CDS 101/110: Lecture 1.3
O.D.E. Behavior & Feedback Characteristics

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Goals:

- Continue/conclude quantitative/qualitative study of LTI O.D.E.s
- Introduce basic concept of a “transfer function”
- Basic characteristics of feedback

Reading:

- Åström and Murray, *Feedback Systems*, Sec 2.1-2.4

Continuing from Last Lecture

Many Engineering & Physical systems can be modeled as *linear, constant coefficient*, (and consequently time invariant) O.D.E.s of the form:

$$\boldsymbol{\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)}$$

Where: $u(t)$ is a known "input" function, and $y(t)$ is the output (to be found)

Goals:

- understand the quantitative and qualitative behavior of such systems
- Know how to solve simple/relevant cases

First-Order O.D.E.s

- simplest linear case: $\dot{x} = ax \quad \rightarrow \quad x(t) = e^{at}x(0)$
- general case: $\dot{x} + p(t)x = g(t) \rightarrow \quad x(t) = \frac{1}{\mu}\int e^{\int p(t)dt} g(t)dt$
- Linear vector case: $\dot{\vec{x}} = A\vec{x} \quad \rightarrow \quad \vec{x}(t) = e^{At}\vec{x}(0)$

2^{nd}-Order Linear O.D.E.

I. Homogeneous: no inputs

$$\frac{d^2}{dt^2}z(t) + a_1\frac{d}{dt}z(t) + a_2 z(t) = 0$$

Assume nontrivial solution $y(t) = Ce^{st}$. Substituting this solution

$$Cs^2e^{st} + Ca_1se^{st} + Ca_2e^{st} = 0 \quad \rightarrow \quad s^2 + a_1s + a_2 = 0$$

$$\rightarrow \quad s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Cases:

- $a_1^2 - 4a_2 > 0$ **:** ***two distinct roots*** $\quad z(t) = e^{-\frac{a_1}{2}t}\left[c_1e^{\frac{1}{2}\sqrt{a_1^2-4a_2}} + c_2e^{-\frac{1}{2}\sqrt{a_1^2-4a_2}}\right]$
- $a_1^2 - 4a_2 < 0$ **:** ***two Complex Conjugate roots***

$$z(t) = e^{-\frac{a_1}{2}t}\left[c_1\sin\left(\frac{t}{2}\sqrt{4a_2 - a_1^2}\right) + c_2\cos\left(\frac{t}{2}\sqrt{4a_2 - a_1^2}\right)\right]$$

- $a_1^2 - 4a_2 = 0$ **:** ***two repeated real roots*** $s_1 = -\frac{a_1}{2}$
  - *Assume* $z(t) = k(t)e^{st}$ *->* $\frac{d^2k}{dt^2} + \underbrace{(2s + a_1)}_{0}\frac{dk}{dt} + \underbrace{(s^2 + a_1s + a_2)}_{0}k = 0$

2^{nd}-Order Linear O.D.E.

- $a_1^2 - 4a_2 = 0$ ***: two repeated real roots*** $s_1 = \frac{a_1}{2}$
  - $\therefore \quad \frac{d^2k}{dt^2} = 0 \quad \rightarrow \quad k(t) = ct + c_0$
  - $z(t) = c_1\, e^{-\frac{a_1}{2}t} + c_2\, te^{-\frac{a_1}{2}t}$

II. Inhomogeneous O.D.E: input, or "forcing function"

$$\frac{d^2}{dt^2}z(t) + a_1\frac{d}{dt}z(t) + a_2 z(t) = f(t)$$

Any solution can be expressed as: $z(t) = z_h(t) + z_p(t)$

- $z_h(t)$ is the *homogeneous solution* (to LHS)
- $z_p(t)$ is the *particular solution*
  - In general, finding $z_p(t)$ is an art, except for special inputs:
  - $f(t)$ polynomial: $f(t) = a_1t^2 + a_2t + a_3 \quad \rightarrow \quad z_p = d_1t^2 + d_2t + d_3$
  - $f(t)$ oscillatory : $f(t) = a_1\cos(st) \quad \rightarrow \quad z_p = d_1\sin(st) + d_2\cos(st)$
  - $f(t)$ exponential: $f(t) = a_1e^{st} \quad \rightarrow \quad z_p = d_1e^{s_1t} + d_2e^{s_2t}$

Converting to 1st –Order Form

2nd and higher-order linear homogeneous o.d.e.s can be converted to 1st order form

- **Key idea:** introduce a *dummy variable*
- **Example:** 2nd-order c.c. o.d.e.

$$\ddot{x}(t) + b\dot{x}(t) + cx(t) = 0$$

- Let $q(\mathrm{t}) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$. Then:

$$\dot{q}(\mathrm{t}) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \mathrm{A}q(\mathrm{t})$$

- Exponential solution: $q(t) = e^{At}q(0)$
- **Note:** that properties of solution depend upon eigenvalues of A, which in turn are specified by the o.d.e. characteristic equation.

Transfer Functions: A First Look

Consider solution to general linear nonhomogeneous c.c. o.d.e

$$\frac{d^n}{dt^n}y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}}y(t) + \cdots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}}u(t) + \cdots + b_n u(t)$$

for special case where $u(t) = e^{st}$.

Look for *particular solution* $y(t) = G(s)e^{st}$

- $\frac{du}{dt} = se^{st}, \qquad \frac{d^2u}{dt^2} = s^2e^{st}, \qquad \cdots$
- $\frac{du}{dt} = sG(s)e^{st}, \qquad \frac{d^2y}{dt^2} = s^2G(s)e^{st}, \qquad \cdots$

Substitute into o.d.e. and rearrange:

$$G(s) = \underbrace{\frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}}_{\text{Transfer Function}} = \frac{b(s)}{a(s)}$$

Transfer Functions: A First Look

General solution has homogeneous and particular parts

$$y(t) = \sum_{k=1}^{m} \underbrace{C_k(t)e^{s_k t}}_{\text{Initial Conditions}} + \overbrace{G(s)e^{st}}^{\text{Effect of Inputs}} \qquad n = \sum_{k=1}^{m}(\deg(C_k) + 1)$$

The transfer function defines how the system responds to different inputs

- Steady State Response: $s = 0 \quad \rightarrow \quad u(t) = 1 \quad \rightarrow \quad G(0) = \frac{b_n}{a_n}$
- Oscillatory Response: $\rightarrow \quad u(t) = \sin(\omega t) = Im(e^{i\omega t})$

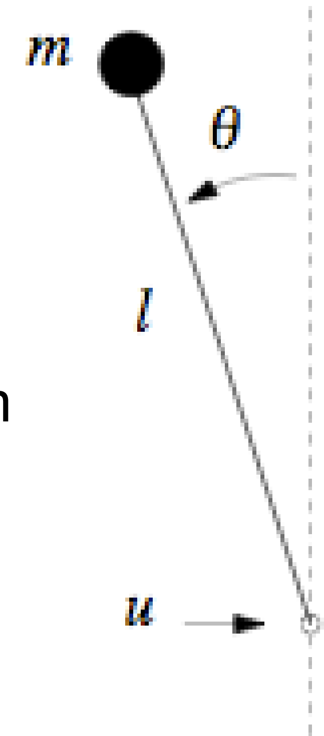
$$y_p(t) = Im\left[G(i\omega)e^{i\omega t}\right] = |G(i\omega)|\ \mathrm{Im}[\mathrm{e}^{\mathrm{i}\,\arg[\mathrm{G}(\mathrm{i}\omega)]}e^{i\omega t}]$$

I.e. oscillatory input is

- *magnified* by $|G(i\omega)|$
- *phase shifted* by $\arg[\mathrm{G}(\mathrm{i}\omega)]$

Some Characteristics of Feedback

To get a "first look" at some of the issues in feedback control, let's look at a simple *inverted pendulum* example problem



- Dynamical Equation:

$$ml^2\ddot{\theta} = -\varepsilon\dot{\theta} + mgl\sin\theta + lu$$

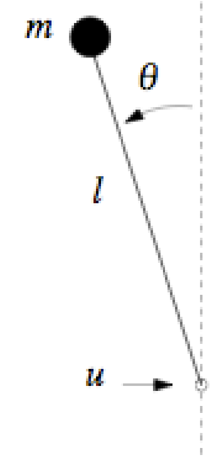
  - Where $\varepsilon \ll 1$ is a small "damping" effect
  - g is the gravitational constant
  - $u(\mathrm{t})$ is a force (which can be varied) applied at bottom
  - We ignore the horizontal dynamics, and only care about stabilizing the pendulum to $\theta = 0$.

- Using a *small angle* approximation for $\sin\theta$

$$\ddot{\theta} + \frac{\varepsilon}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta = \frac{1}{ml}u \quad \rightarrow \quad \ddot{\theta} + \alpha\dot{\theta} - \beta\sin\theta = \gamma u$$

  - Where $\alpha = \frac{\varepsilon}{ml^2}$, $\beta = \frac{g}{l}$, $\gamma = \frac{1}{ml}$

Some Characteristics of Feedback



Characteristic Equation

$$s_{1,2} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 + 4\beta}}{2} \cong -\frac{\alpha}{2} \pm \sqrt{\beta}$$

Since α and β are positive, and $\beta \gg \alpha$, one root is positive. The solution is unstable:

$$\theta(t) = e^{-\frac{\alpha}{2}t}\left(c_1 e^{-\sqrt{\beta}} + c_2 e^{+\sqrt{\beta}}\right)$$

Let's see if *feedback* can improve stability.

- First try *proportional feedback:* $u(t) = -k_p(\theta - \theta_{ref}) = -k_p\theta$

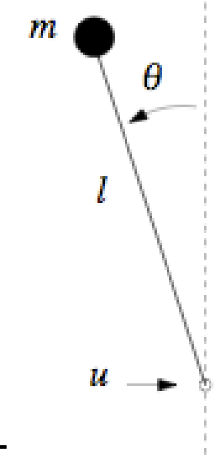
$$\ddot{\theta} + \alpha\dot{\theta} + (\gamma k_p - \beta)\theta = 0 \quad \rightarrow \quad \ddot{\theta} + \alpha\dot{\theta} + \beta'\theta = 0$$

$$s_{1,2} = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta'}}{2}$$

k_p can be chosen to give β' any real value.

- Both roots are *negative* (and thus stable) if $0 < \beta' < \alpha^2/4$
- Roots are stable and oscillatory if $\beta' > \alpha^2/4$
- Magnitudes of roots are *small,* so that response is very *slow*

Some Characteristics of Feedback



Next try *proportional* and *derivative feedback*

- $u(t) = -k_p\left(\theta - \theta_{ref}\right) - k_v(\dot{\theta} - \dot{\theta}_{ref}) = -k_p\theta - k_v\dot{\theta}$

$$\ddot{\theta} + (\alpha + \gamma k_v)\dot{\theta} + \left(\gamma k_p - \beta\right)\theta = 0 \qquad \rightarrow \qquad \ddot{\theta} + \alpha'\dot{\theta} + \beta'\theta =$$

$$s_{1,2} = -\frac{\alpha'}{2} \pm \frac{\sqrt{(\alpha')^2 - 4\beta'}}{2}$$

k_p and k_v can be chosen to place roots *arbitrarily. Hence, any dynamical behavior can, in theory, be designed.*

What can go wrong? *Unmodeled dynamics*