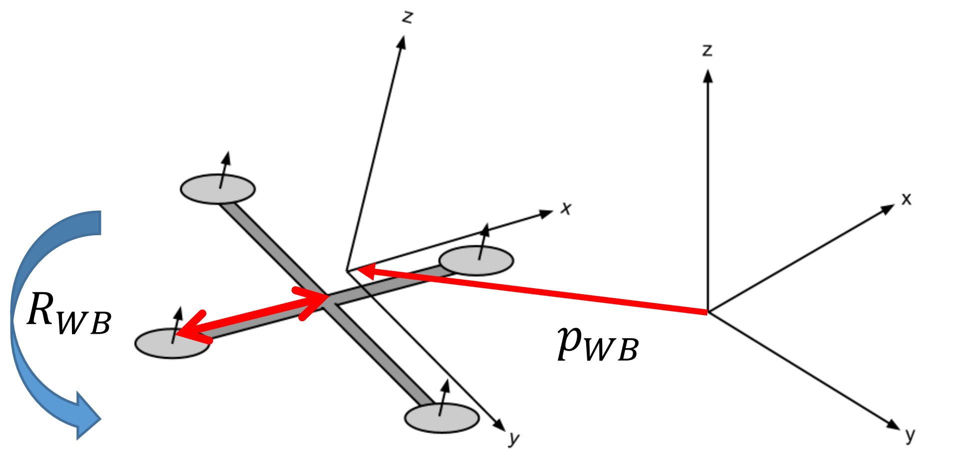
Quick Review of Quadcopter Dynamics

Rigid Body Dynamics in Body-fixed frame (assuming origin at center of mass)

- $\vec{f}^b = m\,\dot{v} + m\omega^b \times v^b + mg\vec{e}^b$ $\quad \overrightarrow{e^b}$ = unit vector parallel to gravity$= -R_{WB}^T\,\vec{z}_W$
- $\vec{\tau}^b = I_{cm}^b\dot{\omega}^b + \omega^b \times I_{cm}^b\omega^b$

Where the force and torque on the quadrotor are given by:

- $\begin{pmatrix} f_z^b \\ \vec{\tau}^b \end{pmatrix} = \begin{pmatrix} c_T & c_T & c_T & c_T \\ 0 & dc_T & 0 & -dc_T \\ -dc_T & 0 & dc_T & 0 \\ -c_Q & c_Q & -c_Q & c_Q \end{pmatrix} \begin{pmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{pmatrix}$
- $\begin{pmatrix} f_z^b \\ \vec{\tau}^b \end{pmatrix} = \mathrm{C}\,\vec{\Omega}^2$



Body-fixed frame

World (inertial) frame

Rigid Body Dynamics in Inertial frame (with frame center at CM) using hybrid velocites

- $$\begin{pmatrix} \vec{f} \\ \vec{\tau} \end{pmatrix} = \begin{pmatrix} mI_{3\times3} & 0 \\ 0 & I_{cm}^s \end{pmatrix} \begin{pmatrix} \ddot{p} \\ \dot{\omega}^s \end{pmatrix} + \begin{pmatrix} 0 \\ \omega^s \times I_{cm}^s \omega^s \end{pmatrix} + mg \begin{pmatrix} \vec{e}_z^s \\ 0 \end{pmatrix}$$
- For quadrotor,
  - $\vec{f} = R_{WB}\, \vec{f}^b = R_{WB}\left(\sum_i c_T \Omega_i^2\right) \vec{z}^b;$
  - $$\vec{\tau} = R_{WB}\, \vec{\tau}^b = R_{WB} \begin{pmatrix} d\, c_T(\Omega_2^2 - \Omega_4^2) \\ d\, c_T(\Omega_3^2 - \Omega_1^2) \\ c_Q(\Omega_2^2 - \Omega_1^2 + \Omega_4^2 - \Omega_3^2) \end{pmatrix}$$

Rigid Body Dynamics (center of mass at position $\vec{c}$) in hybrid velocities

- $$\begin{pmatrix} \vec{f} \\ \vec{\tau} \end{pmatrix} = \begin{pmatrix} mI_{3\times3} & -m\hat{c} \\ +m\hat{c} & I_{cm}^s - m\hat{c}\hat{c} \end{pmatrix} \begin{pmatrix} \ddot{p} \\ \dot{\omega}^s \end{pmatrix} + \begin{pmatrix} m\omega^s \times \omega^s \times \vec{c} \\ \widehat{\omega}^s (I_{cm}^s - m\hat{c}\hat{c})\omega^s \end{pmatrix} + mg \begin{pmatrix} \vec{e}_z{}^s \\ \vec{c} \times \vec{e}_z^s \end{pmatrix}$$

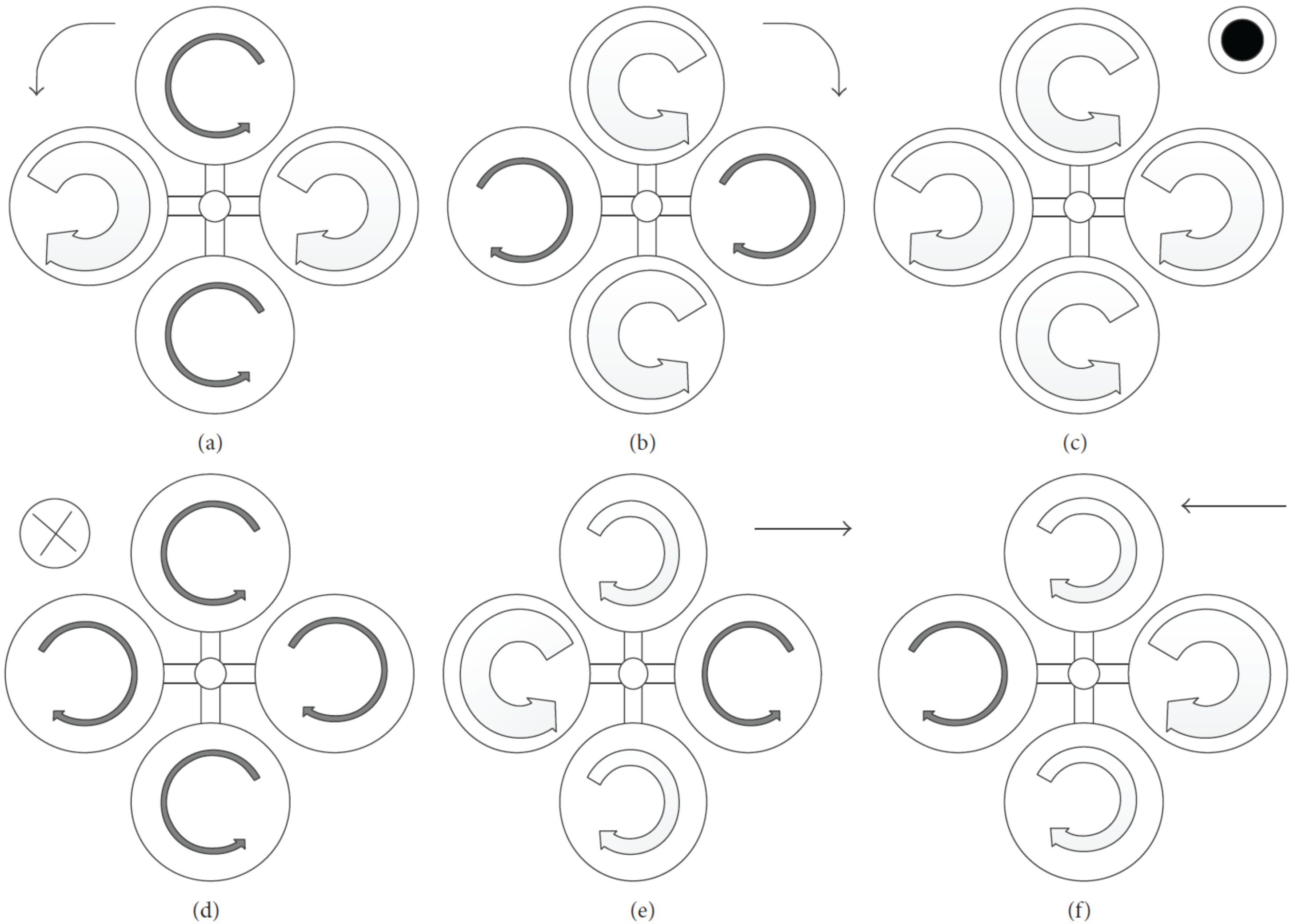


FIGURE 2: Quadrotor dynamics: (a) and (b) difference in torque to manipulate the yaw angle (Ψ); (c) and (d) hovering motion and vertical propulsion due to balanced torques; (e) and (f) difference in thrust to manipulate the pitch angle (θ) and the roll angle (ϕ).

Basics of Quadrotor Control

Control can be roughly decoupled in to *attitude* control and *position* control

Attitude Control: let $u^\tau = \begin{bmatrix}\tau_x & \tau_y & \tau_z\end{bmatrix}^T$ denote the vector of torques about the 3 principle axes

- Propose a *PD control*: $u^\tau = -K_r \vec{e}_r - K_\omega \vec{e}_\omega$
  - $\vec{e}_\omega = \omega - \omega^d$
  - $\vec{e}_r = \frac{1}{2}\left[\left(R_{WB}^d\right)^T R_{WB} - R_{WB}^T R_{WB}^d\right]^\vee$
  - If we use *roll, pitch*, *yaw* angles (x-y-z Euler angles) ϕ, θ, ψ
    - Linearize non-linear error measurement about hover: $\begin{bmatrix} 0 & -\Delta\psi & \Delta\theta \\ \Delta\psi & 0 & -\Delta\phi \\ -\Delta\theta & \Delta\phi & 0 \end{bmatrix}^\vee = \begin{bmatrix}\Delta\phi \\ \Delta\theta \\ \Delta\psi\end{bmatrix}$
- $u^\tau = -K_r \begin{bmatrix}\Delta\phi \\ \Delta\theta \\ \Delta\psi\end{bmatrix} - K_\omega \begin{bmatrix}\omega_x - \omega_x^d \\ \omega_y - \omega_y^d \\ \omega_z - \omega_z^d\end{bmatrix}$

Position Control: let u^f denote the total thrust along the body-fixed z-axis

- $u^f = m\, \vec{z}^b \cdot \left(g\, \vec{z}^s + p^d + K_v\, \vec{e}_v + K_p\, \vec{e}_p\right)$, where $\vec{e}_v = (\dot{p} - \dot{p}^d)$ and $\vec{e}_p = (p - p^d)$

$$\vec{\Omega}^2 = C^{\{-1\}} \begin{pmatrix} u^f \\ u^\tau \end{pmatrix}$$