Kinematics of Motion Capture based on Quaternions

This set of notes derives a technique to estimate the displacement of a rigid body using markers placed on the body, and a camera system to track the marker positions.

1 Least Squares Solution

Assume that a rigid body lies in position #1. A body-fixed reference frame aligns with a fixed observing reference frame in this position. Then body then displaces by a translation $\vec{d}_{12}$ and a rotation $R_{12} \in SO(3)$ to a position #2. Assume that at least three noncolinear marker points can be identified in the body at the first position: $(P_1, P_2, \ldots, P_N)$. After displacements, the three points are located at $(Q_1, Q_2, \ldots, Q_N)$. Clearly,

$$Q_i = \vec{d}_{12} + R_{12} P_i \quad i = 1, 2, 3.$$

In a motion capture context, the points are associated with body-fixed markers, whose positions are readily measured with a camera system. However, we must expect some error in the measurement of the marker locations, which we will model as zero mean noise. Our goal is to estimate $\vec{d}_{12}$ and $R_{12}$ from these measurements. We will use a least squares approach to finding the displacement estimates from the noisy measurements.

Let an error, $e_i$, in the $i^{th}$ point coordinate be defined as follows:

$$e_i = Q_i - \vec{d}_{12} - R_{12} P_i.$$

That is, if the location of points $P_i$ and $Q_i$ where measured by the camera system with no errors, and if we knew $\vec{d}_{12}$ and $R_{12}$ exactly, then the error $e_i$ would be zero. Since measurement errors must be expected, the best estimate of the displacement is found by minimizing the following error:

$$E = \sum_{i=1}^{N} ||e_i||^2 = \sum_{i=1}^{N} ||Q_i - \vec{d}_{12} - R_{12} P_i||^2.$$

That is, the best estimate of $\vec{d}_{12}$ and $R_{12}$ are the ones which minimize this error function.

To simplify the evaluation of this expression, let us introduce the following centroids of the body-fixed points:

$$\bar{P} = \frac{1}{N} \sum_{i=1}^{N} P_i \quad \bar{Q} = \frac{1}{N} \sum_{i=1}^{N} Q_i$$

and express the marker point coordinates with respect to these centroids:

$$P_i' = P_i - \bar{P} \quad Q_i' = Q_i - \bar{Q}.$$

The error term $e_i$ can be expressed in these adjusted coordinates as follows:

$$e_i = Q_i - \vec{d}_{12} - R_{12} P_i = Q_i' + \bar{Q} - \vec{d}_{12} - R_{12}(P_i' + \bar{P}) = Q_i' - R_{12} P_i' - z$$
where \( z = -\vec{d}_{12} + \bar{Q} - R_{12}\bar{P} \). In these adjusted coordinates, the total error takes the form:

\[
E = \sum_{i=1}^{N} ||Q'_i - R_{12}P'_i - z||^2 = \sum_{i=1}^{N} ||Q'_i - R_{12}P'_i||^2 - 2z \cdot (Q'_i - R_{12}P'_i) + z^2 .
\] (2)

Note that the third term, \( z^2 \), can only be minimized if \( z = 0 \), which implies that:

\[
\vec{d}_{12} = \bar{Q} - R_{12}\bar{P} .
\] (3)

That is, once \( R_{12} \) is known, \( \vec{d}_{12} \) can be found from Equation (3), and the expression only depends upon the centroids of the marker points.

The second term of Equation (2) vanishes since \( \sum_{i=1}^{N} Q'_i = \sum_{i=1}^{N} P'_i = 0 \) by the definition of centroid.

\[
2z \cdot \sum_{i=1}^{N} (Q'_i - R_{12}P'_i) = 2z \cdot \left[ \sum_{i=1}^{N} Q'_i - R_{12}\sum_{i=1}^{N} P'_i \right] = 0 .
\]

Thus, \( R_{12} \) is found by minimizing the first term of Equation (2)

\[
R_{12} = \arg \min \sum_{i=1}^{N} ||Q'_i - R_{12}P'_i||^2 .
\] (4)

Note that because rotation matrices preserve the lengths of vectors, each term \( Q'_i - R_{12}P'_i \) is minimized by aligning vector \( R_{12}P'_i \) with vector \( Q'_i \) as closely as possible. Hence, Equation (4) is equivalent to:

\[
R_{12} = \arg \max \sum_{i=1}^{N} Q'_i \cdot (R_{12}P'_i) .
\] (5)

As will be shown below, it is easiest to solve this optimization problem by converting it to use a quaternion representation of the rotation \( R_{12} \).

### 2 Quaternion Review

Recall that a quarterion, \( q \), takes the form

\[
q = q_0 + q_x i + q_y j + q_z k
\]

where basis elements \( i, j, \) and \( k \) obey the rules:

\[
i^2 = j^2 = k^2 = -1 \\
i j = -ji = k \\
i k = -ki = -j \\
j k = -kj = i .
\]
The quaternion can also be simply represented as a 4-tuple, \( q = (q_0, q_x, q_y, q_z) \), with the basis elements implicit. When the context is clear we can interpret the 4-tuple as a \( 4 \times 1 \) vector.

If two quaternions, \( q \) and \( r \), take the form:

\[
q = q_0 + q_x i + q_y j + q_z k \\
r = r_0 + r_x i + r_y j + r_z k
\]

then the product of the two quaternions takes the form:

\[
r \cdot q = (r_0q_0 - r_x q_x - r_y q_y - r_z q_z) + (r_0q_x + r_x q_0 + r_y q_z - r_z q_y)i \\
+ (r_0q_y - r_x q_z + r_y q_0 + r_z q_x)j + (r_0q_z + r_x q_y - r_y q_x + r_z q_0)k
\]

Note that this product can also be represented in the following way

\[
 rq = \begin{bmatrix}
 r_0 & -r_x & -r_y & -r_z \\
 r_x & r_0 & -r_z & r_y \\
 r_y & r_z & r_0 & -r_x \\
 r_z & -r_y & r_x & r_0 \\
\end{bmatrix} q \triangleq \mathcal{R}q \quad (6)
\]

where quaternion \( q \) is treated as a \( 4 \times 1 \) vector. In a similar way

\[
 qr = \begin{bmatrix}
 r_0 & -r_x & -r_y & -r_z \\
 r_x & r_0 & r_z & -r_y \\
 r_y & -r_z & r_0 & r_x \\
 r_z & r_y & -r_x & r_0 \\
\end{bmatrix} q \triangleq \bar{\mathcal{R}}q \quad (7)
\]

Also note that \( r^* q = \mathcal{R}^T q \) and \( qr^* = \bar{\mathcal{R}}^T q \), where \( r^* \) denotes the conjugate of \( r \): \( r^* = (r_0, -r_x, -r_y, -r_z) \).

Finally, let \( \odot \) denote a dot product operator between two quaternions. That is, if we interpret quaternion \( r \) as a \( 4 \times 1 \) vector \( r = \begin{bmatrix} r_0 & r_x & r_y & r_z \end{bmatrix} \) and quaternion \( q \) as the \( 4 \times 1 \) vector \( q = \begin{bmatrix} q_0 & q_x & q_y & q_z \end{bmatrix} \), then

\[
 r \odot q = \begin{bmatrix} r_0 \\
 r_x \\
 r_y \\
 r_z \\
\end{bmatrix} \cdot \begin{bmatrix} q_0 \\
 q_x \\
 q_y \\
 q_z \\
\end{bmatrix} = r_0q_0 + r_xq_x + r_yq_y + r_zq_z.
\]

3 Estimating Displacements using Quaternions

Let \( q_{12} \) be the unit quaternion which represents the same rotation as \( R_{12} \in SO(3) \). Let \( p_i' \) be the pure or vector quaternion that represents the vector \( P_i' \). That is,

\[
P_i' = \begin{bmatrix} P_{i,x}' \\
 P_{i,y}' \\
 P_{i,z}' \\
\end{bmatrix} \quad \Rightarrow \quad p_i' = (0, P_{i,x}', P_{i,y}', P_{i,z}').
\]
Similarly, let \( q'_i = (0, Q'_{i,x}, Q'_{i,y}, Q'_{i,z}) \) be the vector quaternion that represents the vector \( Q'_i \). Recall that the product of the rotation matrix \( R_{12} \in SO(3) \) and the vector \( P'_i \in \mathbb{R}^3 \), \( R_{12}P'_i \), can be represented in terms of quaternions as:

\[
q_{12}p'_i q_{12}^* .
\]

Hence, the least squares estimate of \( R_{12} \) in Equation (5) can be expressed as

\[
q_{12} = \arg \max_n \sum_{i=1}^{N} q'_i \odot (q_{12}p'_i q_{12}^*) .
\]

To solve Equation (8), note that (using Equation (7)) \( q_{12}p'_i q_{12}^* \) can be expressed as \( \hat{Q}_{12}^T(q_{12}p'_i) \). Hence,

\[
q_{12} = \arg \max_n \sum_{i=1}^{N} q'_i \odot (q_{12}p'_i q_{12}^*) = \arg \max_n \sum_{i=1}^{N} q'_i \odot (\hat{Q}_{12}^T q_{12}p'_i) \\
= \arg \max_n \sum_{i=1}^{N} (\hat{Q}_{12}q'_i) \odot (q_{12}p'_i) = \arg \max_n \sum_{i=1}^{N} (q'_i q_{12}) \odot (q_{12}p'_i) \\
= \arg \max_n \sum_{i=1}^{N} (Q'_i q_{12}) \odot (P'_i q_{12}) = \arg \max_n q_{12}^T \sum_{i=1}^{N}(Q'_i)^T P'_i q_{12} \\
\triangleq q_{12}^T \sum_{i=1}^{n} N_i q_{12} \triangleq q_{12}^T N q_{12}
\]

where matrices \( \hat{Q}'_i \) and \( P'_i \) are patterned after Equations (6) and (7):

\[
P'_i = \begin{bmatrix}
0 & -P'_{i,y} & P'_{i,z} \\
P'_{i,x} & 0 & -P'_{i,z} \\
P'_{i,y} & P'_{i,z} & 0
\end{bmatrix} \quad Q'_i = \begin{bmatrix}
0 & Q'_{i,x} & -Q'_{i,y} & -Q'_{i,z} \\
-Q'_{i,x} & 0 & Q'_{i,z} & Q'_{i,y} \\
Q'_{i,z} & -Q'_{i,y} & 0 & Q'_{i,x}
\end{bmatrix}
\]

and

\[
N = \begin{bmatrix}
(S_{xx} + S_{yy} + S_{zz}) & S_{yz} - S_{zy} & S_{zx} - S_{xz} & S_{xy} - S_{yx} \\
S_{yz} - S_{zy} & (S_{xx} - S_{yy} - S_{zz}) & S_{xy} + S_{yx} & S_{xz} + S_{zx} \\
S_{zx} - S_{xz} & S_{xy} + S_{yx} & (S_{xx} + S_{yy} - S_{zz}) & S_{yz} + S_{zy} \\
S_{xy} - S_{yx} & S_{xz} + S_{zx} & S_{yz} + S_{zy} & (S_{xx} - S_{yy} + S_{zz})
\end{bmatrix}
\]

where the \( 3 \times 3 \) matrix \( S \) has the form:

\[
S = \begin{bmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{yy} & S_{yz} \\
S_{zx} & S_{zy} & S_{zz}
\end{bmatrix} = \sum_{i=1}^{N} Q'_i (P'_i)^T .
\]

Note that \( q_{12}^T N q_{12} \) will be maximized with respect to \( q_{12} \) when the \( 4 \times 1 \) vector \( q_{12} \) aligns with the eigenvector associated with the maximum eigenvalue of \( N \).
4 Summary

Let \((P_1, P_2, \ldots, P_N)\) denote the positions of a set of markers attached to a rigid body in the first position (before a displacement). After the rigid body displaces to a second position, the marker locations are described by positions \((Q_1, Q_2, \ldots, Q_N)\). The goal is to estimate the rigid body displacement \((\vec{d}_{12}, R_{12})\), where \(\vec{d}_{12}\) is the translation of the rigid body between the two positions, and \(R_{12}\) denotes the relative orientation of the body in the second position with respect to the first position.

Here is a brief summary of an approach that uses the derivations above:

- Compute the centroids of the points in the first and second positions from Equation (1): \(\bar{P}\) and \(\bar{Q}\).
- Compute the coordinates of the points with respect to the centroids: \(P'_i = P_i - \bar{P}\), \(Q'_i = Q_i - \bar{Q}\), for \(i = 1, \ldots, N\).
- Compute the \(S\)-matrix in Equation (11).
- Compute the \(N\)-matrix, Equation (10).
- Find the eigenvector of \(N\) associated with the largest eigenvalue of \(N\). Normalize the eigenvector to ensure that it is a unit quaternion.
- Find the equivalent rotation matrix \(R_{12}\) to the unit quaternion found in the last step.
- Find the displacement, \(\vec{d}_{12}\), using \(R_{12}\) found in the last step and Equation (3).