Goals:
• Review Bode’s integral formula and the “waterbed” effect
• Show some of the limitations of feedback due to RHP poles and zeros
• More loop shaping examples

Reading:
• Åström and Murray, Feedback Systems, Section 12.6
“Loop Shaping”: Design Loop Transfer Function

Translate specs to “loop shape”

\[ L(s) = P(s)C(s) \]

Design \( C(s) \) to obey constraints

\[ C(s) = k \frac{\prod_{i=1}^{n_z} (s - z_i)}{\prod_{j=1}^{n_p} (s - p_j)} \]

- Poles/Zeros from PID
- Poles/Zeros from
  - Lead
  - Lag

Check the “Gang of Four”

\[ S = \frac{1}{1 + L(s)}; \quad T = \frac{L(s)}{1 + L(s)} \]

\[ PS = \frac{P(s)}{1 + L(s)}; \quad CS = \frac{C(s)}{1 + L(s)} \]
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for \( S(s) = \frac{1}{1+L(s)} = G_{er} = G_{yn} = G_{vd} = -G_{en} \)

- Let \( p_k \) be the unstable poles of \( L(s) \) and assume relative degree of \( L(s) \geq 2 \)
- **Theorem:** the area under the sensitivity function is a conserved quantity:

\[
\int_0^\infty \log_e |S(j\omega)| \, d\omega = \int_0^\infty \log_e \left| \frac{1}{1 + L(j\omega)} \right| \, d\omega = \pi \sum \text{Re } p_k
\]

**Waterbed effect:**
- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in \( \omega \); Bode plots are logarithmic
Example: Magnetic Levitation

System description
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball, \( z \), (from IR sensor)
- States: \( z, \dot{z} \)
- Dynamics: \( F = ma, F = \text{magnetic force generated by wire coil} \)

System Dynamics
\[
m\ddot{z} = mg - k_m(k_Au)^2/z^2
\]
\[
v_{ir} = k_Tz + v_0
\]
where:
- \( u \) = current to electromagnet
- \( v_{ir} \) = voltage from IR sensor

Linearization:
\[
P(s) = \frac{-k}{s^2 - r^2}
\]
- Poles at \( s = \pm r \) \( \Rightarrow \) open loop unstable
Need to create encirclement
- To offset RHP pole
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator
- Produce phase lead at crossover
- Generates loop in Nyquist plot

\[ C(s) = -k \frac{s + a}{s + b} \]
**Performance Limits**

**Nominal design gives low perf**
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

**Bode integral limits improvement**
\[ \int_{0}^{\infty} \log |S(j\omega)| d\omega = \pi r \]
- Must increase sensitivity at some point

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**Sensitivity Function**

![Sensitivity Function Graph](image)

**Step Response**

![Step Response Graph](image)
Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior

- Phase vs. frequency has additional lag (not “minimum”) for a given magnitude
- Can cause output to move opposite from input for a short period of time

**Example:**

\[
H_1(s) = \frac{s + a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad \text{vs} \quad H_2(s) = \frac{s - a}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

![Bode Diagrams](image1)

![Step Response](image2)
Source of non-minimum phase behavior

- To move left, need to make $\theta > 0$
- To generate positive $\theta$, need $f_1 > 0$
- Positive $f_1$ causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

$$
H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}
$$

- Poles: $0, 0, -\sigma \pm i \omega_d$
- Zeros: $\pm \sqrt{mgl}$

### Step Response

- Fan moves right and then moves to the left
Stability in the Presence of (RHP) Zeros

Loop gain limitations

• Poles of closed loop = poles of $1 + L$. Suppose $C(s) = \frac{k n_c(s)}{d_c(s)}$, where $k$ is the controller gain

$$1 + L = 1 + k \frac{n_c n_p}{d_c d_p} = \frac{d_c d_p + k n_c n_p}{d_c d_p}$$

• For large $k$, closed loop poles approach open loop zeros
• RHP zeros limit maximum gain ⇒ serious design constraint!

Root locus interpretation

• Plot location of eigenvalues as a function of the loop gain $k$
• Can show that closed loop poles go from open loop poles ($k = 0$) to open loop zeros ($k = \infty$)

![Root Locus Diagram](image-url)
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of $H_{er}$ over frequency range:

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_{er}(j\omega)|$$
$$M_2 = \max_{0 \leq \omega \leq \infty} |H_{er}(j\omega)|$$

**Theorem:** Suppose that $P(s)$ has a RHP zero at $z$. Then there exist constants $c_1$ and $c_2$ (depending on $\omega_1$, $\omega_2$, $z$) such that $c_1 \log M_1 + c_2 M_2 \geq 0$.

- $M_1$ typically $\ll 1 \Rightarrow M_2$ must be larger than 1 (since sum is positive)
- If we increase performance in active range (make $M_1$ and $H_{er}$ smaller), we must lose performance ($H_{er}$ increases) some place else
- Note that this affects peaks not integrals (different from RHP poles)

$$H(s) = \frac{(s^2 - mg\dot{l})}{s^2(Js^2 + ds + mg\dot{l})}$$
- Poles: $0, 0, -\sigma \pm j \omega_d$
- Zeros: $\pm \sqrt{mg\dot{l}}$

Reduced sensitivity → better performance up to higher frequency

peak increases

Magnitude (dB)
Summary: Limits of Performance

Many limits to performance

• Algebraic: \( S + T = 1 \)
• RHP poles: Bode integral formula
• RHP zeros: Waterbed effect on peak of \( S \)

Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)

\[
\int_0^\infty \log_e |S(j\omega)| \, d\omega = \int_0^\infty \log_e \left| \frac{1}{1 + L(j\omega)} \right| \, d\omega = \pi \sum \text{Re} \, p_k
\]

\[
\int_0^\infty \frac{\log |T(i\omega)|}{\omega^2} \, d\omega = \pi \sum \frac{1}{z_i}
\]

RHP poles
Announcements

**Homework #8:** One problem due on Monday, Dec. 5

**Final exam**
- Out on 5 Dec (Mon.)
- Due on Fri. December 9, by 5 pm:
  - turn in to Sonya Lincoln
  - 250 Gates-Thomas
- Final exam review: December 2 from 2-3 pm, 105 Annenberg
- Office hours during study period
  - 5 Dec (Mon), 3-5 pm
  - 6 Dec (Tue), 3-5 pm