

## CDS 101/110 Homework #5 Solution

**Problem 1 (CDS 101, CDS 110):** (10 points)

Let the state space system be the following form

$$\begin{aligned}\frac{dx}{dt} &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [c_1 \quad c_2] x\end{aligned}\tag{1}$$

where  $x = [x_1 \quad x_2]^T$ .

In this problem, the output  $y$  is the tilt angle  $\phi$ , and the input  $u$  is the steering angle  $\delta$ .

From (1),

$$\begin{aligned}\dot{y} &= c_1 \dot{x}_1 + c_2 \dot{x}_2 \\ &= c_1(-a_1 x_1 - a_2 x_2 + u) + c_2 x_1 \\ \ddot{y} &= c_1(-a_1 \dot{x}_1 - a_2 \dot{x}_2 + \dot{u}) + c_2 \dot{x}_1 \\ &= -a_1 c_1 \dot{x}_1 - a_2 c_1 x_1 + c_1 \dot{u} + c_2(-a_1 x_1 - a_2 x_2 + u) \\ &= -a_1(\dot{y} - c_2 \dot{x}_2) - a_2 c_1 x_1 + c_1 \dot{u} + c_2(-a_1 x_1 - a_2 x_2 + u) \\ &= -a_1(\dot{y} - c_2 x_1) - a_2 c_1 x_1 + c_1 \dot{u} + c_2(-a_1 x_1 - a_2 x_2 + u) \\ &= -a_1 \dot{y} - a_2(c_1 x_1 + c_2 x_2) + c_1 \dot{u} + c_2 u \\ &= -a_1 \dot{y} - a_2 y + c_1 \dot{u} + c_2 u\end{aligned}$$

Compare the given equation in the problem with the above equation

$$a_1 = 0 \quad -a_2 = \frac{mgh}{J} \quad c_1 = \frac{Dv_0}{Jb} \quad c_2 = \frac{mv_0 h}{Jb}$$

Then, the observability matrix is

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \frac{Dv_0}{Jb} & \frac{mv_0 h}{Jb} \\ \frac{mv_0 h}{Jb} & \frac{mgh}{J} \frac{Dv_0}{Jb} \end{bmatrix}$$

The system is observable if the observability matrix is full rank.  $W_o$  is not full rank when

$$\begin{aligned}\det(W_o) &= \left(\frac{Dv_0}{Jb}\right)^2 \frac{mgh}{J} - \left(\frac{mv_0 h}{Jb}\right)^2 = 0 \\ \left(\frac{Dv_0}{Jb}\right)^2 \frac{mgh}{J} &= \left(\frac{mv_0 h}{Jb}\right)^2 \\ D^2 g &= mhJ\end{aligned}$$

Hence, the system is observable except when  $D^2 g = mhJ$ .

**Problem 2 (CDS 110):** (35 points)

The dynamics is given by

$$\begin{aligned} J_1 \frac{d^2 \phi_1}{dt^2} + c \left( \frac{d\phi_1}{dt} - \frac{d\phi_2}{dt} \right) + k(\phi_1 - \phi_2) &= k_I I \\ J_2 \frac{d^2 \phi_2}{dt^2} + c \left( \frac{d\phi_2}{dt} - \frac{d\phi_1}{dt} \right) + k(\phi_2 - \phi_1) &= T_d \end{aligned}$$

Let  $x_1 = \phi_1$ ,  $x_2 = \phi_2$ ,  $x_3 = \omega_1/\omega_0$ , and  $x_4 = \omega_2/\omega_0$  where  $\omega_0 = \sqrt{k(J_1 + J_2)/(J_1 J_2)}$ . Then,

$$\begin{aligned} J_1 \ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) &= k_I I \\ J_1 \ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) &= T_d \end{aligned}$$

So,

$$\begin{aligned} \frac{dx_1}{dt} &= \dot{\phi} = \omega_1 = \omega_0 x_3 \\ \frac{dx_2}{dt} &= \omega_0 x_4 \\ \frac{dx_3}{dt} &= \frac{\dot{\omega}_1}{\omega_0} = \frac{\ddot{x}_1}{\omega_0} \\ &= \frac{1}{\omega_0 J_1} (c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) + k_I I) \\ &= -\frac{k}{\omega_0 J_1} x_1 + \frac{k}{\omega_0 J_1} x_2 - \frac{c}{J_1} x_3 + \frac{c}{J_1} x_4 + \frac{k_I}{\omega_0 J_1} I \\ \frac{dx_4}{dt} &= \frac{k}{\omega_0 J_2} x_1 - \frac{k}{\omega_0 J_2} x_2 + \frac{c}{J_2} x_3 - \frac{c}{J_2} x_4 + \frac{1}{\omega_0 J_2} T_d \end{aligned}$$

In state space form,

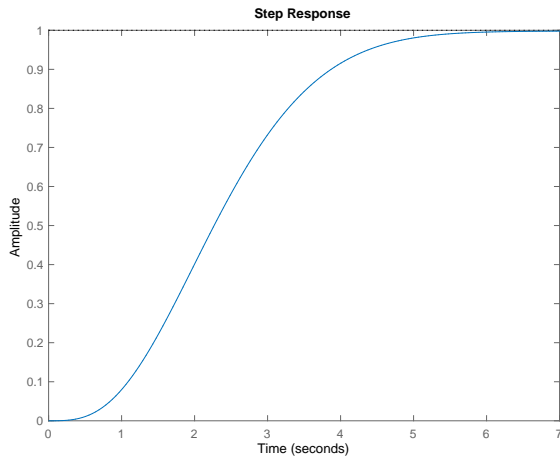
$$\dot{x} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ -\frac{k}{\omega_0 J_1} & \frac{k}{\omega_0 J_1} & -\frac{c}{J_1} & \frac{c}{J_1} \\ \frac{k}{\omega_0 J_2} & -\frac{k}{\omega_0 J_2} & \frac{c}{J_2} & -\frac{c}{J_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{k_I}{\omega_0 J_1} \\ 0 \end{bmatrix} I + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\omega_0 J_2} \end{bmatrix} T_d$$

From Exercise 7.12,  $J_1 = 10/9$ ,  $J_2 = 10$ ,  $c = 0.1$ ,  $k = 1$ , and  $k_I = 1$ .

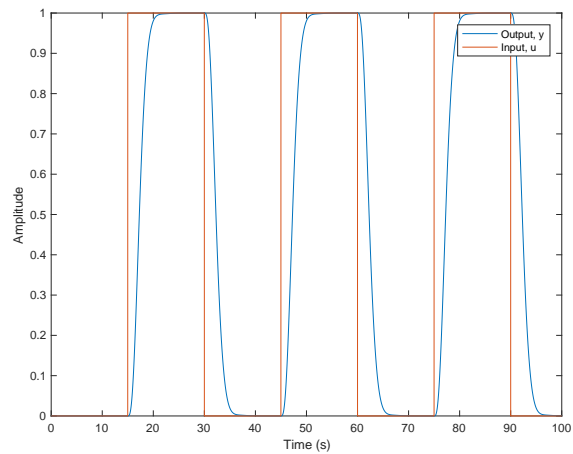
Let  $u = -Kx + K_r x$ . Then,  $K = [8.93 \quad 35.51 \quad 5.44 \quad 101.22]$  and  $K_r = 44.44$ .

The observer gain is given by  $L = [646.9 \quad 9.9 \quad 297.9 \quad 38.0]^T$ .

The simulation for two type of inputs are shown below:



(a) Step input



(b) Square input

The code is given below:

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%Parameters
J1=10/9;J2=10;k=1;kd=0.1;ki=1;
w0=sqrt(k*(J1+J2)/J1/J2);a1=J2/(J1+J2);a2=J1/(J1+J2);
b1=kd/(w0^2*J1);b2=kd/(w0^2*J2);
g1=ki/(w0^2*J1);g2=1/(w0^2*J2);

%System matrices
A=[0 0 1 0;0 0 0 1;-a1 a1 -b1 b1;a2 -a2 b2 -b2]*w0;
B=[0;0;g1;0];
C=[0 1 0 0];
D=0;

% State feedback gain
P=[-1 -2 -1+1i -1-1i]; %Desired closed loop poles
K=place(A,B,P);

% Reference tracking gain
Kr = -1/(C/(A-B*K)*B);
Bref=Kr*B;

% Observer gain
P=[-4 -2 2*(-1+1i) 2*(-1-1i)];
L=place(A',C',P)';

% Closed loop system
LC = L*C;
BK = B*K;
ALCBK=A-LC-BK;
Atot=[A -BK; LC ALCBK];
Btot=[Bref; Bref];
Ctot=[0 1 0 0 0 0 0];
Dtot=0;
systot=ss(Atot,Btot,Ctot,Dtot);

%Simulate the system under step input
figure(1)

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step(systot)

%Simulate the system under square input
[usquare,tsquare]=gensig('square',30,100,0.01);
[Ytot, Ttot, Xtot]=lsim(systot,usquare(:,1),tsquare);
graphtot=[Ytot usquare(:,1)];
figure(2)
plot(Ttot, graphtot);
ylabel('Amplitude')
xlabel('Time (s)')
legend('Output, y', 'Input, u')

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**Problem 3 (CDS 101, CDS 110):** (5 points)

Transfer function is given by

$$G(s) = C(sI - A)^{-1}B$$

$$(sI - A)^{-1} = \frac{1}{s^2 - mgl/J_t} \begin{bmatrix} s & 1 \\ mgl/J_t & s \end{bmatrix}$$

$$G(s) = \frac{1}{J_t s^2 - mgl}$$

**Problem 4 (CDS 101, CDS 110):** (20 points)

(a)  $G(s)$  can be simplified into

$$G(s) = k \frac{R_2 C s + 1}{C(kR_1 + R_2)s + 1}$$

Note that  $s = j\omega$ .

When  $\omega \rightarrow 0$ ,

$$G(s) = k$$

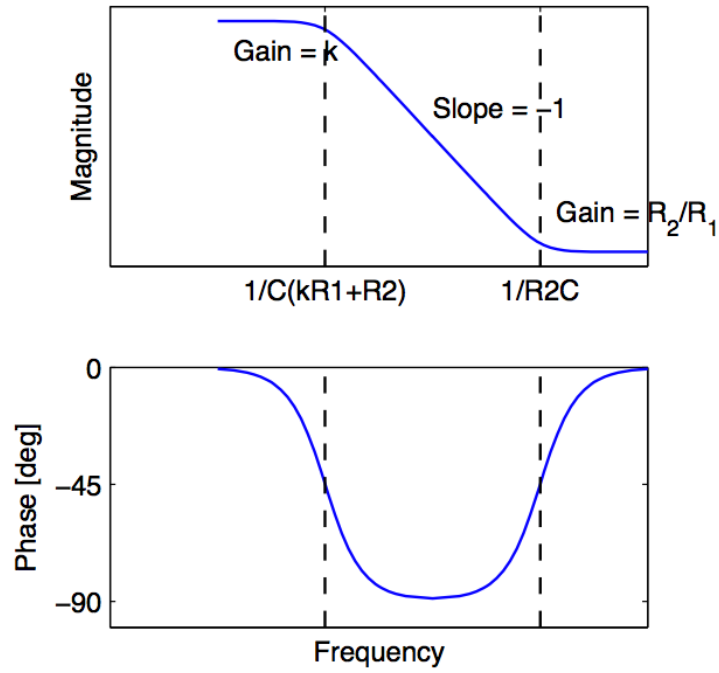
When  $\omega \rightarrow \infty$ ,

$$G(s) = k \frac{R_2 C}{C(kR_1 + R_2)}$$

Because  $k \gg R_1 > R_2$ ,

$$G(s) = \frac{R_2}{R_1}$$

The Bode plot is given by



(b) Replace  $k$  in  $G(s)$  with  $\frac{k}{1+sT}$ . Then,

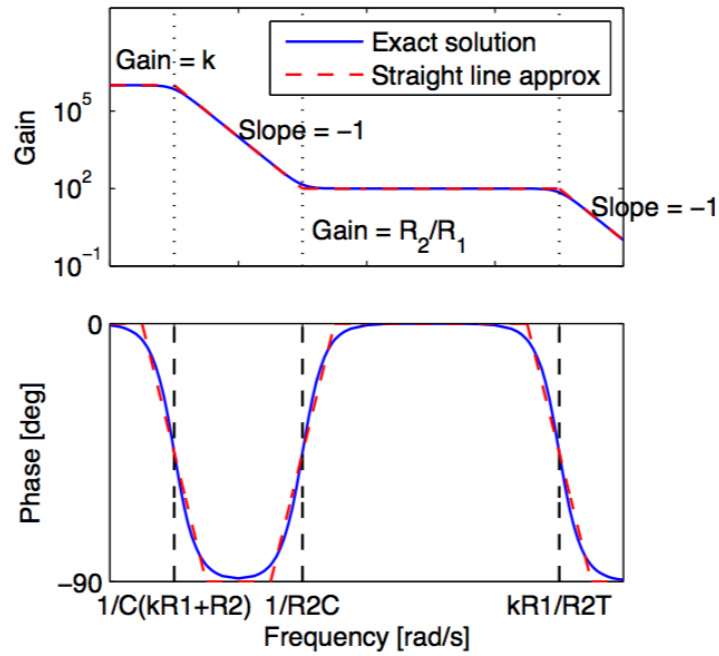
$$G(s) = k \frac{R_2 C s + 1}{C R_2 T s^2 + (k C R_1 + C R_2 + T) s + 1}$$

Plug in all the given values

$$G(s) = 10^8 \frac{s + 1}{s^2 + 1000101s + 100}$$

Then, the zero of  $G(s)$  is  $-1$ , and the poles of  $G(s)$  are  $-10^6$  and  $-10^{-4}$ .

(c) The Bode plot is given by



**Problem 5 (CDS 110):** (15 points) Rewrite the dynamics in frequency domain

$$m_w x_w s^2 = -F + k_t(x_r - x_w) \implies F = k_t x_r - (k_t + m_w s^2) x_w$$

$$F = k(x_w - x_b) + c(x_w - x_b)s \implies x_w = \frac{F + (k + cs)x_b}{k + cs}$$

Combine both equation together.

$$F = k_t x_r - \frac{k_t + m_w s^2}{k + cs} F - (k_t + m_w s^2) x_b$$

$$F = \frac{k_t x_r - (k_t + m_w s^2) x_b}{1 + (k_t + m_w s^2)/(k + cs)}$$

Use  $F = m_b a = m_b \ddot{x}_b = m_b s^2 x_b$ .

$$x_b = \frac{a}{s^2}$$

We get

$$m_b a = \frac{k_t x_r - (k_t + m_w s^2) a/s^2}{1 + (k_t + m_w s^2)/(k + cs)}$$

Simplify

$$a = \frac{k_t}{m_b \left(1 + \frac{k_t + m_w s^2}{k + cs}\right) + \frac{k_t}{s^2} + m_w} x_r$$

So,

$$G_{ax_r}(s) = \frac{k_t}{m_b(1 + \frac{k_t + m_w s^2}{k + cs}) + \frac{k_t}{s^2} + m_w}$$

When  $s = iw_t = i\sqrt{k_t/m_w}$ ,  $k_t + m_w s^2 = 0 \implies G_{ax_r}(iw_t) = k_t/m_b$ .