Global Bilinearization and Controllability of Control-Affine Nonlinear Systems: A Koopman Spectral Approach

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Abstract— **This paper considers the problem of global bilinearization of the drift and control vector fields of a control-affine system. While there are linearization techniques like Carleman linearization for embedding a finite-dimensional nonlinear system into an infinite-dimensional space, they depend on the analytic property of the vector fields and work only on polynomial space. The proposed method utilizes the Koopman Canonical Transform to transform the dynamics and ensures bilinearity from the projection of the Koopman operator associated with the control vector fields on the eigenspace of the drift Koopman operator. The resulting bilinear system is then subjected to controllability analysis using the Myhill semigroup method and Lie algebraic structures. The results are supported by a numerical example.**

I. INTRODUCTION

Traditionally, dynamical systems are described in terms of trajectories defined by a flow function or iterative map on finite dimensions. However, there is an alternative framework, an operator-theoretic approach, that relies on the (linear) operators of infinite-dimensional function spaces. One such operator is the Koopman operator proposed by B.O. Koopman in 1931 [1].

The Koopman operator's action on an observable function is to describe its evolution along the trajectory of the original system. Despite being implicitly used in Lyapunov stability theory for a century, operator-theoretic approach has found its way into the description of system outputs very recently [2], [3]. Being a linear operator, Koopman operator can be used in spectral analysis of nonlinear flows. Koopman eigenfunctions are directly related to the geometry of the system dynamics, e.g., periodic partitions in an ergodic system [2]. The nonlinear flow can be characterized by its dominant Koopman modes as shown in [3]. The Koopman modes may be approximated from snapshots of the system without having knowledge of the underlying dynamics, using the Krylov-subspace method for discrete-time systems [3]. For continuous time systems, the Koopman modes may be approximated by Taylor series expansion or Bernstein polynomials [4].

Operator-theoretic methods essentially work by embedding finite-dimensional dynamics in an infinite-dimensional function space in which functions evolve under a linear operator. The Koopman operator offers effective methods to characterize a nonlinear system in terms of stability [4] and linearization [5]. In [4], a framework is developed for formulating the global stability properties of the fixed points and limit cycles. A numerical approach to estimate the basin of attraction of those attractors uses the Koopman method. In addition, Koopman methods are related to numerical algorithms like Dynamic Mode Decomposition [3], [6].

The application of the Koopman method to actuated systems has proven to be difficult, because the actuation signals change the spectral properties of the Koopman operator. [7] introduces a method to incorporate the control input in Koopman framework. [8] bridges the gap of the analysis and simulation by providing a method to determine the spectral property of the Koopman operator of the underlying unforced system from the data of the actuated dynamics. However, neither of these methods permit the linearization of full-scale actuated systems with Koopman spectra. Recently however, [9] proposed a framework for designing an observer for a discrete-time unactuated nonlinear system. [10] extends the same framework into continuous time with control-affine dynamics by introducing the Koopman Canonical Transform (KCT) using Koopman eigenfunctions, which transform the (nonlinear) dynamics into an observer form.

This manuscript utilizes the KCT to transform a control-affine system into a bilinear one. The sufficient conditions are given for such a reduction. The bilinearization thus obtained is global and does not rely on the neighborhood of the operating point or trajectory. The bilinear system is then used for controllability analysis and designing stabilizing control. For this we resorted to Myhill semigroup theory and the Lie algebraic methods described in [11], [12].

The contributions of this work are (1) proposing a method using the Koopman Canonical Transform (KCT) to globally bilinearize a control-affine nonlinear system; (2) sufficient conditions for complete bilinearization using the eigenspaces of the Koopman operators; and (3) analysis of the controllability and design of the stabilizing control using the resulting bilinear system. Numerical simulations demonstrate effective bilinearization of a control-affine nonlinear system. The bilinearized system thus obtained is simpler than the original system in terms of controllability and control design.

The manuscript is organized as follows. Section II provides a brief overview of Koopman spectral theory, describes the motivation behind spectral bilinearization and explains the Koopman Canonical Transform and its properties. Section III explores the sufficient conditions for bilinearizability for a control-affine nonlinear system under various assumptions. Section IV gives the reachability analysis of the Koop-

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man bilinear form using a Lie algebraic setup. Section V supports the analytical conclusions with numerical examples. Section VI summarizes the manuscript and discusses possible future work.

II. Mathematical Background

A. Overview of Koopman Theory

Consider a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{X} \subseteq \mathbb{R}^d$ and $\mathbf{f} : \mathbb{X} \to \mathbb{X}$. Let $\mathbf{\Phi}(t, \mathbf{x})$ be the flow map of the system (1). Let $\mathcal{F}$ be the space of all complex-valued observables $\varphi : \mathbb{X} \to \mathbb{C}$. The continuous time Koopman operator is defined as $\mathcal{K}^t : \mathcal{F} \to \mathcal{F}$ such that

$$(\mathcal{K}^t\varphi)(\cdot) = \varphi \circ \mathbf{\Phi}(t, \cdot). \tag{2}$$

Being linear over its arguments, the Koopman operator can be characterized by its eigenvalues and eigenfunctions. A function $\phi : \mathbb{X} \to \mathbb{C}$ is an eigenfunction of $\mathcal{K}^t$ if

$$(\mathcal{K}^t\phi)(\cdot) = e^{\lambda t}\phi(\cdot), \tag{3}$$

with eigenvalue $\lambda \in \mathbb{C}$. It can be shown [4] that the infinitesimal generator of $\mathcal{K}^t$, i.e., $\lim_{t\to 0} \dfrac{\mathcal{K}^t - I}{t}$ is $\mathbf{f} \cdot \nabla = L_\mathbf{f}$, where $L_\mathbf{f}$ is the Lie derivative with respect to $\mathbf{f}$. The infinitesimal generator satisfies the eigenvalue equation

$$L_\mathbf{f}\phi = \lambda\phi. \tag{4}$$

Hence, the time-varying observable $\psi(t, \mathbf{x}) \triangleq \mathcal{K}^t\varphi(\mathbf{x})$ is the solution of the PDE [4]

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= L_\mathbf{f}\psi, \\ \psi(0, \mathbf{x}) &= \varphi(\mathbf{x}_0), \end{aligned} \tag{5}$$

where $\mathbf{x}_0$ is the initial condition for the system (1).

Despite its linearity, the Koopman operator is infinite dimensional. If ϕ_1 and ϕ_2 are eigenfunctions of $\mathcal{K}^t$ with eigenvalues λ_1 and λ_2 respectively, then $\phi_1^k\phi_2^l$ is also an eigenfunction with eigenvalue $k\lambda_1 + l\lambda_2$ for any $k, l \in \mathbb{N}$. Moreover, the Koopman operator, being infinite dimensional, may contain continuous and residual spectra with a generalized eigendistribution [13]. The discussions in this paper are restricted to the point spectra of the Koopman operator.

Let $\mathbf{g}(\cdot) \in \mathcal{F}^p$, $p \in \mathbb{N}$ be a vector-valued observable. $\mathbf{g}$ can be expressed in terms of Koopman eigenfunctions $\phi_i(\cdot)$ as follows:

$$\mathbf{g}(\cdot) = \sum_{i=1}^{\infty} \phi_i(\cdot)\mathbf{v}_i^\mathbf{g}, \tag{6}$$

where $\mathbf{v}_i^\mathbf{g} \in \mathbb{R}^p, i = 1, 2, \ldots$ are called the *Koopman modes* of the observable $\mathbf{g}(\cdot)$. Koopman modes form the projection of the observable on the span of Koopman eigenfunctions [14]. The Koopman eigenvalues and the eigenfunctions are properties of the dynamics only, whereas the Koopman modes depend on the observable.

B. Motivation for bilinearization: Koopman Canonical Transform

A control affine system is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{f}_i(\mathbf{x})u_i \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}), \end{aligned} \tag{7}$$

where $\mathbf{x} \in \mathbb{X} \subseteq \mathbb{R}^d$, $u_i \in \mathbb{R}$, for $i = 1, \ldots, m$ and $\mathbf{y} \in \mathbb{R}^p$. Let $\psi(t, \mathbf{x})$ be defined as in (5). Note that $\psi(t, \mathbf{x})$ gives the evolution of an observable quantity $\varphi(\cdot)$ with time along the trajectory. Applying Eq. (5) for the system (7), the evolution PDE is given by

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= L_{\mathbf{f}_0}\psi + \sum_{i=1}^{m} u_i L_{\mathbf{f}_i}\psi, \\ \psi(0, \mathbf{x}) &= \varphi(\mathbf{x}_0), \end{aligned} \tag{8}$$

where $L_{\mathbf{f}_i} \triangleq \mathbf{f}_i \cdot \nabla$, $i = 0, \ldots, m$ are the corresponding Lie derivatives and hence are linear operators on the space of ψ. The system of PDE (8) looks quite similar to $\dot{\mathbf{x}} = A\mathbf{x} + \sum_{i=1}^{m} B_i\mathbf{x}u_i$, i.e., the usual bilinear system. The system (8) differs only in the fact that $L_{\mathbf{f}_i}$ are infinite dimensional operators operating over function space. The motivation comes from the possibility of projecting these operators on a finite-dimensional space by choosing suitable functional basis in terms of observables.

The natural choice of the basis functions is to use the Koopman eigenfunctions, since these functions, when operated on by the Koopman infinitesimal generator, are multiplied by a scalar only. For this transformation, we use Koopman Canonical Transform (KCT) defined in [10]. The KCT relies on the point spectra of the Koopman operator related to the drift vector field, and it suffices for most systems because the continuous spectrum is empty for most of them near an attractor [4]. For the system (7) we investigate the Koopman eigenvalues and eigenfunctions of the unactuated dynamics, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}), \tag{9}$$

and the flow associated with it. Let $\lambda_i, \phi_i(\cdot)$ for $i = 1, 2, \ldots$ be the eigenvalue-eigenfunction pairs of the Koopman operator associated with the system (9). KCT [10] transforms the dynamics (7) using the eigenfunctions ϕ_i in a possibly higher dimensional space. To enable use of KCT, [10] mentions the following assumption.

Assumption 1: $\exists\ \phi_i, i = 1, 2, \ldots, n$ such that

$$\mathbf{x} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^\mathbf{x}, \quad \mathbf{h}(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^\mathbf{h},$$

where $\mathbf{v}_i^\mathbf{x} \in \mathbb{C}^d$ and $\mathbf{v}_i^\mathbf{h} \in \mathbb{C}^p$. This assumption in turn tells us that the state vector and the output function can be described in terms of a finite number of Koopman eigenfunctions. With sufficiently large n, Assumption 1 is likely to be satisfied. If it is not, then $\mathbf{x}$ and $\mathbf{h}(\mathbf{x})$ may be well approximated by n eigenfunctions as we have in the case of a Fourier series.

KCT consists of the transformation $T(\mathbf{x})$ defined as follows [10]:

$$\begin{aligned} T(\mathbf{x}) &= [\tilde{\phi}_1(\mathbf{x}), \ldots, \tilde{\phi}_n(\mathbf{x})]^T \\ \tilde{\phi}_i(\mathbf{x}) &= \phi_i(\mathbf{x}), \text{ if } \phi_i : \mathbb{X} \to \mathbb{R} \\ (\tilde{\phi}_i(\mathbf{x}), \tilde{\phi}_{i+1}(\mathbf{x}))^T &= (2Re(\phi_i(\mathbf{x})), -2Im(\phi_i(\mathbf{x})))^T, \\ & \quad \text{if } \phi_i : \mathbb{X} \to \mathbb{C} \\ & \quad \text{and assuming } \phi_{i+1} = \overline{\phi}_i. \end{aligned} \tag{10}$$

Following the transformation $\mathbf{z} = T(\mathbf{x})$, the system (7) in the new coordinates is [10]

$$\begin{aligned} \mathbf{x} &= C^{\mathbf{x}}\mathbf{z}, \\ \dot{\mathbf{z}} &= D\mathbf{z} + \sum_{i=1}^{m} L_{\mathbf{f}_i} T(\mathbf{x}) u_i |_{\mathbf{x}=C^{\mathbf{x}}\mathbf{z}}, \\ \mathbf{y} &= C^{\mathbf{h}}\mathbf{z}, \end{aligned} \tag{11}$$

where $C^{\mathbf{x}} = [\tilde{\mathbf{v}}_1^{\mathbf{x}}| \ldots |\tilde{\mathbf{v}}_n^{\mathbf{x}}]$ and $C^{\mathbf{h}} = [\tilde{\mathbf{v}}_1^{\mathbf{h}}| \ldots |\tilde{\mathbf{v}}_n^{\mathbf{h}}]$ with $\tilde{\mathbf{v}}_i^{\mathbf{x}} = \mathbf{v}_i^{\mathbf{x}}$ if ϕ_i is real-valued, and $[\tilde{\mathbf{v}}_i^{\mathbf{x}}, \tilde{\mathbf{v}}_{i+1}^{\mathbf{x}}]$ is $[\operatorname{Re} \mathbf{v}_i^{\mathbf{x}}, \operatorname{Im} \mathbf{v}_i^{\mathbf{x}}]$ if ϕ_i is complex-valued. $\tilde{\mathbf{v}}_i^{\mathbf{h}}$ are defined similarly. $D \in \mathbb{R}^{n \times n}$ is a block diagonal matrix with diagonal entry $D_{i,i} = \lambda_i$ if ϕ_i is a real-valued eigenfunction, or $\begin{bmatrix} D_{i,i} & D_{i,i+1} \\ D_{i+1,i} & D_{i+1,i+1} \end{bmatrix} = |\lambda_i| \begin{bmatrix} \cos(\angle\lambda_i) & \sin(\angle\lambda_i) \\ -\sin(\angle\lambda_i) & \cos(\angle\lambda_i) \end{bmatrix}$ if ϕ_i is complex.

The transformed system (11) is bilinearizable with certain conditions on the control vector fields so that their Lie-derivative operators may be represented in terms of the Koopman eigenfunctions of the drift vector field.

III. Bilinearizability of the KCT

To establish the bilinearizability of the system (11), we need to analyze the control vector fields of the original system. In the transformed system, the control enters through the transformed vector field $L_{\mathbf{f}_i} T(\mathbf{x})|_{\mathbf{x}=C^{\mathbf{x}}\mathbf{z}}$. Note that $L_{\mathbf{f}_i}$ is the infinitesimal Koopman operator with respect to control vector field $\mathbf{f}_i$.

Theorem 1: The systems (7) and (11) are bilinearizable in a countable (possibly infinite) basis if the eigenspace of $L_{\mathbf{f}_0}$, i.e., the Koopman operator corresponding to the drift vector field, is an invariant subspace of $L_{\mathbf{f}_i}, i = 1, \ldots, m$, i.e., the Koopman operators related to the control vector fields.

Proof: If the hypothesis is true, then choose eigenfunctions of $L_{\mathbf{f}_0}$, $\{\phi_j : j = 1, 2, \ldots\}$, such that $L_{\mathbf{f}_i}\phi_k \in \operatorname{span}\{\phi_j : j = 1, 2, \ldots\}, \forall\, i = 1, \ldots, m;\ k = 1, 2, \ldots$. This choice is guaranteed, because $\operatorname{span}\{\phi_j : j = 1, 2, \ldots\}$, i.e., the eigenspace of $L_{\mathbf{f}_0}$, is invariant under $L_{\mathbf{f}_i}, i = 1, \ldots, m$. So, $\forall\, k = 1, 2, \ldots$, we have $L_{\mathbf{f}_i}\phi_k = \sum_{j=1}^{\infty} v_j^{\mathbf{f}_i}\phi_j$, where $v_j^{\mathbf{f}_i} \in \mathbb{R}$. Now taking $T(\mathbf{x})$ as in Eq. (10), but without imposing the finite n condition, we get

$$L_{\mathbf{f}_i} T(\mathbf{x}) = \sum_{j=1}^{\infty} \mathbf{v}_j^{\mathbf{f}_i} \phi_j(\mathbf{x}) = \sum_{j=1}^{\infty} \tilde{\mathbf{v}}_j^{\mathbf{f}_i} \tilde{\phi}_j(\mathbf{x}),$$

where $\tilde{\mathbf{v}}_j^{\mathbf{f}_i} \in \mathbb{R}^d$ and $\tilde{\phi}_j$ are defined as in Eq. (10). Define $B_i = [\tilde{\mathbf{v}}_1^{\mathbf{f}_i}|\tilde{\mathbf{v}}_2^{\mathbf{f}_i}| \ldots]$. Then, with $\mathbf{z} = T(\mathbf{x})$, the system (11) becomes

$$\dot{\mathbf{z}} = D\mathbf{z} + \sum_{i=1}^{m} B_i \mathbf{z} u_i. \tag{12}$$

Since the system (11) is just a transformation of the (7), bilinearization the former implies the same for the latter. ■

Although Theorem 1 gives the condition for the bilinearizability of the control-affine system using KCT with a countable number of eigenfunctions, it still does not solve the problem with infinitely many eigenfunctions. However, for an approximate result we can truncate the number of eigenfunctions to only the dominant ones [9]. This linear approximation, unlike the Jacobian approach, is global, i.e., it is valid over the manifold $\mathbb{X}$ on which the dynamics (7) is defined.

Corollary 1: The systems (7) and (11) are bilinearizable if the drift vector field $\mathbf{f}_0 \equiv 0$, i.e., it is a pure control-affine system.

Proof: The proof follows from the fact that every function $\phi(\cdot) \in \mathcal{F}$ is an eigenfunction of $L_{\mathbf{f}_0}$ with $\mathbf{f}_0 \equiv 0$ corresponding to the zero eigenvalue. Hence the whole space $\mathcal{F}$ is the eigenspace of $L_{\mathbf{f}_0}$, which is of course invariant under $L_{\mathbf{f}_i}$, $\forall\, i = 1, \ldots, m$. Therefore, from Theorem 1, the system is bilinearizable. ■

Theorem 1 and Corollary 1 essentially embed the finite-dimensional nonlinear dynamics (7) in a higher, possibly infinite-dimensional linear system (12). There are other embedding techniques that deal with Hermite polynomials, e.g., Carleman embedding [15], but that works only on analytic nonlinearities. The method with Koopman eigenfunctions works on a wide varieties of systems, and can be characterized in terms of the range and eigenspace of the corresponding Koopman operator.

For a finite-dimensional bilinearization of the system (7), we need a stronger assumption than invariance of the eigenspace of $L_{\mathbf{f}_0}$: the invariant subspace must be spanned by a finite number of Koopman eigenfunctions.

Theorem 2: Suppose $\exists\, \{\phi_j : j = 1, \ldots, n\}$, $n \in \mathbb{N}$, $n < \infty$ such that ϕ_j, $j = 1, \ldots, n$ are the Koopman eigenfunctions of the unactuated system (9) and $\operatorname{span}\{\phi_1, \ldots, \phi_n\}$ forms an invariant subspace of $L_{\mathbf{f}_i}$, $i = 1, \ldots, m$. Then the system (7) and, in turn system (11), are bilinearizable with an n dimensional state space.

Proof: The hypothesis dictates that $L_{\mathbf{f}_i}\phi_k \in \operatorname{span}\{\phi_j : j = 1, \ldots, n\}\, \forall\, i = 1, \ldots, m;\ k = 1, \ldots, n$. Therefore, we conclude

$$L_{\mathbf{f}_i}\phi_k = \sum_{j=1}^{n} v_j^{\mathbf{f}_i}\phi_j,\ k = 1, \ldots, n,$$

where $v_j^{\mathbf{f}_i} \in \mathbb{R}$. Now consider $T(\mathbf{x})$ as defined in Eq. (10). Its Lie derivatives with respect to the control vector fields are

$$L_{\mathbf{f}_i} T(\mathbf{x}) = \sum_{j=1}^{n} \mathbf{v}_j^{\mathbf{f}_i} \phi_j(\mathbf{x}) = \sum_{j=1}^{n} \tilde{\mathbf{v}}_j^{\mathbf{f}_i} \tilde{\phi}_j(\mathbf{x}),$$

where $\tilde{\mathbf{v}}_j^{\mathbf{f}_i} \in \mathbb{R}^d$ and $\tilde{\phi}_j$ are defined as in Eq. (10). Now, as in Theorem 1, let us define $B_i \triangleq [\tilde{\mathbf{v}}_1^{\mathbf{f}_i}|\tilde{\mathbf{v}}_2^{\mathbf{f}_i}| \ldots |\tilde{\mathbf{v}}_n^{\mathbf{f}_i}]$. The difference from the B_i in Theorem 1 is that this B_i is not only countable but a finite-dimensional operator. Now with coordinates $\mathbf{z} = T(\mathbf{x})$, the transformed system is

$$\dot{\mathbf{z}} = D\mathbf{z} + \sum_{i=1}^{m} B_i \mathbf{z} u_i, \tag{13}$$

with $\mathbf{z} \in \mathbb{R}^n$, $n < \infty$. ■

Though the hypothesis of Theorem 2 is difficult to satisfy, we can always include more eigenfunctions ϕ_j in the span so that $\|L_{\mathbf{f}_i}\phi_j - \sum_{j=1}^{n} v_j^{\mathbf{f}_i}\phi_j\|$ becomes sufficiently small. But the bilinear form (13) will not be unique in that scenario due to the lack of invertibility of KCT in most cases. Note that usually $n \gg d$, i.e., this method of bilinearization lifts the original dynamics (7) to a higher dimensional state-space. The resulting bilinear system is relatively easier to work with in terms of controllability analysis and designing a stabilizing control, as illustrated in the next section. The bilinear system defined by (13) will be referred as the Koopman Bilinear Form (KBF) in the sequel.

IV. Reachability of KBF

To analyze the controllability of the KBF in (13), we borrow the concatenation semigroup structure of control signals defined in [11].

Let $\mathbf{u}^i$ be an m-dimensional piecewise control signal. We can form a semigroup from the set $\{\mathbf{u}^i(\cdot) | \mathbf{u}^i : \mathbb{R}^+ \to \mathbb{R}^m, \mathbf{u}^i \text{ piecewise continuous}\}$ with the concatenation operation. The concatenation operation looks like

$$\mathbf{u}^1 \circ \mathbf{u}^2 = \begin{cases} \mathbf{u}^1(t), & t \in [0, t_1) \\ \mathbf{u}^2(t - t_1), & t \in [t_1, t_2) \end{cases} \tag{14}$$

Denote this semigroup as U^m. Each $\mathbf{u} \in U^m$, when applied to the dynamics (13), generates a one-to-one continuous map from $\mathbb{R}^n$ into $\mathbb{R}^n$ in terms of a flow map. Let T^n be the semigroup of all such maps with composition operation. The system (13) defines a homomorphism from U^m into T^n [11]. Let $H : U^m \to T^n$ be the homomorphism. The image of U^m under H is called the Myhill semigroup [11] of the system. The maps of the Myhill semigroup are, in fact, the flow maps of the system with a particular piecewise-continuous control signal $\mathbf{u} \in U^m$, and therefore give all the information of the dynamics.

In general, i.e., for an arbitrary nonlinear system these maps are difficult to obtain, and yield no practical use. But for the bilinear system (13), the Myhill semigroup maps are the matrices $Z \in \mathbb{R}^{n \times n}$, satisfying the matrix differential equation

$$\begin{aligned} \dot{Z}(t) &= DZ(t) + \sum_{i=1}^{m} B_i Z(t) u_i, \\ Z(0) &= I, \end{aligned} \tag{15}$$

with $\mathbf{z}(t) = Z(t)\mathbf{z}(0)$ for any $\mathbf{z}(0) \in \mathbb{R}^n$. Therefore, given any initial state $\mathbf{z}_0$, the states reachable from $\mathbf{z}_0$ are given by all the points in $\mathbb{R}^n$ that can be generated by $Z(t)\mathbf{z}_0$ for some $t \geq 0$, where $Z(t)$ satisfies (15). Consequently, the controllability of the system (13) can be characterized by the controllability of the matrix differential equation (15). The controllability of a bilinear matrix system has been studied widely [11], [12], [16], [17], exploiting the characteristic of matrices as operators and the corresponding Lie algebraic structures.

The Lie bracket of $\mathbb{R}^{n \times n}$ matrices is defined as

$$[\cdot, \cdot] : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n},$$
$$[X, Y] \mapsto XY - YX.$$

Any space of matrices closed under the Lie bracket operation forms a Lie algebra. The matrix exponentials of all elements of a matrix Lie algebra along with usual matrix multiplication forms a matrix Lie group associated with the algebra. Denote $\{X_i : i = 1, \ldots, n\}_A$ as the smallest Lie algebra containing $\{X_i : i = 1, \ldots, n\}$ and $\{\exp\{X_i\} : i = 1, \ldots, n\}_G$ as the smallest Lie group containing $\{\exp\{X_i\} : i = 1, \ldots, n\}$. Also $\forall\, A, B \in \mathbb{R}^{n \times n}$ and $k = 0, 1, \ldots$ we define $\mathrm{ad}_A^{k+1} B \triangleq [A, \mathrm{ad}_A^k B]$ with $\mathrm{ad}_A^0 B \triangleq B$. The controllability results are stated below using the notation of Lie groups and algebras.

Theorem 3: Consider the drift-free matrix differential equation in $\mathbb{R}^{n \times n}$,

$$\dot{Z}(t) = \sum_{i=1}^{m} B_i Z(t) u_i(t),\ Z(0) = I \tag{16}$$

which corresponds to the system (13) with $\mathbf{f}_0 \equiv 0$ and B_i as defined in Theorem 2. $Z_1 \in \mathbb{R}^{n \times n}$ is in the reachable space of (16) if and only if $Z_1 \in \{\exp\{\{B_i : i = 1, \ldots, m\}_A\}\}_G$, i.e., it lies within the smallest group generated by the matrix exponential of the elements of the smallest algebra generated by the control matrices.

The proof of Theorem 3 goes according to Theorem 5 in [12]. For brevity we omit the proof.

Theorem 3 gives the reachable set for systems without drift in terms of the Koopman modes of $L_{\mathbf{f}_i} T(\mathbf{x})$, because the matrices B_i are the column-wise collection of the Koopman modes $\mathbf{v}_j^{\mathbf{f}_i}$. So if the system is approximately bilinearized, i.e., if $\|L_{\mathbf{f}_i}\phi_j - \sum_{j=1}^{n} v_j^{\mathbf{f}_i}\phi_j\|$ is sufficiently small, then the reachable set of the bilinearized dynamics (12) is a subset of the reachable set for the original dynamics (7). However, with drift, the problem becomes more difficult and we need more assumptions to get a reachable set. For systems with drift, the reachable set is specified below.

Theorem 4: Consider the matrix differential equation in $\mathbb{R}^{n \times n}$,

$$\dot{Z}(t) = DZ(t) + \sum_{i=1}^{m} B_i Z(t) u_i(t),\ Z(0) = I \tag{17}$$

where D and B_i are defined as in the proof of Theorem 2. Assume $[\mathrm{ad}_D^k B_i, B_j] = 0$ for $i, j = 1, \ldots, m$ and $k = 0, 1, \ldots, n^2 - 1$. Let $\mathcal{L} = \mathrm{span}\{\mathrm{ad}_D^k B_i : i = 1, \ldots, m,\ k = 0, 1, \ldots, n^2 - 1\}$. Then Z_1 is reachable at time t_1 through continuous controls if and only if $\exists\, L \in \mathcal{L}$ such that

$$Z_1 = \exp(t_1 D)\exp(L).$$

The proof of Theorem 4 relies on defining a new system of differential equation on $Y(t) = e^{-Dt} Z(t)$. For the

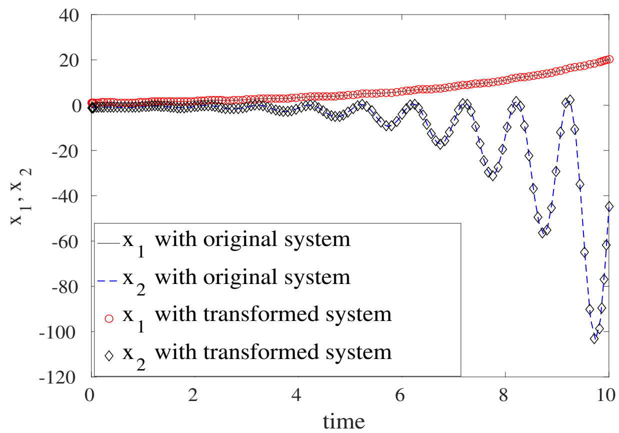

Fig. 1: Original and transformed system response for complete bilearization

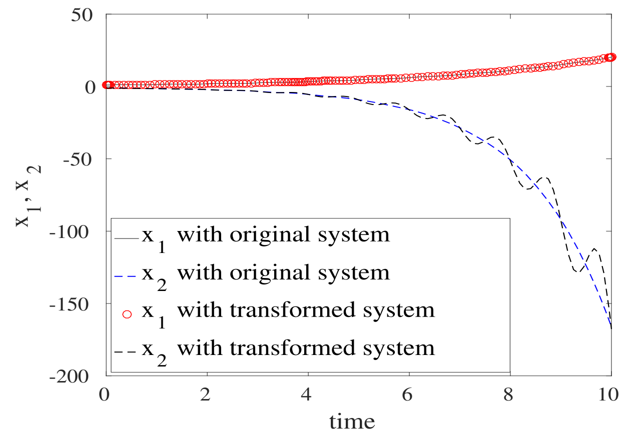

Fig. 2: Original and transformed system response for approximate bilearization

complete proof see [12].

The condition of Theorem 4 seems conservative, but is usually satisfied with sparse B_i when $n \gg m$.

V. Numerical Simulation

To demonstrate the effectiveness of the bilinearization technique described consider the system

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2, \tag{18}$$

where the drift $\mathbf{f}_0$ is

$$\mathbf{f_0}(\mathbf{x}) = \begin{pmatrix} \lambda x_1 \\ \mu x_2 + (2\lambda - \mu)cx_1^2. \end{pmatrix}$$

This choice of $\mathbf{f}_0$ is inspired from [3] so that the eigenfunctions may be obtained by inspection. For the demonstration, we choose various $\mathbf{g}_1$ and $\mathbf{g}_2$ for different cases.

It can be verified that the Koopman eigenvalue-eigenfunction pairs for $L_{\mathbf{f}_0}$ are as follows:

- $\phi_1(\mathbf{x}) = x_1$ with eigenvalue λ,
- $\phi_2(\mathbf{x}) = x_2 - cx_1^2$ with eigenvalue μ,
- $\phi_3(\mathbf{x}) = x_1^2$ with eigenvalue 2λ, and
- $\phi_4(\mathbf{x}) = 1$ with eigenvalue 0.

Any multiplicative combination of these eigenfunctions will yield another eigenfunction with a suitable eigenvalue. However, for our discussion it is sufficient to consider only these four. ϕ_4 is the trivial constant eigenfunction with zero eigenvalue, introduced to deal with constant control vector fields.

The Koopman canonical transformation is

$$\begin{aligned}\mathbf{z} = T(\mathbf{x}) &= \begin{bmatrix}\phi_1(\mathbf{x}) & \phi_2(\mathbf{x}) & \phi_3(\mathbf{x}) & \phi_4(\mathbf{x})\end{bmatrix}^T \\ &= \begin{bmatrix}x_1 & x_2 - cx_1^2 & x_1^2 & 1\end{bmatrix}^T\end{aligned}$$

and matrix D is given by $D = \mathrm{diag}(\lambda, \mu, 2\lambda, 0)$.

A. Completely bilinearizable system

Now let us choose $\mathbf{g}_1$ and $\mathbf{g}_2$ such that the system becomes completely bilinearizable in four dimensions according to Theorem 2. Let

$$\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix}1 & x_1^2\end{bmatrix}^T \text{ and } \mathbf{g}_2(\mathbf{x}) = \begin{bmatrix}0 & 1\end{bmatrix}^T$$

Then

$$L_{\mathbf{g}_1}T(\mathbf{x}) = \begin{bmatrix}1 & -2cx_1 + x_1^2 & 2x_1 & 0\end{bmatrix}^T = B_1\mathbf{z},$$

and $L_{\mathbf{g}_2}T(\mathbf{x}) = B_2\mathbf{z}$, where

$$B_1 = \begin{bmatrix}0 & 0 & 0 & 1 \\ -2c & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \; B_2 = \begin{bmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}.$$

For this simulation, we set $\lambda = 0.3$, $\mu = 0.2$ and $c = -0.5$. We applied $u_1 = \cos(2\pi t)$, a sinusoidal excitation, and $u_2 = -x_2 = -(z_2 + cz_1^2) = -(z_2 + cz_3)$, a state feedback. Fig. 1 shows the original system response and the response from the bilinearized system after transforming back to the original coordinates. The responses are identical.

B. Approximately bilinearized system

Now choose $\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix}1 & \cos x_1\end{bmatrix}^T$ and keep everything else the same as Section V.A. But now $L_{\mathbf{g}_1}T(\mathbf{x})$ does not lie in the span of $\phi_i, i = 1, \ldots, 4$. So we can only approximately bilinearize the system (18) by taking the projection of $L_{\mathbf{g}_1}T(\mathbf{x})$ into the span of these four eigenfunctions. Here

$$L_{\mathbf{g}_1}T(\mathbf{x}) = \begin{bmatrix}1 & -2cx_1 + \cos x_1 & 2x_1 & 0\end{bmatrix}^T.$$

From cosine series expansion, approximate $\cos x_1 = 1 - \frac{x_1^2}{2} = \phi_4(\mathbf{x}) - \frac{1}{2}\phi_3(\mathbf{x})$. With this approximation

$$B_1 = \begin{bmatrix}0 & 0 & 0 & 1 \\ -2c & 0 & -\frac{1}{2} & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}.$$

The transformed bilinearized system and the original system do not give exactly the same response, but they closely follow each other. The resultant responses are shown in

Fig. 2. The accuracy can be increased by including higher-order eigenfunctions, thereby increasing the number of terms in the cosine series. The bilinearization by KBF performs better compared to classical bilinearization techniques since the former is valid globally. Due to the lack of space, the comparison results are omitted here.

C. Controllability of the system

The controllability and reachable sets of the system (7) may be characterized by the transformed bilinearized system with D, B_1, and B_2 from the completely bilinearizable system in Section V.A. However, because the transformed KBF has more dimensions, it may not achieve complete controllability even when the original system does. It can be shown that the bilinearized system $\dot{\mathbf{z}}(t) = D\mathbf{z}(t) + B_1\mathbf{z}(t)u_1(t) + B_2\mathbf{z}(t)u_2(t)$ satisfies the hypothesis of Theorem 4. So we resort to finding $Z(t) \in \mathbb{R}^{n\times n}$, where $Z(t)$ satisfies the matrix differential equation (15) with $m = 2$ and $\mathbf{z}(t) = Z(t)\mathbf{z}(0)$. According to Theorem 4, $Z(t)$ must take the form

$$Z(t) = \exp(tD)\exp(L),$$

where $L \in \mathcal{L} = \mathrm{span}\{\mathrm{ad}_D^k B_i : i = 1,\ldots,m,\, k = 0,1,\ldots,n^2-1\}$. By explicitly calculating $\exp(tD)\exp(L_1)$, where $L_1 = c_1B_1 + c_2B_2 \in \mathrm{span}\{\mathrm{ad}_D^k B_i : i = 1,\ldots,m,\, k = 0,1,\ldots,n^2-1\}$, the resultant $Z(t)$ is

$$\begin{bmatrix} e^{\lambda t} & 0 & 0 & c_1 e^{\lambda t} \\ -c_1 e^{\mu t}\left(2c - \frac{c_1}{2}\right) & e^{\mu t} & c_1 e^{\mu t} & e^{\mu t}\left(c_2 - c c_1^2 + \frac{c_1^3}{3}\right) \\ 2c_1 e^{2\lambda t} & 0 & e^{2\lambda t} & c_1^2 e^{2\lambda t} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

From any $\mathbf{z}_0$, we can achieve $\mathbf{z}(t) = Z(t)\mathbf{z}(0)$ and, therefore, any $z_1(t)$ and $z_2(t)$ can achieved by varying the scalars c_1 and c_2 as $z_4 \equiv 1$. So we have global controllability for the original system (18). However, the transformed system is not globally controllable because we have no control authority over $z_4(t) \equiv 1$. To stabilize the system we choose $u_1(t) = -(\lambda + 0.5)\, z_1(t) = -(\lambda + 0.5)\, x_1(t)$ and $u_2(t) = -(2cz_1(t) - z_3(t))\, u_1(t) - (\mu + 0.5)\, z_2(t) = (2cx_1(t) - x_1^2(t))\, u_1(t) - (\mu + 0.5)(x_2(t) - cx_1^2(t))$. This effectively reduces the transformed system into $\dot{\mathbf{z}} = A\mathbf{z}$ where $A = \mathrm{diag}(-0.5, -0.5, -0.5, 1)$. This input in turn feedback linearizes the original system (18) giving a strong connection between KBF and feedback linearizability of the system. The system response under this feedback is shown in Fig. 3.

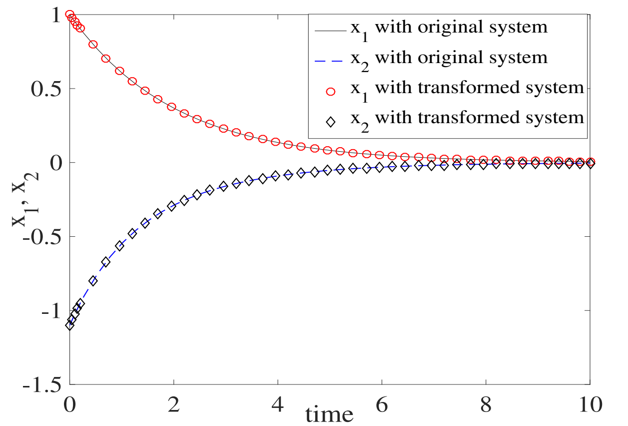

Fig. 3: Original and transformed system response for complete bilinearization with feedback

VI. CONCLUSION

This paper presents an effective method to globally bilinearize control-affine nonlinear systems using the Koopman Canonical Transform. The sufficient conditions for bilinearizablity in both countable and finite bases have been provided. The Koopman Bilinear Form is analyzed for controllability and reachability using the Myhill semigroup formation and Lie algebraic methods. The theoretical justification of the bilinearizability and controllability has been demonstrated using numerical simulations. The future work includes developing an optimal control strategy using KBF and the investigation its relation with feedback linearization.

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