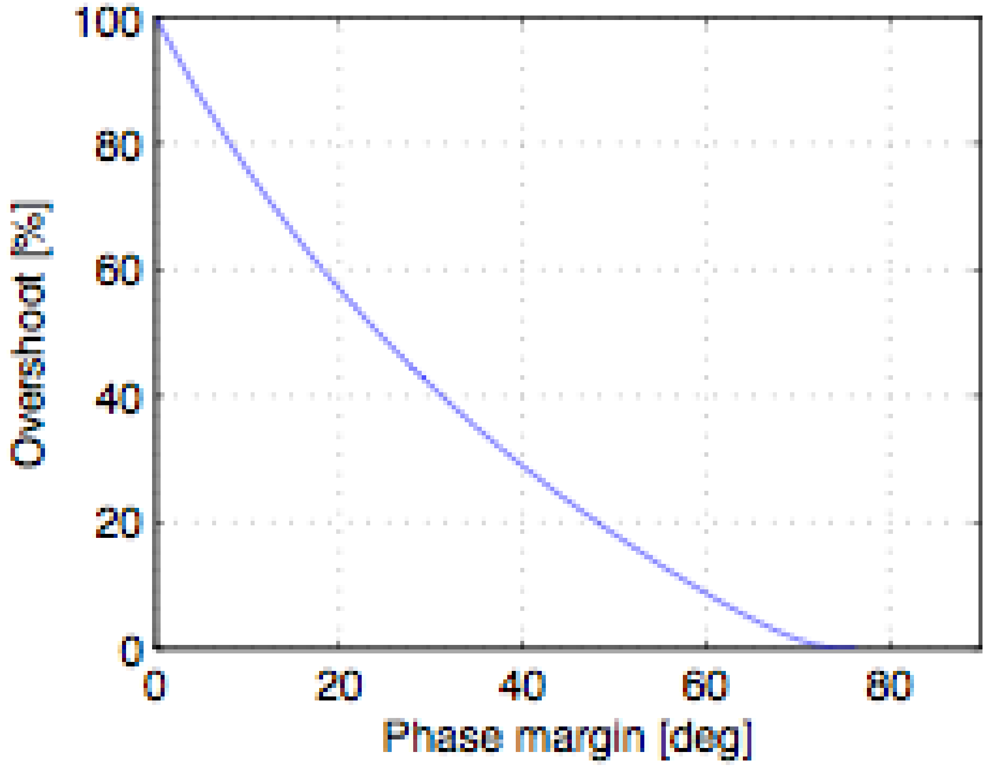




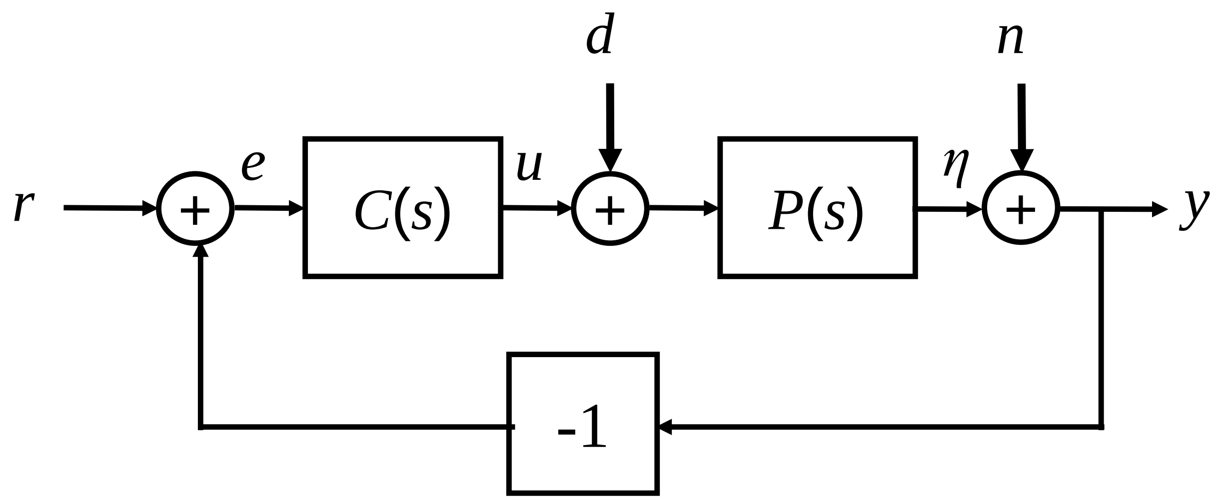
Overshoot $< Z\%$ $\Rightarrow$ Phase Margin $> f(Z)$

Phase/Gain margins (Specified Directly)

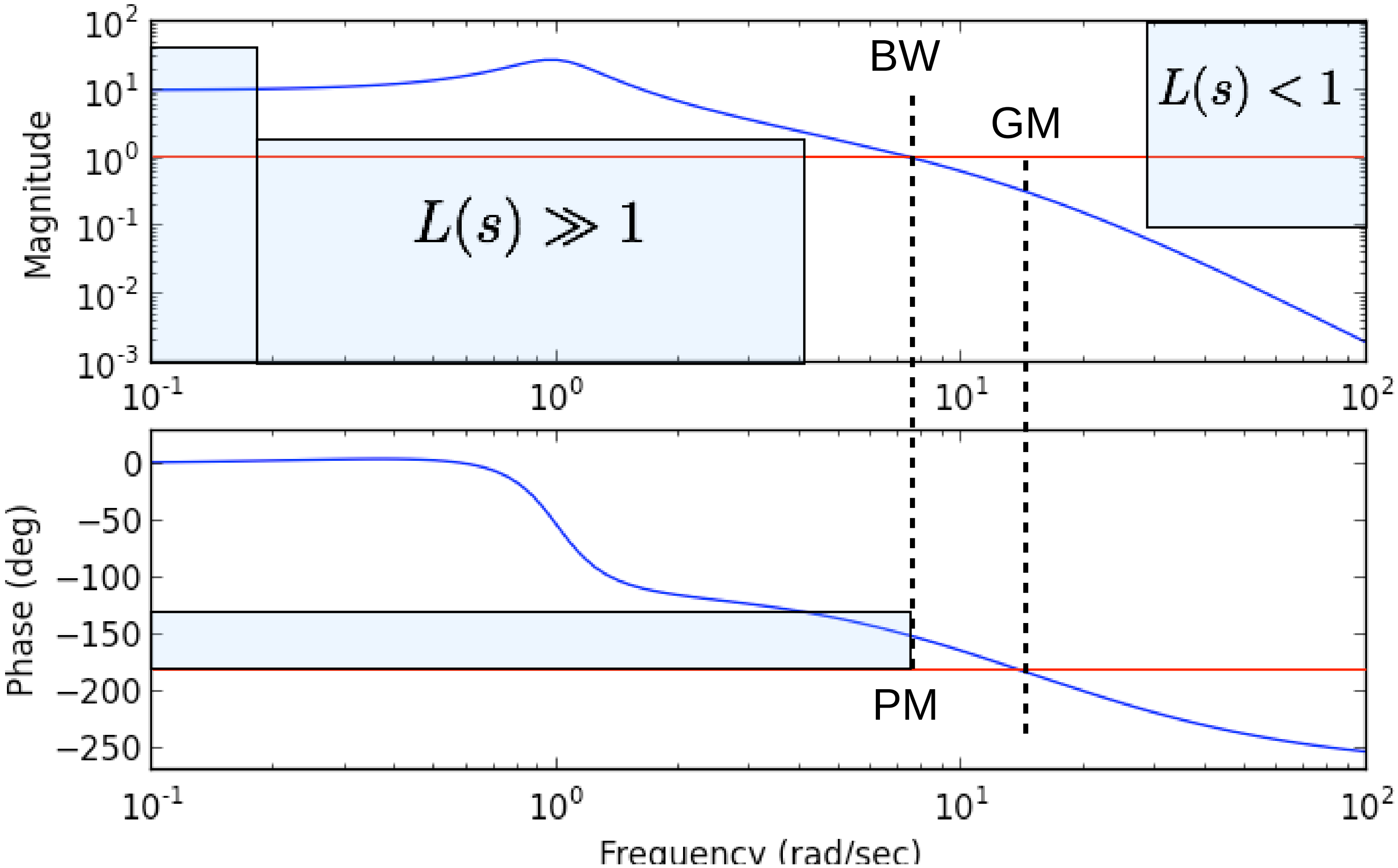
- For robustness
- Typically, at least gain margin of 2 (6 dB)
- Usually, phase margin of 30-60 degrees



“Loop Shaping”: Design Loop Transfer Function



$$H_{er} = \frac{1}{1+L} \qquad H_{\eta n} = \frac{-L}{1+L}$$



Translate specs to “loop shape”

$$L(s) = P(s)C(s)$$

Design L(s) to obey constraints

- High gain at low frequency
  - Good Steady-state error
  - Good disturbance rejection at low freqs.
  - Decent tracking in bandwidth

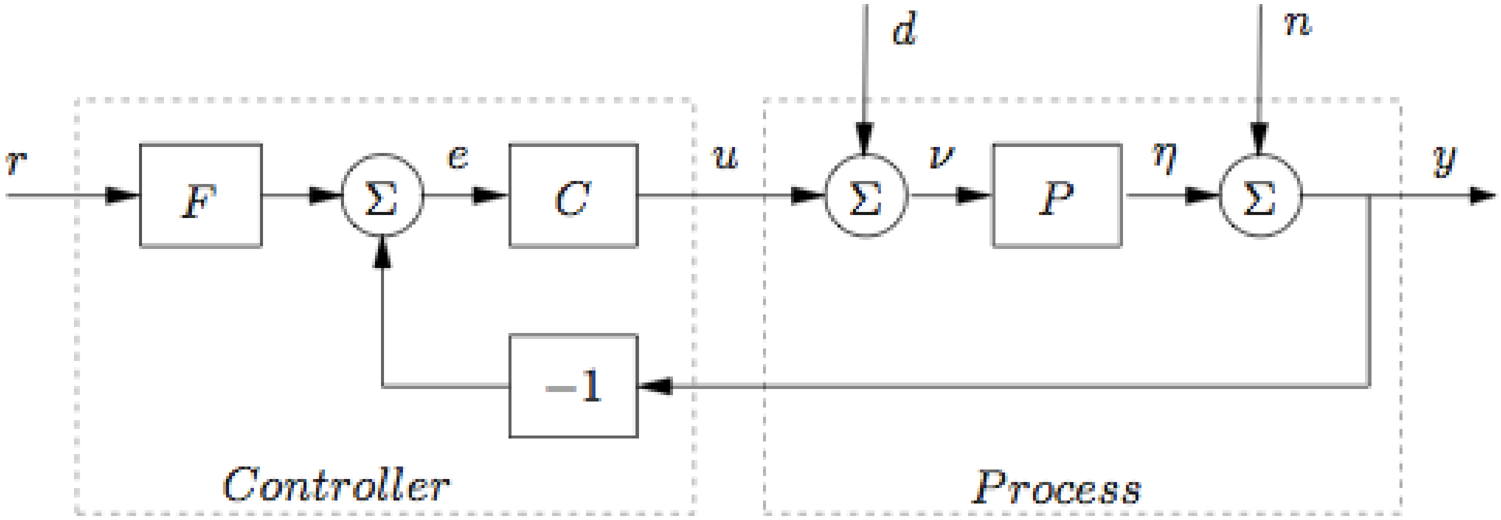
Low gain at high frequency
  - Avoid amplifying noise

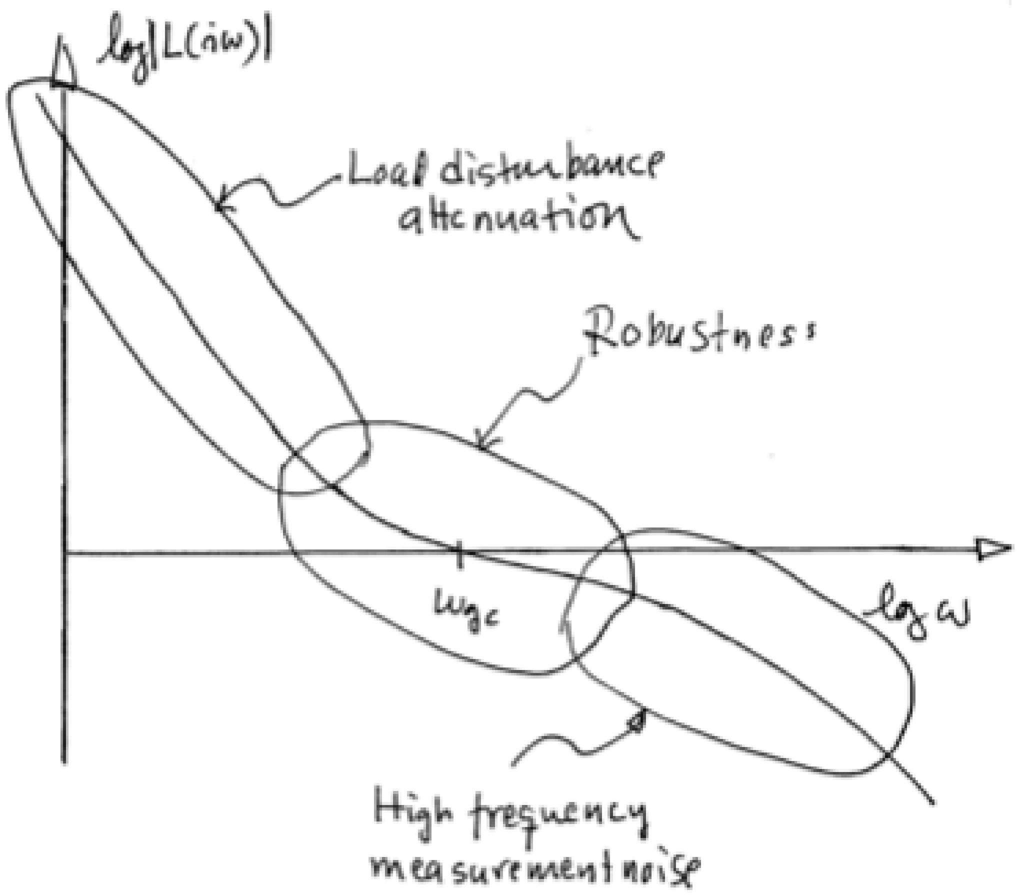
Sufficiently high bandwidth
  - Good rise/settling time

Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Loop shaping is *trial and error*

Additional Loop Shaping Concepts





Disturbance rejection $H_{ed} = \frac{-P}{1+L}$

- Would like H_{ed} to be small make ⇒ large L(s)
- Typically require this in low frequency range

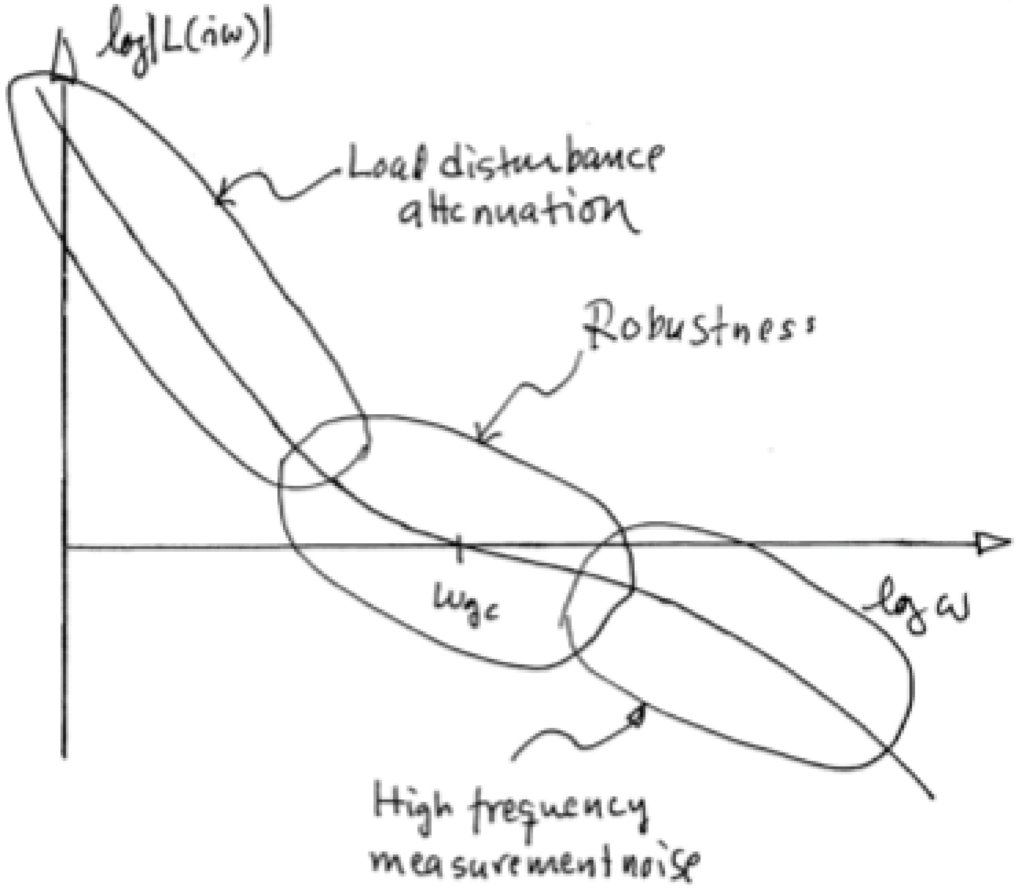
High frequency measurement noise $H_{un} = \frac{-C(s)}{1+P(s)C(s)}$

- Want to make sure that H_{un} is small (avoid amplifying noise)
- Typically generates constraints in high frequency range

Robustness: gain and phase margin

- Focus on gain crossover region: make sure the slope is “gentle” at gain crossover
- Fundamental tradeoff: transition from high gain to low gain through crossover

Design Method #1: Process Inversion



Simple trick: invert out process

- Write performance specs in terms of desired loop transfer function
- Choose L(s) to satisfy specfications
- Choose controller by *inverting* P(s)

$$C(s) = L(s)/P(s)$$

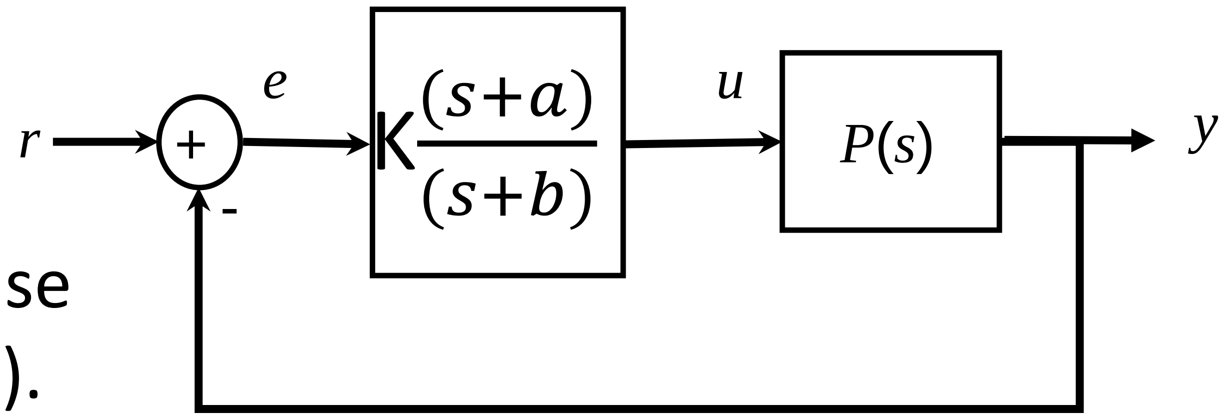
Pros

- Simple design process
- L(s) = k/s often works very well
- Can be used as a first cut, with additional tuning

Cons

- High order controllers (at least same order as plant)
- Requires “perfect” process model (due to inversion)
- Can generate non-proper controllers (order(num) > order(den))
  - Difficult to implement, plus amplifies noise at high frequency (C(∞) = ∞)
  - Fix by adding high frequency poles to roll off control response at high frequency
- Does not work if right half plane poles or zeros (*internal instability*)

Lead & Lag Compensators



Lead: K > 0, a < b

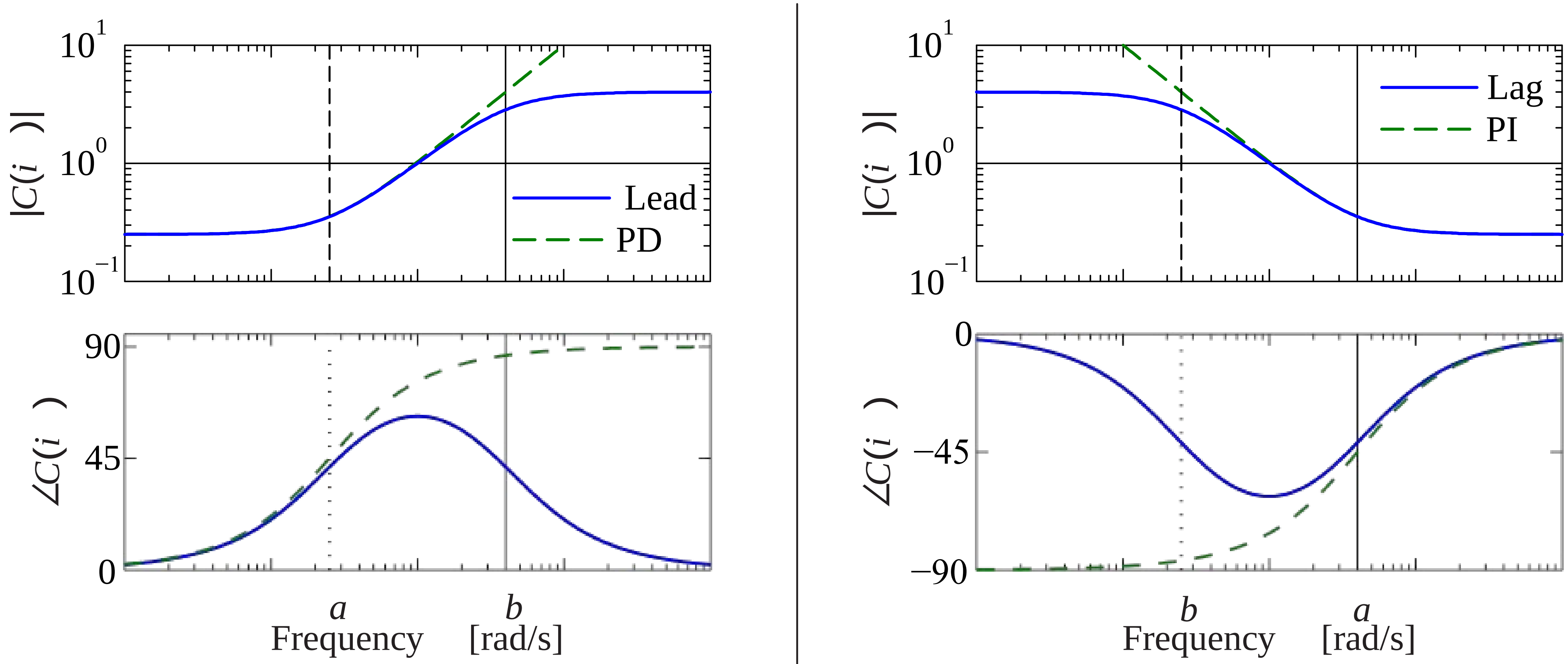
- Add phase near crossover
- Improve gain & phase margins, increase bandwidth (better transient response).

Lag: K > 0, a> b

- Add gain in low frequencies
- Improves steady state error

Lead/Lag:

- Better transient and steady state response



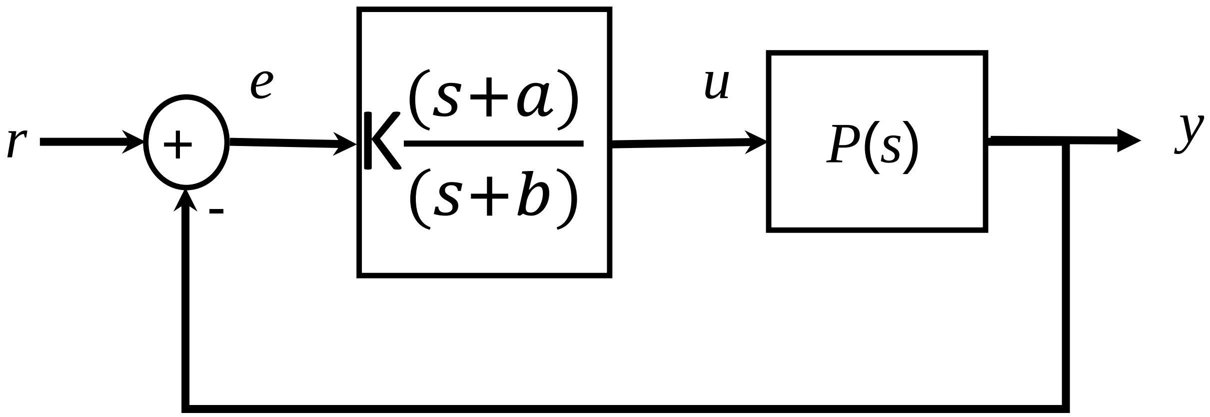
(a) Lead compensation, $a < b$

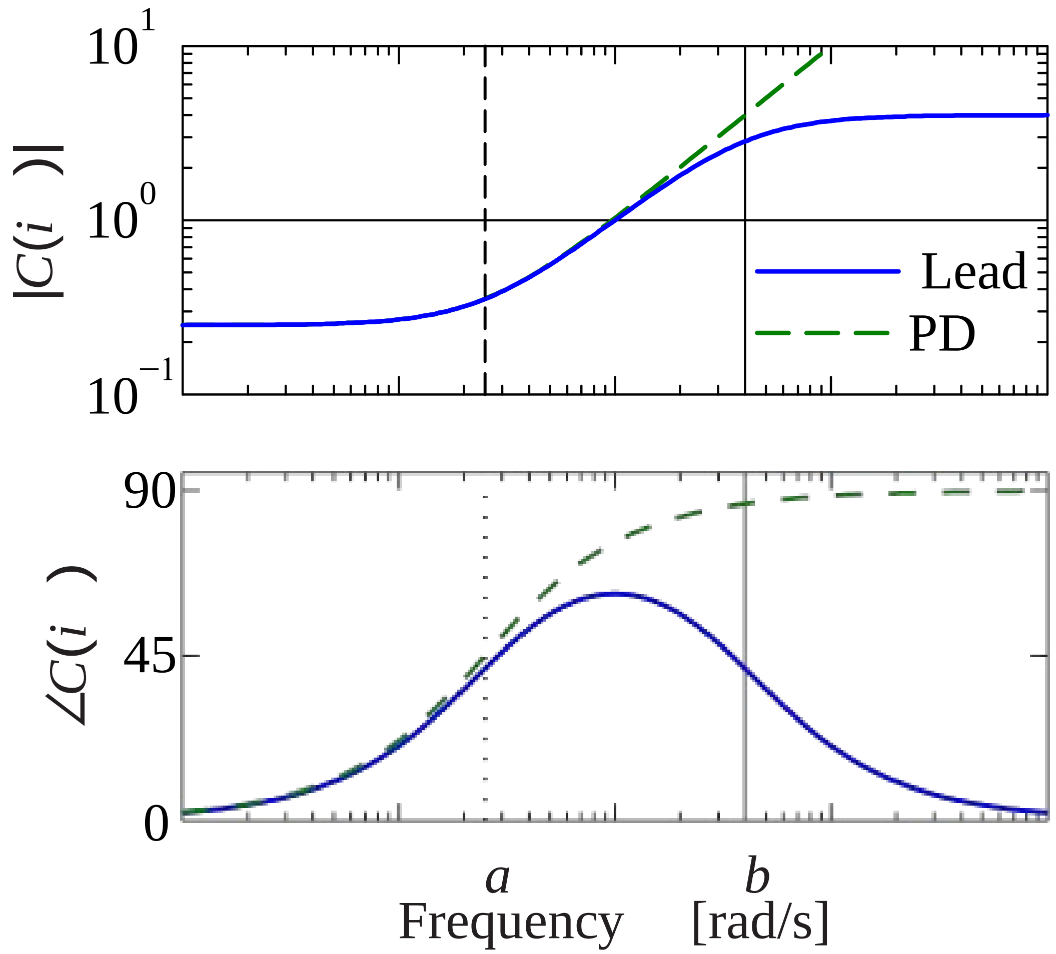
(b) Lag compensation, $b < a$

Lead & Lag Compensators

Lead: adds phase, ϕ_m at:

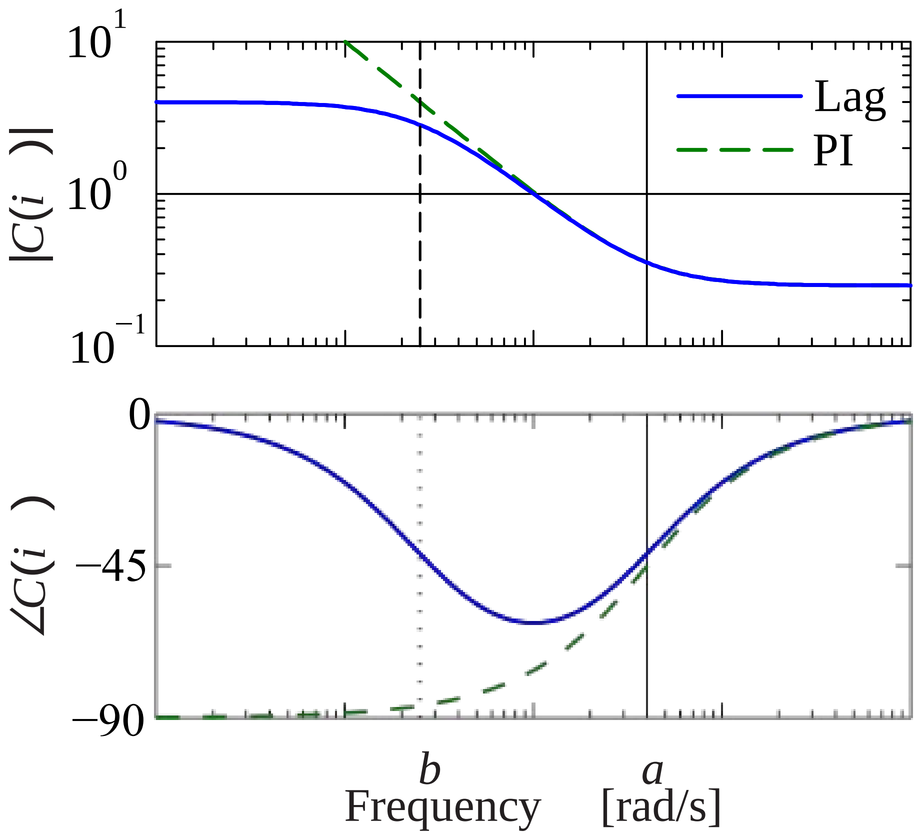
- $\omega = \sqrt{ab}$
- $\phi_m = 90^o - 2\tan^{-1}\sqrt{\frac{a}{b}}$





(a) Lead compensation, $a < b$

Lag: reduces steady state error by factor of a/b

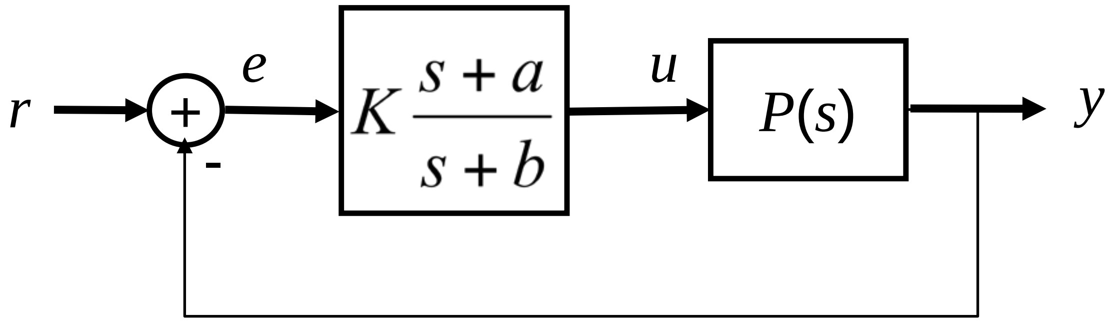


(b) Lag compensation, $b < a$

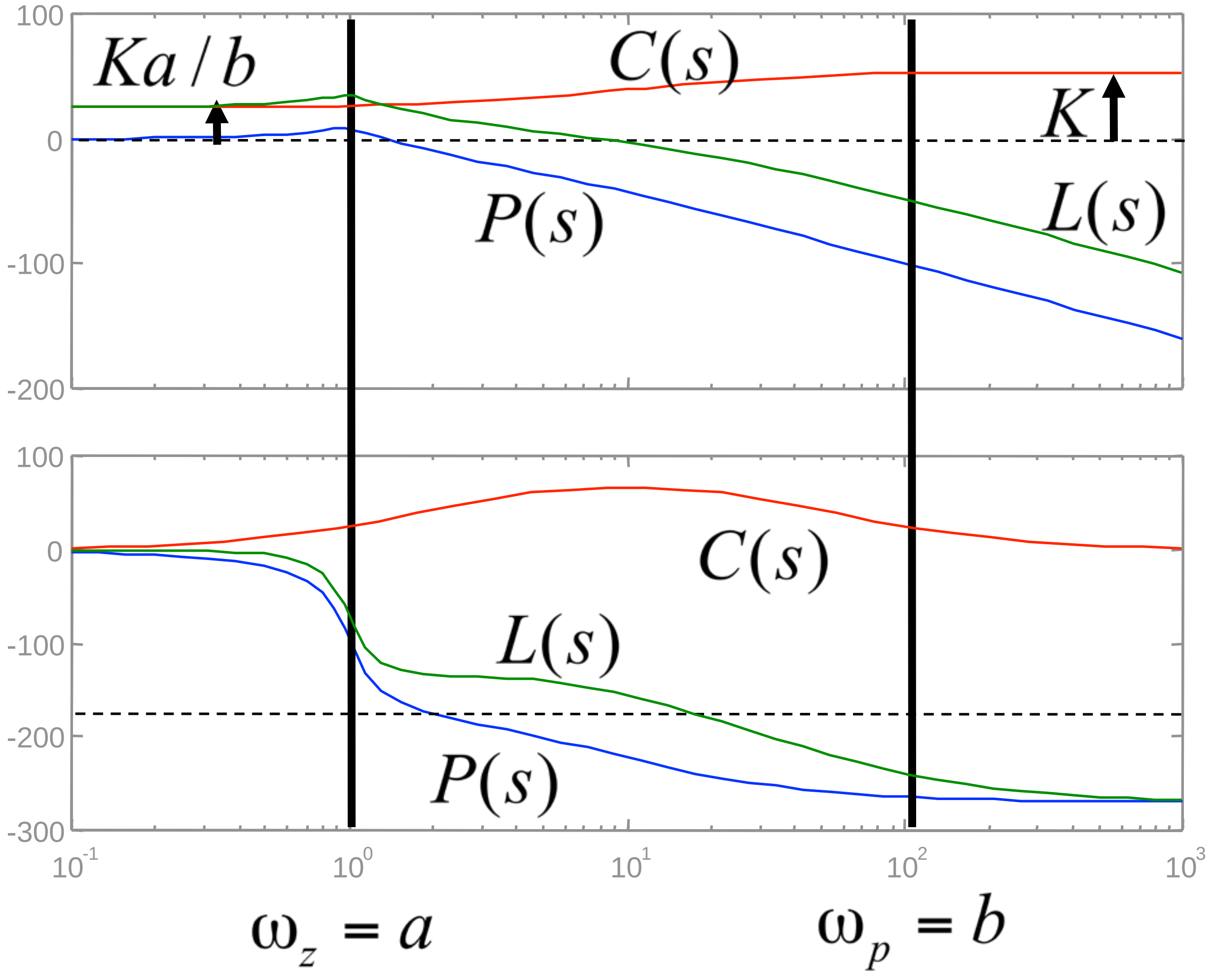
Design Method #2: Add Lead, Lag, Lead/Lag compensation

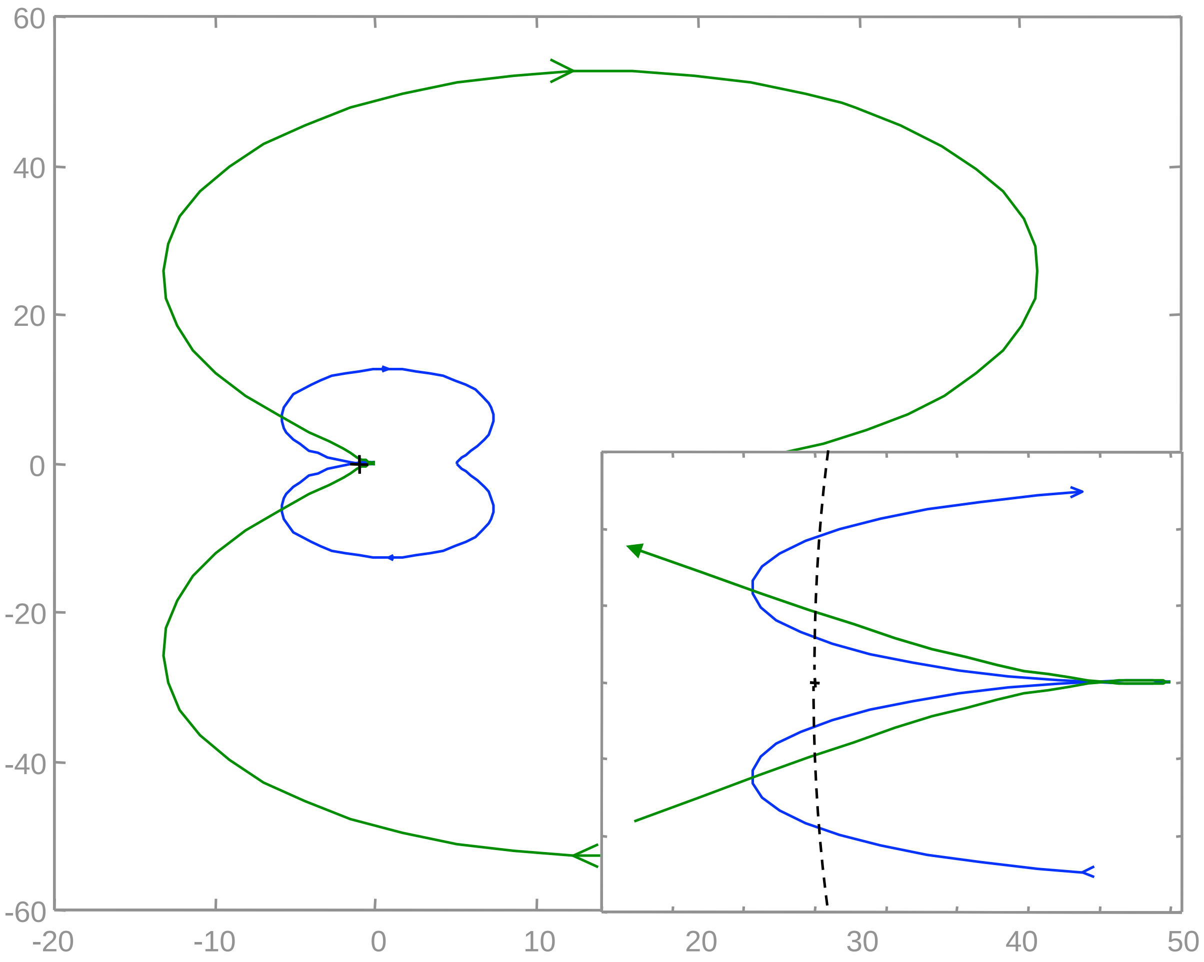
Lead: increases phase in frequency band

- Effect: lifts phase by increasing gain at high frequency
- Increases PM
- Bode: add phase between zero and pole
- Nyquist: increase phase margin



$$a < b \qquad K > 0$$





Example: Lead Compensation for Second Order System

System description

$$P(s) = \frac{p_1 p_2}{(s+p_1)(s+p_2)}$$

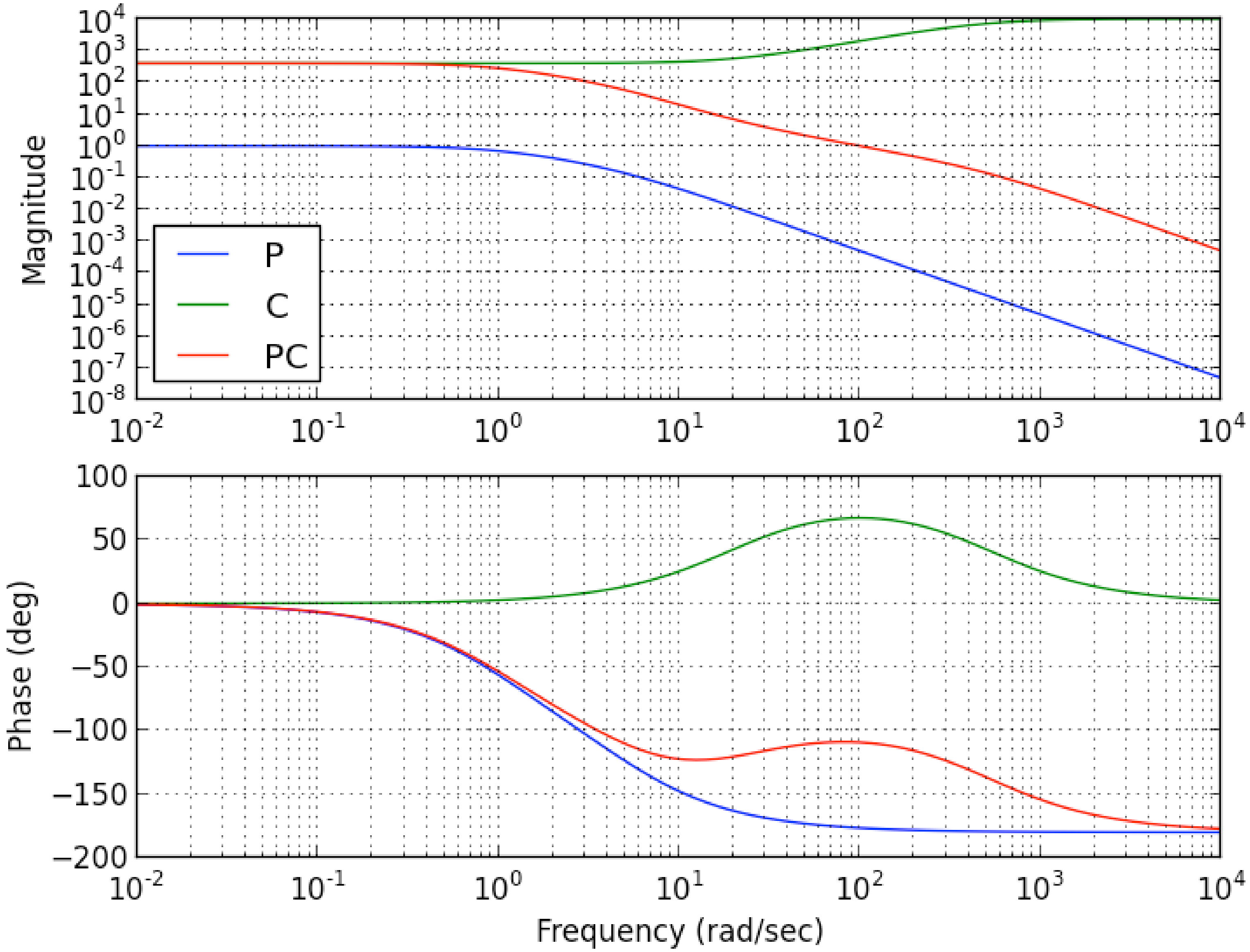
- Poles: p_1 = 1, p_2 = 5

Control specs

- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%
  - Gives PM of ~60 deg

Try a lead compensator

$$C(s) = K\frac{s+a}{s+b}$$



- Want gain cross over at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error ⇒ $|L(0)| > 100$
- a = 20, b = 500, K = 10,000 (gives $|C(0)| = |L(0)| = 400$)

Example: Third Order System

System description

$$P(s) = \frac{1}{(s+1)^3}$$

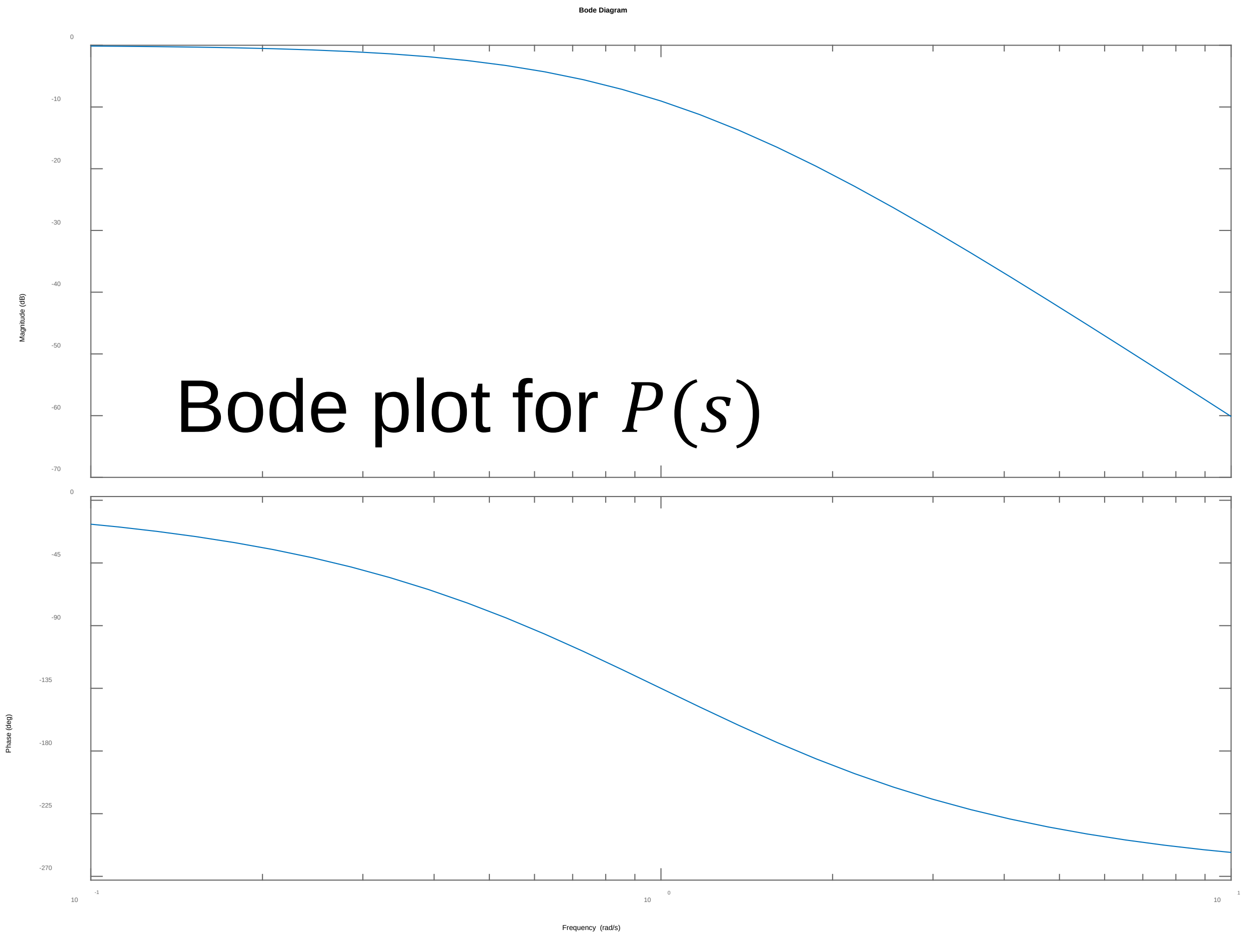
- Poles: $p_{1,2,3}$ = -1

Control specs

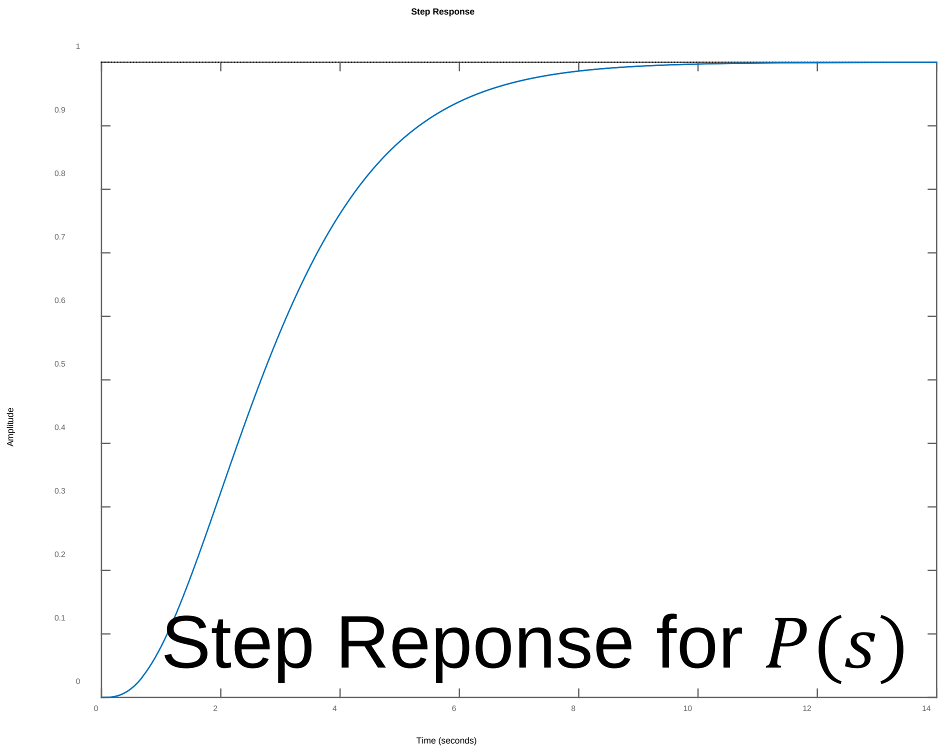
- Steady state error < 1%
- <10% tracking error up to 10 rad/s
- 0.8 sec settling time
- Overshoot less than 10%
  - Gives PM of ~60 deg

First Cut:

- Need to boost low frequency gain
- Need to increase bandwidth
- Need to increase phase at higher frequency

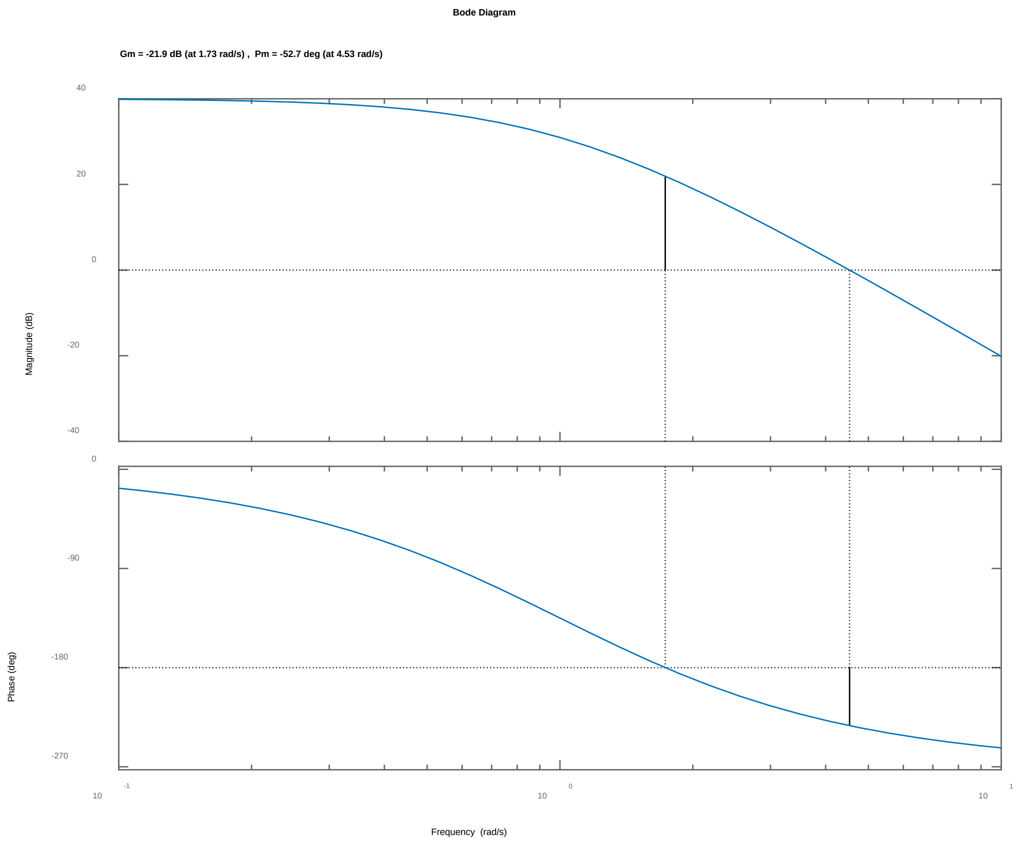


Bode plot for $P(s)$



Step Reponse for $P(s)$

Example: Third Order System (2)

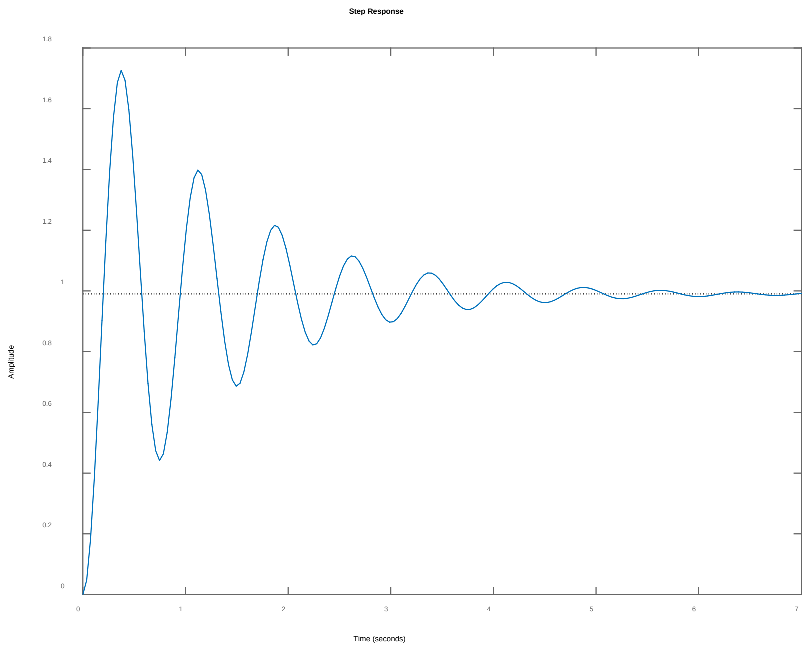


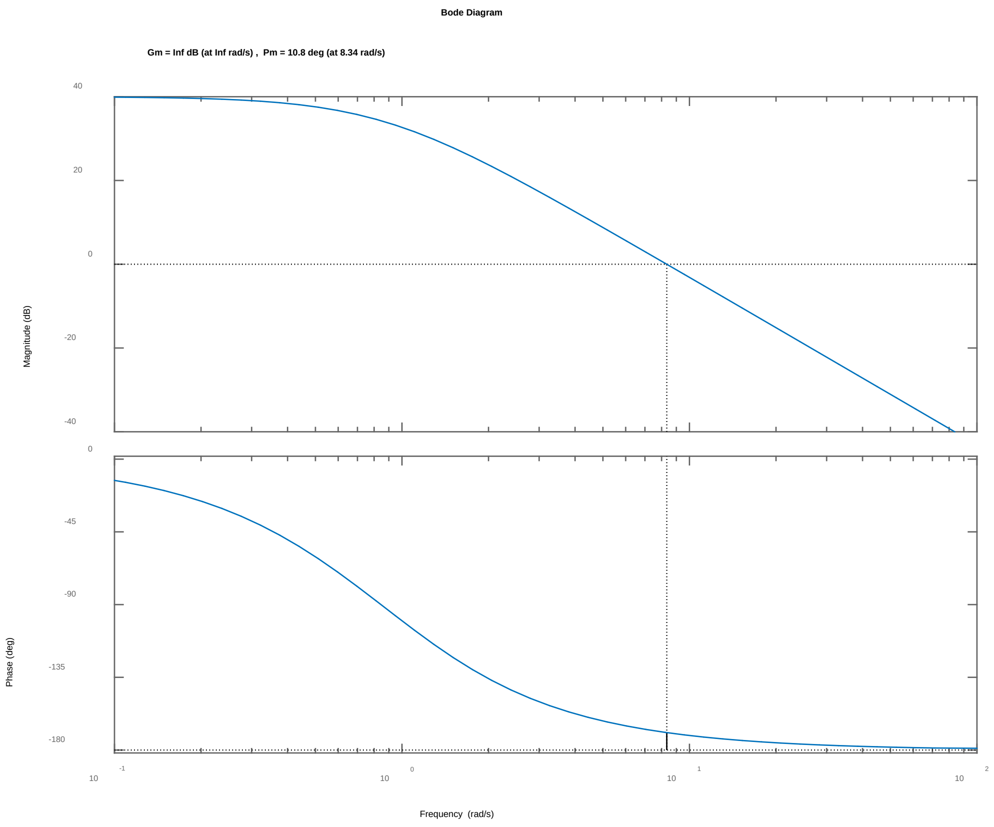
Start with”P” of PID control

- $k_p = 100$
- Closed loop system is ***unstable***
- Bandwidth too low

Next: PD Control

- $k_p = 100, k_d = 70$
- Closed loop system is ***stable***
- Phase margin 10^o at 8.3 rad/sec
    - Oscillatory step response
- Tracking requirement not met





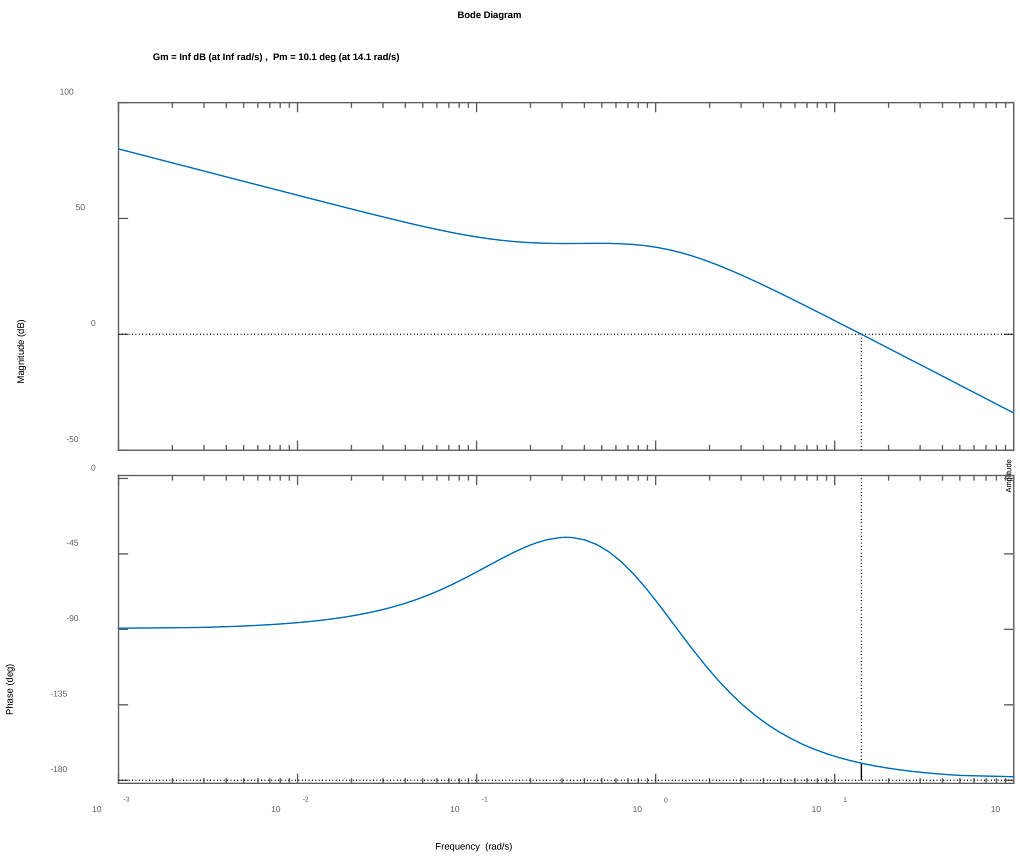
Can “I” in PID help?

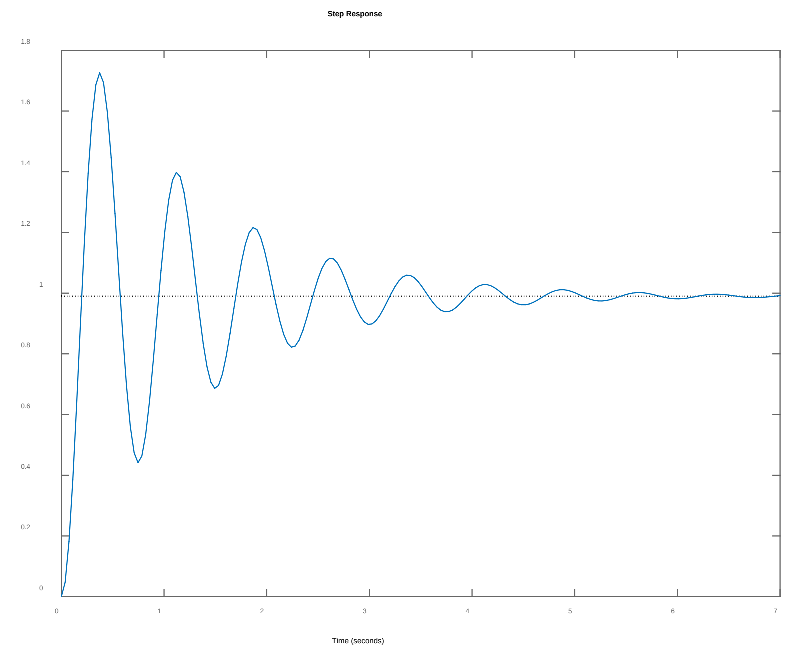
- adds -90^o phase!

Example: Third Order System (3)

Can “I” in PID help?

- Can create stable system which meets steady-state spec., but not phase margin (and overshoot) & bandwidth



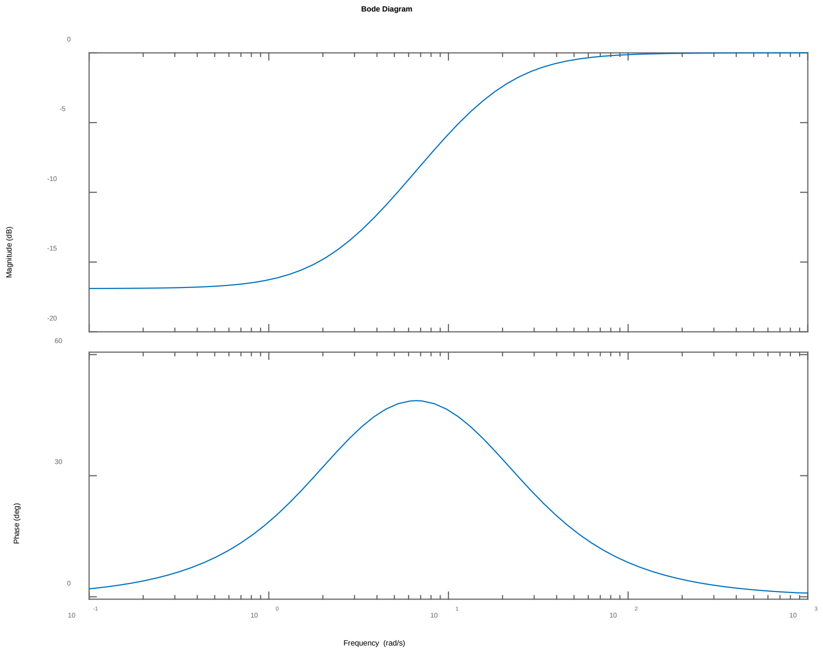


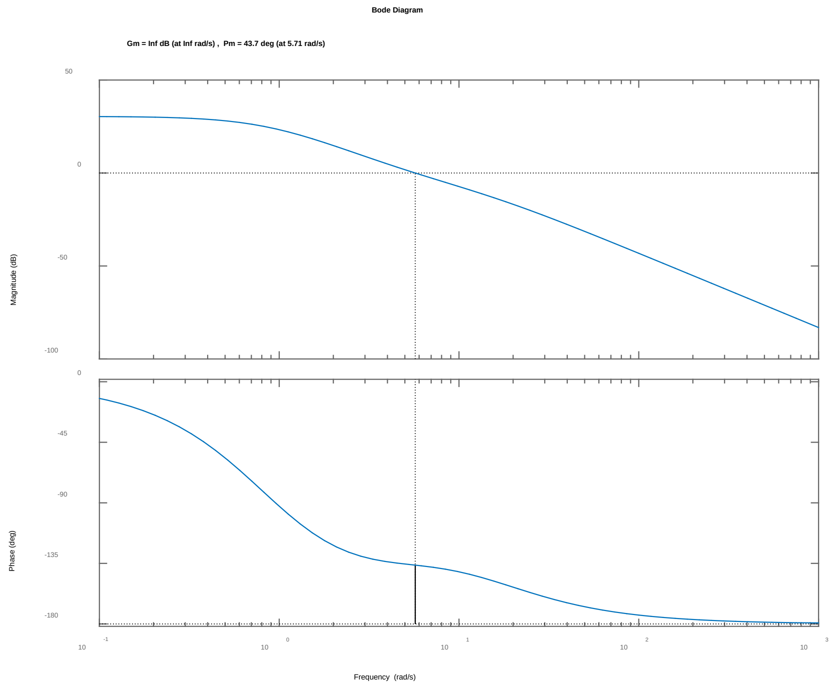
PD + ***Lead*** compensator?

- $k_p = 100,\ k_d = 70$
- Lead compensator reduces low frequency gain (& bandwidth)

$$C_{lead}(s) = \frac{s+a}{s+b}$$

$$a = 2.5,\ b = 17.5$$





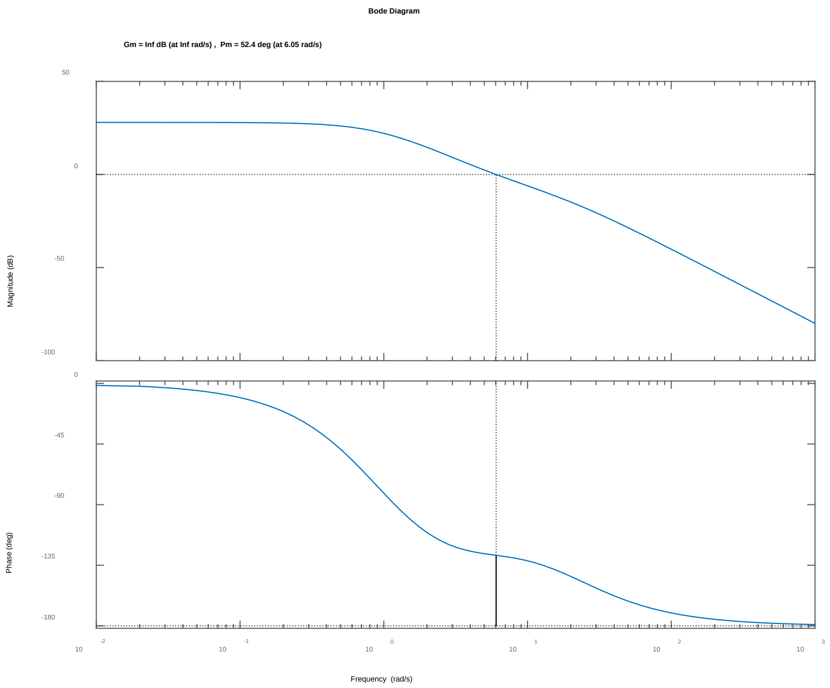
Bode plot for Lead

Bode plot for $L(s)$

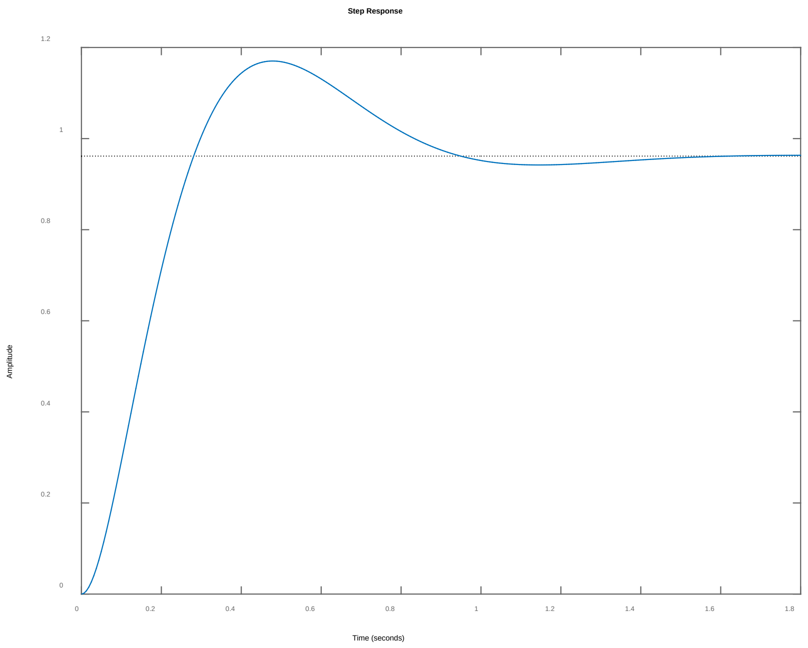
Example: Third Order System (4)

PD + Lead compensator

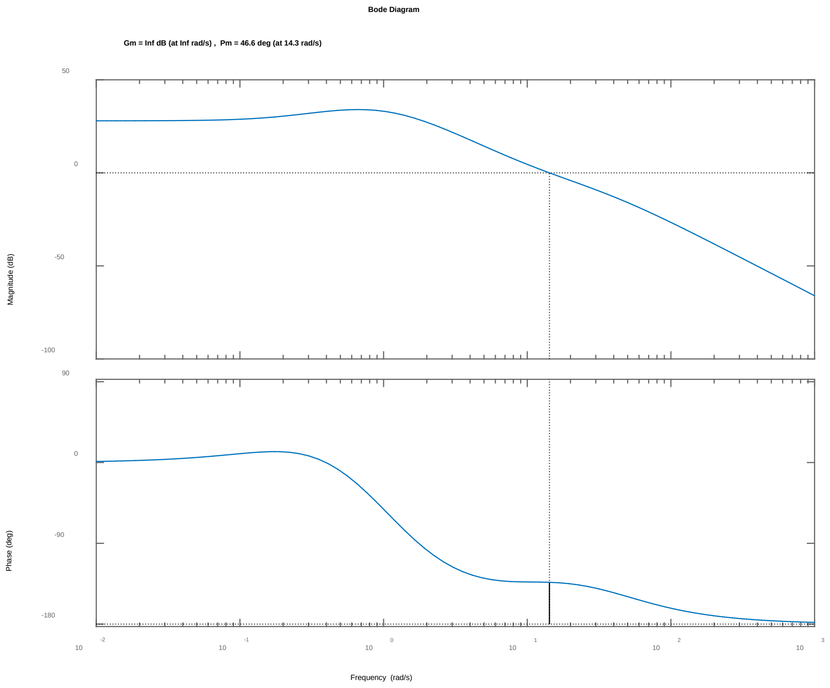
- Increase proportional gain to compensate for Lead at low frequency.
- Not enough bandwidth to meet tracking spec.
- Live with it, or keep searching

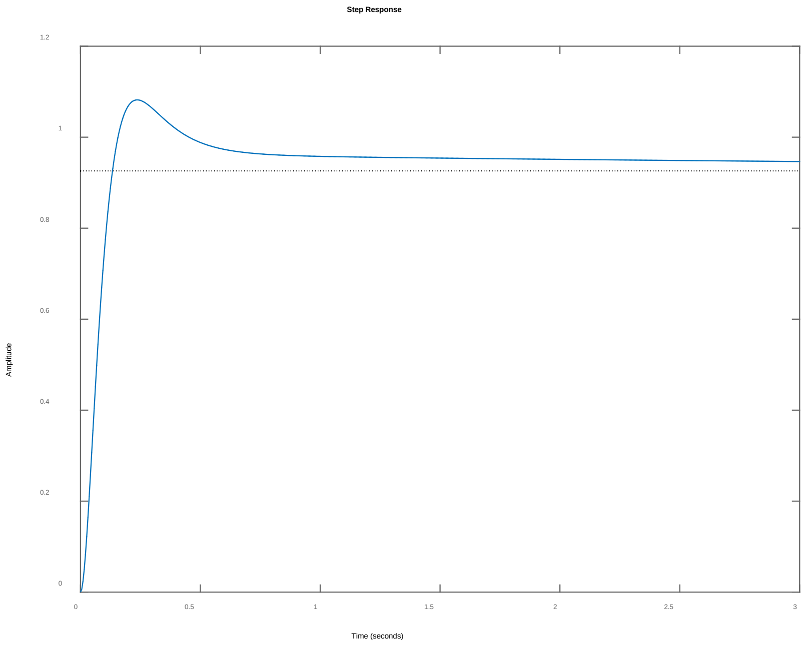


Bode plot for $L(s)$



- Increase “D” to add gain at high frequency
  - $k_p = 100,\ k_d = 500$
- Shift phase lead center frequency (a=5, b=40);
- Almost meets spec.s

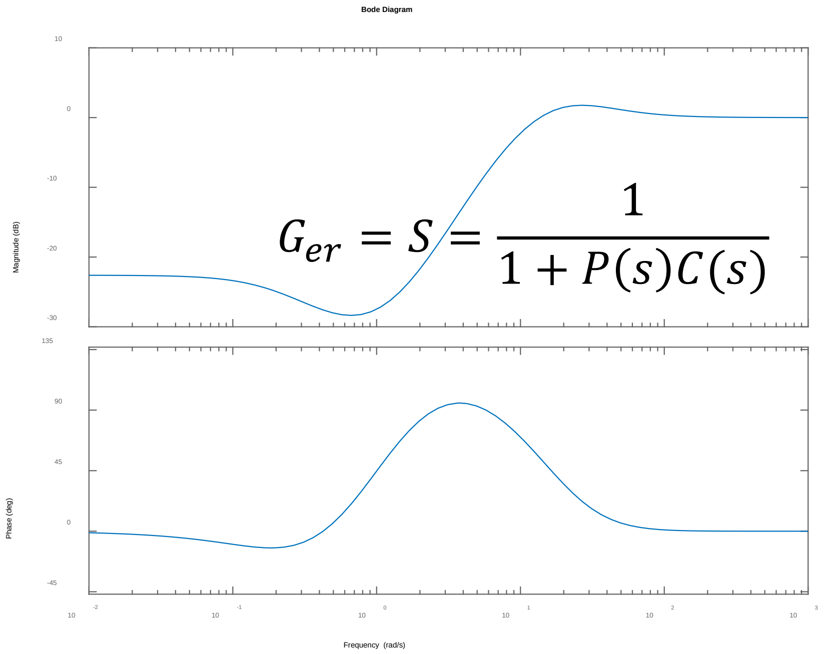




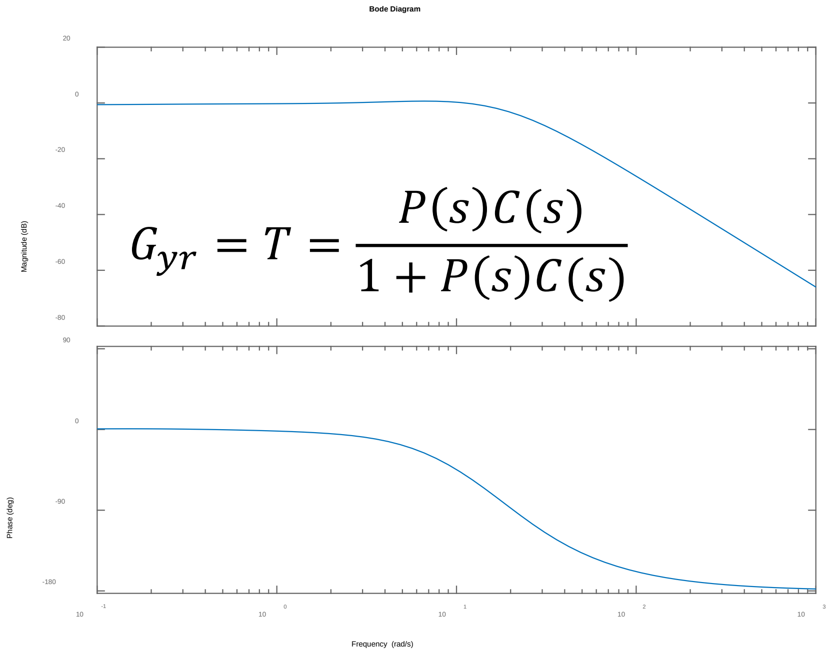
Example: Third Order System (5)

Before trying more esoteric designs, check the basics

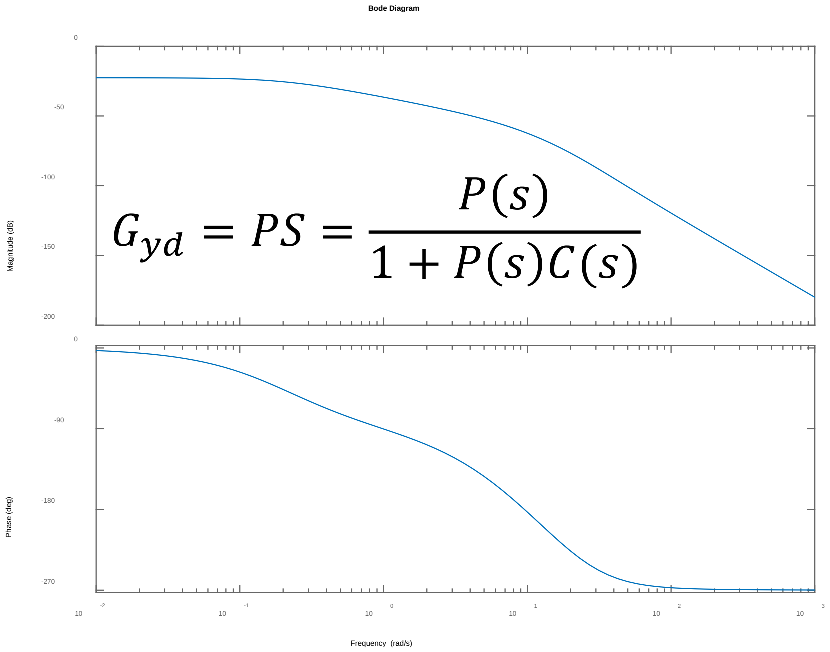
- Nyquist
- Gang of Four



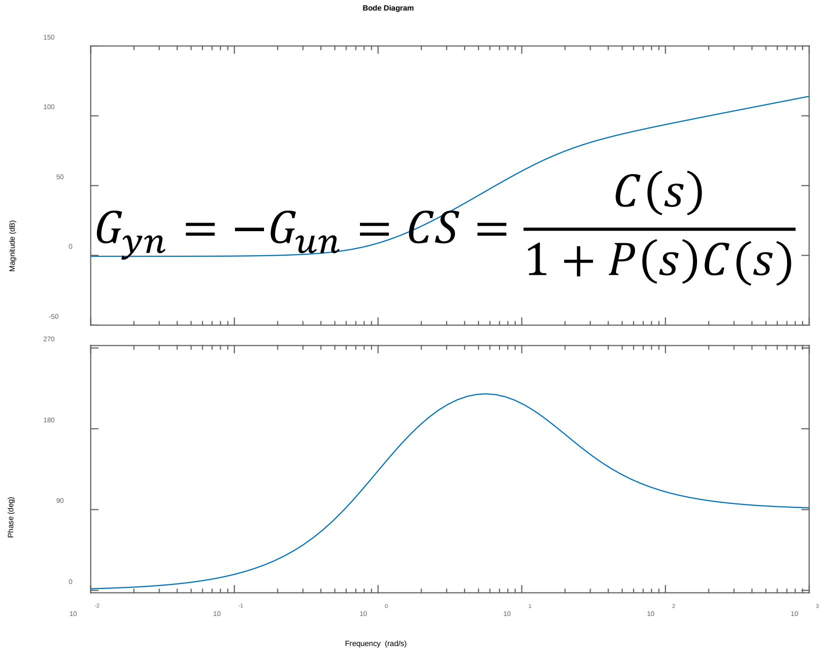
$$G_{er} = S = \frac{1}{1 + P(s)C(s)}$$



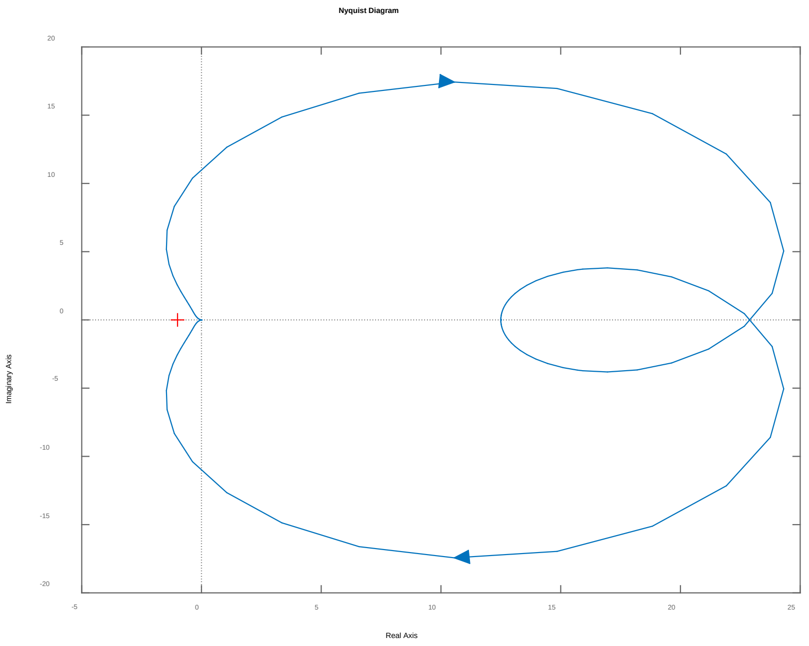
$$G_{yr} = T = \frac{P(s)C(s)}{1 + P(s)C(s)}$$



$$G_{yd} = PS = \frac{P(s)}{1 + P(s)C(s)}$$



$$G_{yn} = -G_{un} = CS = \frac{C(s)}{1 + P(s)C(s)}$$



$$C(s) = \frac{500s^2 + 2600s + 500}{s + 40}$$

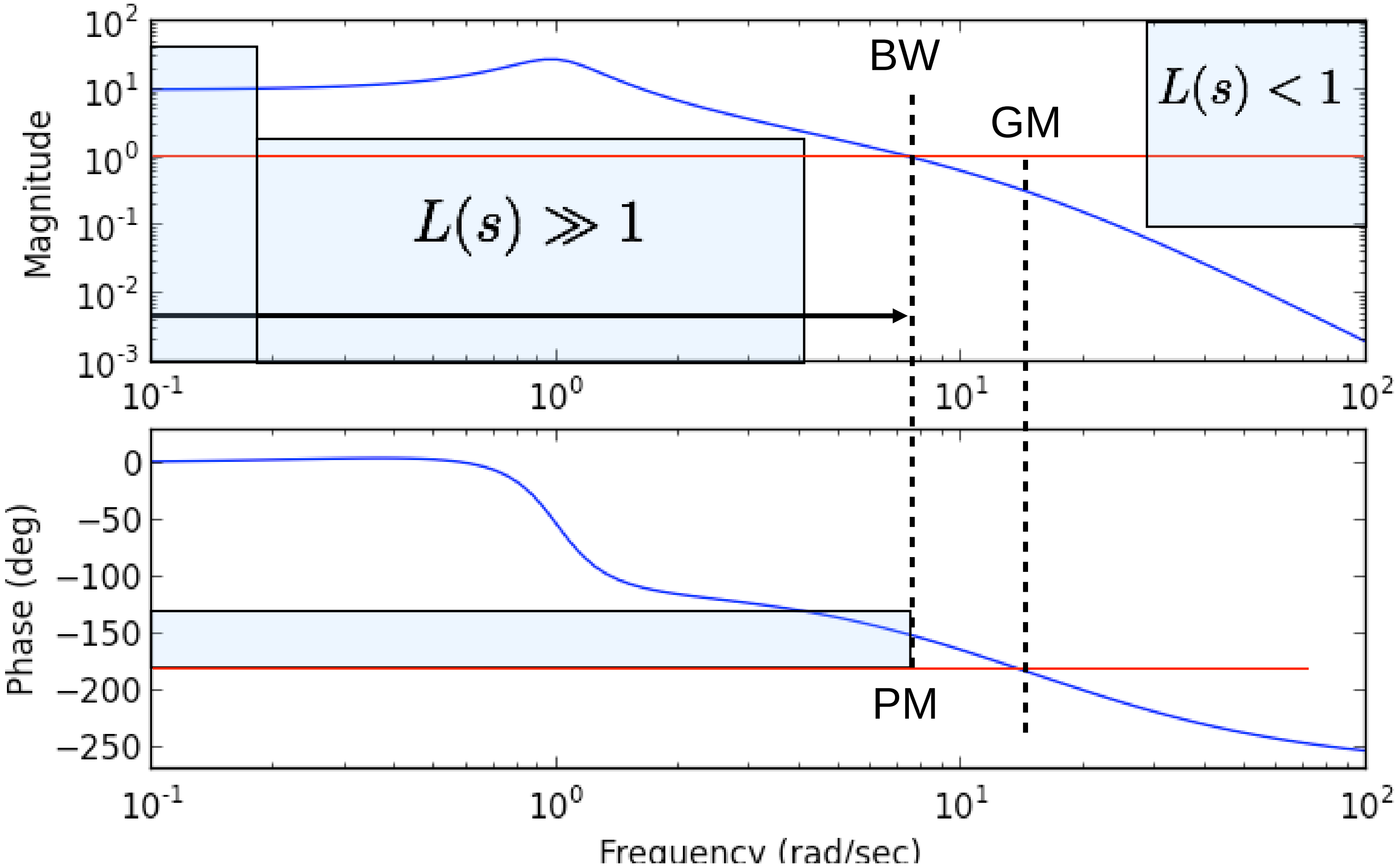
This controller amplifies high frequency noise, which will lead to a significant actuator activity

Fix by adding “roll-off” pole(s) at higher freq.

Summary: Loop Shaping

Loop Shaping for Stability & Performance

- Steady state error, bandwidth, tracking response
- Specs can be on any input/output response pair



Things to remember (for homework and exams)

- Always plot Nyquist to verify stability/robustness
- Check gang of 4 to make sure that noise and disturbance responses also look OK

Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI