Localization Using the KF

Assumption: there are $N$ known landmarks, with known locations

$$\{x_1^L, x_2^L, \ldots, x_N^L\}$$

Use the Kalman filter to update robot’s position estimate. The robot’s state is $x^R$. The dynamics are generally of the form:

$$x^R_{k+1} = f(x^R_k, u_k) + \nu_k$$

And the measurements are of the form:

$$y_{k+1} = h(x^R, x_1^L, \ldots x_N^L) + \omega_{k+1}$$

Pseudo-Code: (1) Initialize $\hat{x}_{0|0}^R$ and $P_{0|0}, k = 0$

(2) Move to $x_{k+1}$, estimate $\hat{x}_{k+1|k}, P_{k+1|k}$

(3) Measure $y_{k+1}$, update $\hat{x}_{k+1|k+1}, P_{k+1|k+1}$

(4) $k=k+1$; go to step (2)
Localization Using the KF
(Example)

Turtlebot kinematics:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} = \frac{R}{2} \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\dot{\phi}_1 + \dot{\phi}_2 \\
0 \\
(\dot{\phi}_1 - \dot{\phi}_2)/W
\end{bmatrix}
\]

We cannot integrate these equations exactly for a discrete time system. Approximate integration:

\[
\begin{bmatrix}
(x_{k+1} - x_k)/\Delta t \\
(y_{k+1} - y_k)/\Delta t \\
(\theta_{k+1} - \theta_k)/\Delta t
\end{bmatrix} = \frac{R}{2} \begin{bmatrix}
\cos \theta_k & -\sin \theta_k & 0 \\
\sin \theta_k & \cos \theta_k & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\Delta \phi_{1,k} + \Delta \phi_{2,k} \\
0 \\
(\Delta \phi_{1,k} + \Delta \phi_{2,k})/W
\end{bmatrix}
\]

Rearranging:

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1} \\
\theta_{k+1}
\end{bmatrix} = I_{3\times3} \begin{bmatrix}
x_k \\
y_k \\
\theta_k
\end{bmatrix} + \frac{R}{2} \begin{bmatrix}
\cos \theta_k & \cos \theta_k \\
\sin \theta_k & \sin \theta_k \\
1/W & -1/W
\end{bmatrix} \begin{bmatrix}
\Delta \phi_{1,k} \\
\Delta \phi_{2,k}
\end{bmatrix}
\]

\[\hat{z}_{k+1} = A_k \hat{z}_k + B_k u_k\]
Localization Using the KF
(Example)

Range Measurement: range to $j^{th}$ landmark at $t_k$
\[
y_{j,k} = r_{j,k} = \sqrt{(x^R_k - x^L_j)^2 + (y^R_k - y^L_j)^2} = h_j(x^R, x^L_j)
\]

Bearing Measurement: bearing to $j^{th}$ landmark at $t_k$
\[
y_{j,k} = \varphi_{j,k} = \text{Atan2}\left[(y^R_k - y^L_k), (x^R_k - x^L_k)\right] - \theta_k
\]

Overall Measurement equations:
\[
\mathbf{\hat{y}}_{k+1} = \begin{bmatrix} r_{1,k+1} \\ r_{2,k+1} \\ \vdots \\ \varphi_{1,k+1} \\ \varphi_{2,k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1(x^R_{k+1}, x^L_1) \\ h_2(x^R_{k+1}, x^L_2) \\ \vdots \\ h_{N+1}(x^R_{k+1}, x^L_1) \\ h_{N+2}(x^R_{k+1}, x^L_2) \\ \vdots \end{bmatrix} + \mathbf{\omega}_{k+1}
\]
If the measurements of the different landmarks are independent, we can use a sequential updating method for the measurement update:

- **Init:** \( \bar{x}_{k+1|k} = \hat{x}_{k+1} ; \quad \bar{P}_{k+1|k} = P_{k+1|k} \)
- **Iterate for all viewed landmarks, \( i = 1, \ldots, N_{view} \):**
  - \( S_{k+1}^i = H_{k+1}^i \bar{P}_{k+1|k} (H_{k+1}^i)^T + R_{k+1}^i \)
  - \( K_{k+1}^i = \bar{P}_{k+1|k} (H_{k+1}^i)^T (S_{k+1}^i)^{-1} \)
  - \( \bar{x}_{k+1|k} = \bar{x}_{k+1|k} + K_{k+1}^i (y_{k+1} - h(\bar{x}_{k+1|k})) \)
  - \( \bar{P}_{k+1|k} = (I - K_{k+1}^i H_{k+1}) \bar{P}_{k+1|k} \)
  - \( \hat{x}_{k+1|k+1} = \bar{x}_{k+1|k} ; \quad \hat{P}_{k+1|k+1} = \bar{P}_{k+1|k} \)

Where:

\[
S_{k+1|k}^i = E \left[ (y_{k+1} - h(\hat{x}_{k+1|k})) (y_{k+1} - h(\hat{x}_{k+1|k}))^T \right] = \text{innovation covariance}
\]
The EKF

For system: \[ x_{k+1} = f(x_k, u_k) + \eta_k \]
\[ y_{k+1} = h(x_{k+1}) + \omega_k \]
\[ \eta_k \sim N(0, Q_k) \]
\[ \omega_k \sim N(0, R_k) \]

Just like the KF, the EKF has a 2-step structure:

– Dynamic (time) update)
  
  • \( \hat{x}_{k+1|k} = f(\hat{x}_{k+1|k}, u_k) \);

  • \( P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k \)

  \[ A_k = \frac{\partial f(x)}{\partial x} \Big|_{\hat{x}_{k+1|k}} \]

– Measurement Update

  • \( \hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - h(\hat{x}_{k+1|k})) \)

  • \( P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T \left( H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1} \right)^{-1} H_{k+1} P_{k+1|k} \)

  \[ = \left( I - K_{k+1} H_{k+1} \right) P_{k+1|k} \]

Where: \( K_{k+1} = P_{k+1|k} H_{k}^T \left( H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1} \right)^{-1} \);

\[ H_k = \frac{\partial h(x)}{\partial x} \Big|_{\hat{x}_{k+1|k+1}} \]
In practice we may not exactly know the “identity” of the landmark. Hence, we must solve the data association problem, which tries to assign a label or index to each landmark.

- Innovation: $v_{ij} = y_{k+1}^i - h^j(x_{k+1|k})$
  - measurement error if $j^{th}$ landmark generates $i^{th}$ measurement
- Gate: Eliminate all possible measurement-landmark pairings

$$S_{k+1|k} = H_{k+1}P_{k+1|k}H_{k+1} + R_{k+1}$$
= predicted measurement error

$R_{k+1}$ (measure noise)
$P_{k+1|k}$ (Pose Uncertainty)
Data Association (2)

In practice we may not exactly know the “identity” of the landmark. Hence, we must solve the data association problem, which tries to assign a label or index to each landmark.

- Innovation: \( \nu_{ij} = y^i_{k+1} - h^j(x_{k+1|k}) \)
  - measurement error if \( j^{th} \) landmark generates \( i^{th} \) measurement
- \( S_{ij} = \) innovation uncertainty \( \tilde{x}^m \)

Optimal Data Association (or pairing of measurement with landmark index):

- Associate measurement \( i \) to the landmark \( j \) which minimizes \( \nu_{ij}^T S_{ij}^{-1} \nu_{ij} \) over all \( j \) that survived the “gate”
EKF SLAM

**Map:** a set of landmarks, which can be modeled as a set of $N$ landmarks and their positions:

$$\tilde{x}^m = [x_1^L, x_2^L, \ldots, x_N^L]$$

- **Option 1:** Positions not known in advance, but $N$ is known
- **Option 2:** Even $N$ is not known.

Use Localization EKF with modification that state is: $[\tilde{x}^r, \tilde{x}^m]^T$

- **Dynamics:**
  $$\begin{bmatrix} x_{k+1}^r \\ x_{k+1}^m \end{bmatrix} = \begin{bmatrix} f(x_k^r, u_k) \\ x_k^m \end{bmatrix} + \begin{bmatrix} \eta_k \\ 0 \end{bmatrix}$$

- **Measurement:**
  $$y_{k+1}^i = h(\tilde{x}_{k+1}^r, \tilde{x}^i) + \omega_{k+1}^i \quad \text{(for } i^{th} \text{ landmark)}$$

**Key issues:**
- How to properly “initialize” the filter state and covariance when a new landmark is first seen?
- How to properly “extend” the state when new landmarks arise?
**EKF SLAM/Tracking**

**Map+ Targets:** a set of *static* landmarks, and a set of moving targets:

\[
\tilde{x}^m = [x_1^L, x_2^L, ..., x_N^L]; \quad \tilde{x}_k^t = [x_{1,k}^t, x_{2,k}^t, ..., x_{N_t,k}^t]
\]

Use EKF with state: \([\tilde{x}^r, \tilde{x}^t, \tilde{x}^m]^T\)

- **Dynamics:**

\[
\begin{bmatrix}
    x_{k+1}^r \\
    x_{k+1}^t \\
    x_{k+1}^m
\end{bmatrix}
= \begin{bmatrix}
    f^r(x_k^r, u_k) \\
    f^t(x_k^r, x_k^t) \\
    x_{k+1}^m
\end{bmatrix} + \begin{bmatrix}
    \eta_k^r \\
    \eta_k^t \\
    0
\end{bmatrix}
\]

- **Measurement:**

\[
y_{k+1} = h(\tilde{x}_{k+1}^r, \tilde{x}_{k+1}^t, \tilde{x}_{k+1}^m) + \omega_{k+1}
\]

**Key issues:**

- Dynamics of target?

\[
x_{k+1}^t = \begin{bmatrix}
    I & \Delta t I \\
    0 & I
\end{bmatrix} x_k^t + \begin{bmatrix}
    0 \\
    \eta_k
\end{bmatrix} \quad (\eta_k \text{ captures possible accelerations})
\]

- Confusion of Targets (use multiple hypothesis tracking for data association)
EKF SLAM: Landmark Initialization

First time you see a landmark: need to

- Initialize its position (after dynamic update)
  \[
  \begin{bmatrix}
  x_k^L_i \\
  y_k^L_i
  \end{bmatrix} =
  \begin{bmatrix}
  \hat{x}^r_{k+1|k} \\
  \hat{y}^r_{k+1|k}
  \end{bmatrix} +
  \begin{bmatrix}
  r^i_{k+1} \cos(\phi^i_{k+1} + \hat{\theta}^t_{k+1}) \\
  r^i_{k+1} \sin(\phi^i_{k+1} + \hat{\theta}^t_{k+1})
  \end{bmatrix}
  \]

- Initialize landmark position uncertainty. If \( N \) known, let \( F^i \) be:
  \[
  H^i_{k+1} =
  \begin{bmatrix}
  \frac{\partial h^i}{\partial x^r} & \frac{\partial h^i}{\partial x^L_i} \\
  \end{bmatrix}
  \begin{bmatrix}
  I & 0 & \ldots & 0 & \ldots & 0 \\
  0 & 0 & \ldots & I & \ldots & 0 \\
  \end{bmatrix}
  \]

- Assuming independent landmark measurements (not quite true…), sequential update:
  \[
  \begin{align*}
  &K^i_{k+1} = \bar{P}_{k+1|k}(H^i_{k+1})^T \left( H^i_{k+1} \bar{P}_{k+1|k}(H^i_{k+1})^T + R_{k+1} \right)^{-1} \\
  &\bar{x}_{k+1|k} = \bar{x}_{k+1|k} + K^i_{k+1}(y_{k+1} - h(\bar{x}_{k+1|k})) \\
  &\bar{P}_{k+1|k} = (I - K^i_{k+1}H_{k+1})\bar{P}_{k+1|k} \\
  &\hat{x}_{k+1|k+1} = \bar{x}_{k+1|k} ; \quad \hat{P}_{k+1|k+1} = \bar{P}_{k+1|k}
  \end{align*}
  \]
Gmapping: how is it related?

Notation: $x_{1:T} = \{x_1, x_2, \ldots, x_T\}$

Localization: $p(x_{k+1}^r | x_{1:k}^r, y_{1:k+1})$
  - In KF/EKF, $\hat{x}_{k+1}^r, P_{k+1|k+1}$ are sufficient statistics

EKF SLAM: $p(x_{k+1}^r, x_{k+1}^m | x_{1:k}^r, x_k^m, y_{1:k+1})$

Occupancy Grid: “map” is a grid of “cells”: $\{x_i^m\}$
  - $x_{i,j}^m = 1$ if cell (i,j) is empty; $x_{i,j}^m = 0$ if cell (i,j) is occupied
  - $p \left( x_{k+1}^r, \{x_{i,j}^m\}_{k+1} | x_{1:k}^r, \{x_{i,j}^m\}_k, y_{1:k+1} \right)$ (estimates cell occupancy probability)

Gmapping:
  - Uses a Rao-Blackwellized particle filter for estimator
  - Actually computes $p \left( x_{1:T}^r, \{x_{i,j}^m\} | x_{1:k}^r, x_k^m, y_{1:k+1} \right)$