Goals:
• Continue/conclude quantitative/qualitative study of LTI O.D.E.s
• Introduce basic concept of a “transfer function”
• Basic characteristics of feedback

Reading:
• Åström and Murray, *Feedback Systems*, Sec 2.1-2.4
Continuing from Last Lecture

Many Engineering & Physical systems can be modeled as linear, constant coefficient, (and consequently time invariant) O.D.E.s of the form:

\[
\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)
\]

Where: \( u(t) \) is a known “input” function, and \( y(t) \) is the output (to be found)

**Goals:**
- Understand the quantitative and qualitative behavior of such systems
- Know how to solve simple/relevant cases

**First-Order O.D.E.s**
- simplest linear case: \( \dot{x} = ax \) \( \rightarrow \) \( x(t) = e^{at} x(0) \)
- general case: \( \dot{x} + p(t)x = g(t) \) \( \rightarrow \) \( x(t) = \frac{1}{\mu} \int e^{\int p(t)dt} g(t)dt \)
- Linear vector case: \( \dot{x} = Ax \) \( \rightarrow \) \( x(t) = e^{At} x(0) \)
2\textsuperscript{nd}-Order Linear O.D.E.

I. Homogeneous: no inputs

\[
\frac{d^2}{dt^2}z(t) + a_1 \frac{d}{dt}z(t) + a_2 z(t) = 0
\]

Assume nontrivial solution \(y(t) = Ce^{st}\). Substituting this solution

\[
Cs^2 e^{st} + Ca_1 se^{st} + Ca_2 e^{st} = 0 \rightarrow s^2 + a_1 s + a_2 = 0
\]

\[
\rightarrow s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}
\]

Cases:

- \(a_1^2 - 4a_2 > 0\) : two distinct roots

\[
z(t) = e^{-\frac{a_1}{2}t} \left[ c_1 e^{\frac{1}{2}\sqrt{a_1^2 - 4a_2}} + c_2 e^{-\frac{1}{2}\sqrt{a_1^2 - 4a_2}} \right]
\]

- \(a_1^2 - 4a_2 < 0\) : two Complex Conjugate roots

\[
z(t) = e^{-\frac{a_1}{2}t} \left[ c_1 \sin \left( \frac{t}{2} \sqrt{4a_2 - a_1^2} \right) + c_2 \cos \left( \frac{t}{2} \sqrt{4a_2 - a_1^2} \right) \right]
\]

- \(a_1^2 - 4a_2 = 0\) : two repeated real roots

\[
s_1 = -\frac{a_1}{2}
\]

- Assume \(z(t) = k(t)e^{st}\) \(-> \frac{d^2k}{dt^2} + (2s + a_1) \frac{dk}{dt} + (s^2 + a_1 s + a_2)k = 0\)
**2nd-Order Linear O.D.E.**

- \( a_1^2 - 4a_2 = 0 \) : *two repeated real roots* \( s_1 = \frac{a_1}{2} \)
- \( \therefore \frac{d^2 k}{dt^2} = 0 \quad \rightarrow \quad k(t) = ct + c_0 \)
- \( z(t) = c_1 e^{-\frac{a_1}{2}t} + c_2 te^{-\frac{a_1}{2}t} \)

**II. Inhomogeneous O.D.E:** input, or “forcing function”

\[
\frac{d^2}{dt^2} z(t) + a_1 \frac{d}{dt} z(t) + a_2 z(t) = f(t)
\]

Any solution can be expressed as: \( z(t) = z_h(t) + z_p(t) \)

- \( z_h(t) \) is the *homogeneous solution* (to LHS)
- \( z_p(t) \) is the *particular solution*

  - In general, finding \( z_p(t) \) is an art, except for special inputs:
  - \( f(t) \) polynomial: \( f(t) = a_1 t^2 + a_2 t + a_3 \quad \rightarrow \quad z_p = d_1 t^2 + d_2 t + d_3 \)
  - \( f(t) \) oscillatory: \( f(t) = a_1 \cos(st) \quad \rightarrow \quad z_p = d_1 \sin(st) + d_2 \cos(st) \)
  - \( f(t) \) exponential: \( f(t) = a_1 e^{st} \quad \rightarrow \quad z_p = d_1 e^{s_1 t} + d_2 e^{s_2 t} \)
Converting to 1st –Order Form

2nd and higher-order linear homogeneous o.d.e.s can be converted to 1st order form

- **Key idea:** introduce a *dummy variable*
- **Example:** 2nd-order c.c. o.d.e.
  \[ \ddot{x}(t) + b\dot{x}(t) + cx(t) = 0 \]

- Let \( q(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \). Then:

  \[
  \dot{q}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = Aq(t)
  \]

- Exponential solution: \( q(t) = e^{At} q(0) \)
- **Note:** that properties of solution depend upon eigenvalues of A, which in turn are specified by the o.d.e. characteristic equation.
Transfer Functions: A First Look

Consider solution to general linear nonhomogeneous c.c. o.d.e

\[
\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)
\]

for special case where \( u(t) = e^{st} \).

Look for particular solution \( y(t) = G(s)e^{st} \)

- \( \frac{du}{dt} = se^{st} \), \( \frac{d^2u}{dt^2} = s^2e^{st} \), \( \vdots \)
- \( \frac{du}{dt} = sG(s)e^{st} \), \( \frac{d^2y}{dt^2} = s^2G(s)e^{st} \), \( \vdots \)

Substitute into o.d.e. and rearrange:

\[
G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{b(s)}{a(s)}
\]
Transfer Functions: A First Look

General solution has homogeneous and particular parts

\[ y(t) = \sum_{k=1}^{m} C_k(t)e^{skt} + G(s)e^{st} \quad n = \sum_{k=1}^{m}(\text{deg}(C_k) + 1) \]

The transfer function defines how the system responds to different inputs

- Steady State Response: \( s = 0 \) → \( u(t) = 1 \) → \( G(0) = \frac{b_n}{a_n} \)
- Oscillatory Response: → \( u(t) = \sin(\omega t) = \text{Im}(e^{i\omega t}) \)

\[ y_p(t) = \text{Im}[G(i\omega)e^{i\omega t}] = |G(i\omega)| \text{Im}[e^{i\arg[G(i\omega)]}e^{i\omega t}] \]

I.e. oscillatory input is

- magnified by \( |G(i\omega)| \)
- phase shifted by \( \arg[G(i\omega)] \)
Some Characteristics of Feedback

To get a “first look” at some of the issues in feedback control, let’s look at a simple inverted pendulum example problem.

- Dynamical Equation:
  
  \[ ml^2 \ddot{\theta} = -\varepsilon \dot{\theta} + mgl \sin \theta + lu \]

  - Where \( \varepsilon \ll 1 \) is a small “damping” effect
  - \( g \) is the gravitational constant
  - \( u(t) \) is a force (which can be varied) applied at bottom
  - We ignore the horizontal dynamics, and only care about stabilizing the pendulum to \( \theta = 0 \).

- Using a small angle approximation for \( \sin \theta \)

  \[ \ddot{\theta} + \frac{\varepsilon}{ml^2} \dot{\theta} - \frac{g}{l} \sin \theta = \frac{1}{ml} u \quad \rightarrow \quad \ddot{\theta} + \alpha \dot{\theta} - \beta \sin \theta = \gamma u \]

  - Where \( \alpha = \frac{\varepsilon}{ml^2}, \beta = \frac{g}{l}, \gamma = \frac{1}{ml} \)
Some Characteristics of Feedback

Characteristic Equation
\[ s_{1,2} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 + 4\beta}}{2} \approx -\frac{\alpha}{2} \pm \sqrt{\beta} \]

Since \( \alpha \) and \( \beta \) are positive, and \( \beta \gg \alpha \), one root is positive. The solution is unstable:
\[ \theta(t) = e^{-\frac{\alpha}{2}t} \left( c_1 e^{-\sqrt{\beta}} + c_2 e^{+\sqrt{\beta}} \right) \]

Let’s see if feedback can improve stability.

• First try proportional feedback: \( u(t) = -k_p (\theta - \theta_{ref}) = -k_p \theta \)
  \[ \ddot{\theta} + \alpha \dot{\theta} + (\gamma k_p - \beta) \theta = 0 \quad \rightarrow \quad \ddot{\theta} + \alpha \dot{\theta} + \beta' \theta = 0 \]
  \[ s_{1,2} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta'}}{2} \]

\( k_p \) can be chosen to give \( \beta' \) any real value.
  • Both roots are negative (and thus stable) if \( 0 < \beta' < \alpha^2 / 4 \)
  • Roots are stable and oscillatory if \( \beta' > \alpha^2 / 4 \)
  • Magnitudes of roots are small, so that response is very slow
Some Characteristics of Feedback

Next try proportional and derivative feedback

\[ u(t) = -k_p(\theta - \theta_{ref}) - k_v(\dot{\theta} - \dot{\theta}_{ref}) = -k_p\theta - k_v\dot{\theta} \]

\[ \ddot{\theta} + (\alpha + \gamma k_v)\dot{\theta} + (\gamma k_p - \beta)\theta = 0 \quad \rightarrow \quad \ddot{\theta} + \alpha'\dot{\theta} + \beta'\theta = \]

\[ s_{1,2} = -\frac{\alpha'}{2} \pm \frac{\sqrt{(\alpha')^2 - 4\beta'}}{2} \]

\[ k_p \text{ and } k_v \text{ can be chosen to place roots arbitrarily. Hence, any dynamical behavior can, in theory, be designed.} \]

What can go wrong? Unmodeled dynamics