CDS 101/110 Homework #4 Solution

Problem 1  (CDS 101, CDS 110):  (35 points)

(a) Let \( z = (x, y, \dot{x}, \dot{y}) \). Rewrite the system as
\[
\dot{z} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
9w^2 & 0 & 0 & 2w \\
0 & -4w^2 & -2w & 0
\end{bmatrix} z
\]
Find the eigenvalues of the matrix which are \(-0.3465 \pm 0.5548i\) and \(0.3465 \pm 0.5548i\). There is an eigenvalue with a positive real part. Therefore, the equilibrium point is unstable.

(b) \( k_1 = 2.0138 \quad k_2 = 0.6933 \quad k_3 = 2.7039 \quad k_4 = 2.8166 \)

Code:
```matlab
w = 2*pi/29;
A = [0 1 0 0 ;
     0 0 0 1 ;
     9*w^2 0 0 2*w ;
     0 -4*w^2 -2*w 0 ];
B = [0 0 0 1]';
eig(A)
place(A, B, [-3*w -4*w -3*w+3*w*1i -3*w-3*w*1i])
```

(c) (This problem will not be graded because the phase plot \((x, y)\) can be any arbitrary phase plot.)

Problem 2  (CDS 101, CDS 110):  (5 points)

Set \( z(t) = x(t) - e^{At}x(0) \). Then, \( z(t) \) can reach any states from zero initial state according to the argument in the textbook on page 7-3. When \( z(t) \) can reach any state, we can represent \( z(t) \) by some basis vectors (i.e. \( W_r \)) that span the entire state space. We can also write \( x(0) \) using the same set of basis vectors (i.e. \( W_r \)). So, \( x(t) = e^{At}x(0) + z(t) \) can be written using the basis vectors in \( W_r \) as well. Hence, \( x(t) \) is reachable from a non-zero initial state.

Problem 3  (CDS 110):  (25 points)

Let \( z = (\phi, \delta, \dot{\phi}, \dot{\delta}) \).

Feedback gain \( K \):
\[
k_1 = 0.3247 \quad k_2 = 8.4043 \quad k_3 = -1.3455 \quad k_4 = 0.1071
\]

Reference gain:
\[
K_r = -0.5429
\]
run('Bicycle_whipple.m')

% Desired closed loop eigenvalues
P=[1 [1 1i -1-1i] 2 10];
K=place(A,B,P);
Acl=A-B*K;
C=[0 1 0 0];
Kr=-1/(C/Acl*B);
disp('Feedback gain K=');disp(K)
disp('Reference gain Kr=');disp(Kr)

% Simulate
bike_cls=ss(Acl,B*Kr,C,0);
t=0:0.01:8;
[y,t,x]=step(bike_cls,t);
u=Kr*0.002-K*x';

subplot(211);
p=plot(t,x(:,2)/500,'b-',t,0.002*ones(1,length(t)),'k--');
set(p,'Linewidth',1.5);grid on;
axis([0 8 -0.005 0.005]);
ylabel('$\delta$','Interpreter','latex');

subplot(212);
p=plot(t,u/500,'b-');
set(p,'Linewidth',1.5);grid on;
xlabel('t');ylabel('u');

Plot: The dashed line is the reference at 0.002.
Problem 4 (CDS 110): (15 points)

Let $\lambda_k$ be an eigenvalue of $A$. Then,

$$\lambda(\lambda_k) = \lambda_k^n + a_1\lambda_k^{n-1} + \cdots + a_{n-1}\lambda_k + a_n = 0.$$ 

Let $\Lambda$ be a diagonal matrix that has the eigenvalues of $A$ at its diagonal. Then,

$$\Lambda^n + a_1\Lambda^{n-1} + \cdots + a_{n-1}\Lambda + a_n I = 0.$$ 

Because $A$ is diagonalizable, $A = T\Lambda T^{-1}$. Furthermore, $A^k = T\Lambda^k T^{-1}$. So,

$$\lambda(A) = A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_n I$$
$$= T\Lambda^n T^{-1} + a_1 T\Lambda^{n-1} T^{-1} + \cdots + a_{n-1} T\Lambda T^{-1} + a_n I$$
$$= T(\Lambda^n + a_1\Lambda^{n-1} + \cdots + a_{n-1}\Lambda + a_n I) T^{-1}$$
$$= 0$$