CDS 101/110: Lecture 10.1
Limits on Performance

November 28, 2016

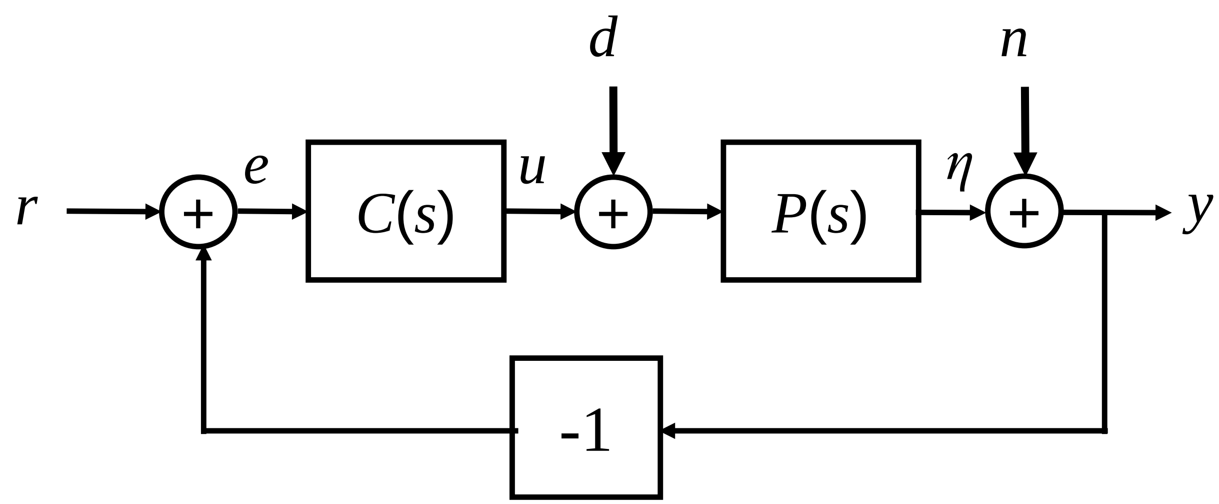
Goals:

- Introduce concept of limits on performance of feedback systems
- Introduce Bode's integral formula and the "waterbed" effect
- Show some of the limitations of feedback due to RHP poles and zeros

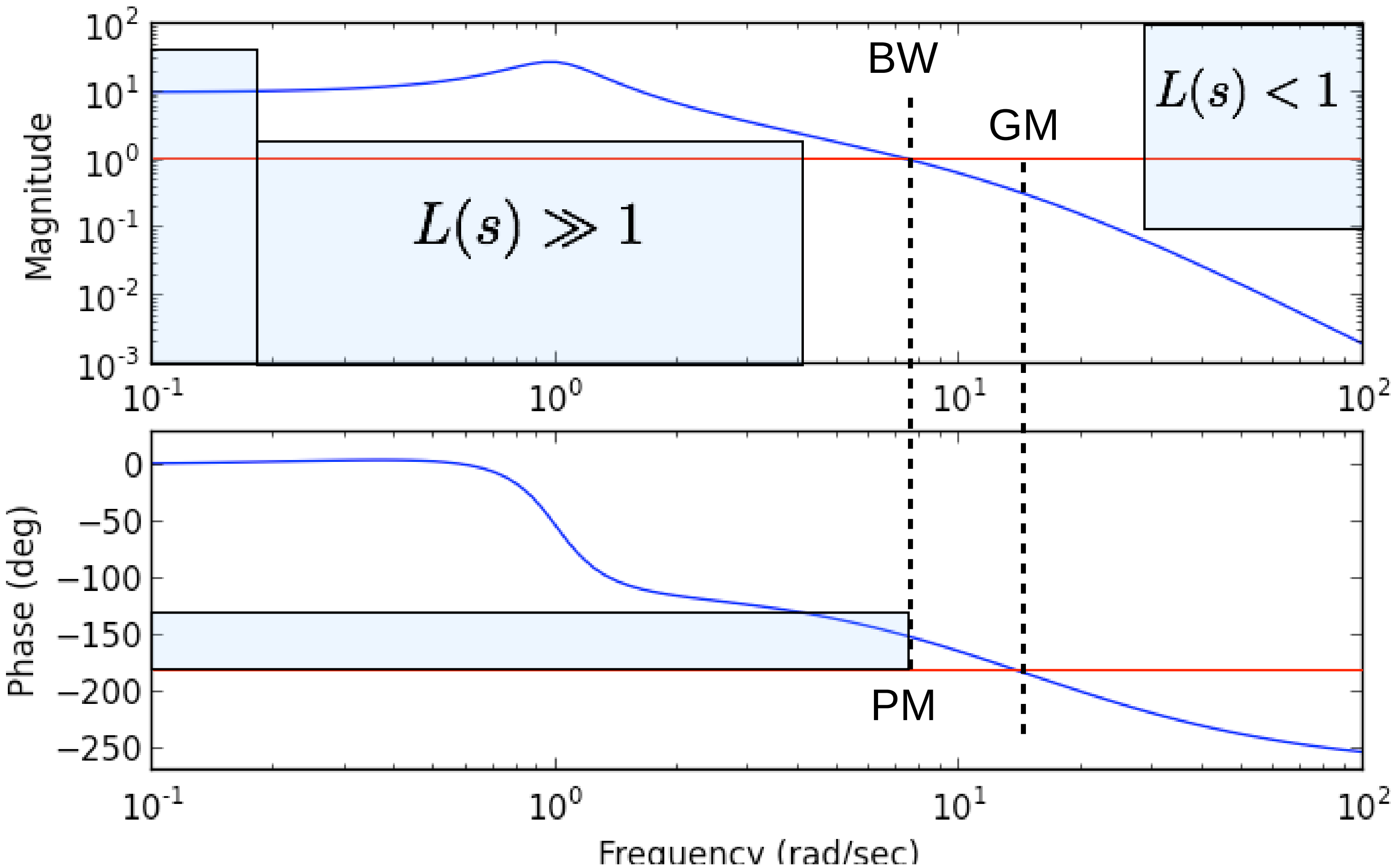
Reading:

- Åström and Murray, Feedback Systems, Section 12.6

"Loop Shaping": Design Loop Transfer Function



$$H_{er} = \frac{1}{1+L} \qquad H_{\eta n} = \frac{-L}{1+L}$$



Translate specs to "loop shape"

$$L(s) = P(s)C(s)$$

Design C(s) to obey constraints

$$C(s) = k\frac{\prod_{i=1}^{n_z}(s - z_i)}{\prod_{j=1}^{n_p}(s - p_j)}$$

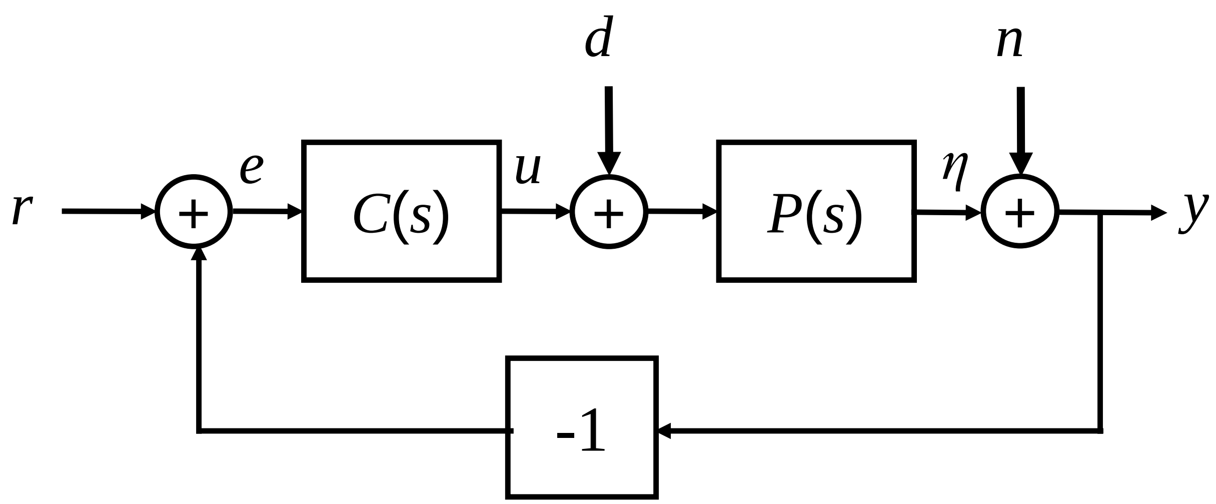
- Poles/Zeros from PID
- Poles/Zeros from
  - Lead
  - Lag

Check the "Gang of Four"

$$S = \frac{1}{1+L(s)}; \quad T = \frac{L(s)}{1+L(s)}$$

$$PS = \frac{P(s)}{1+L(s)}; \quad CS = \frac{C(s)}{1+L(s)}$$

Algebraic Constraints on Performance



$$H_{er} = \frac{1}{1 + PC} =: S$$

Sensitivity function

$$H_{yn} = \frac{PC}{1 + PC} =: T$$

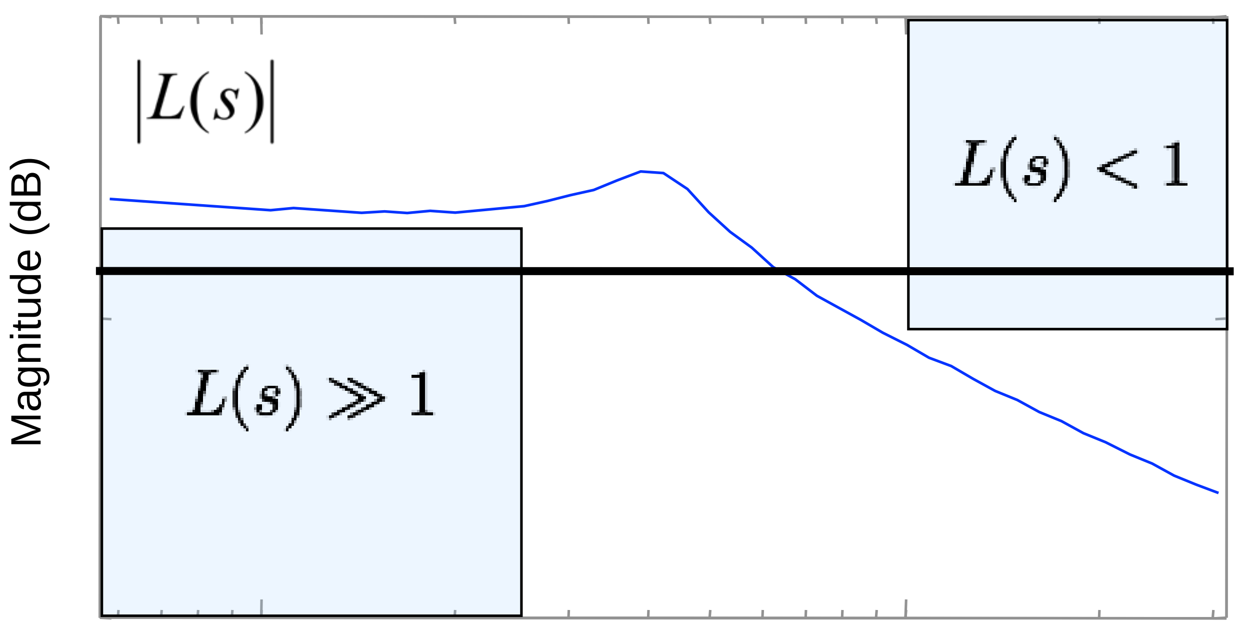
Complementary sensitivity function

Goal: keep S & T small

- S small ⇒ low tracking error
- T small ⇒ good noise rejection (and robustness)

Problem: S + T = 1

- Can't make both S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop gain interpretation: keep L large at low frequency, and small at high frequency



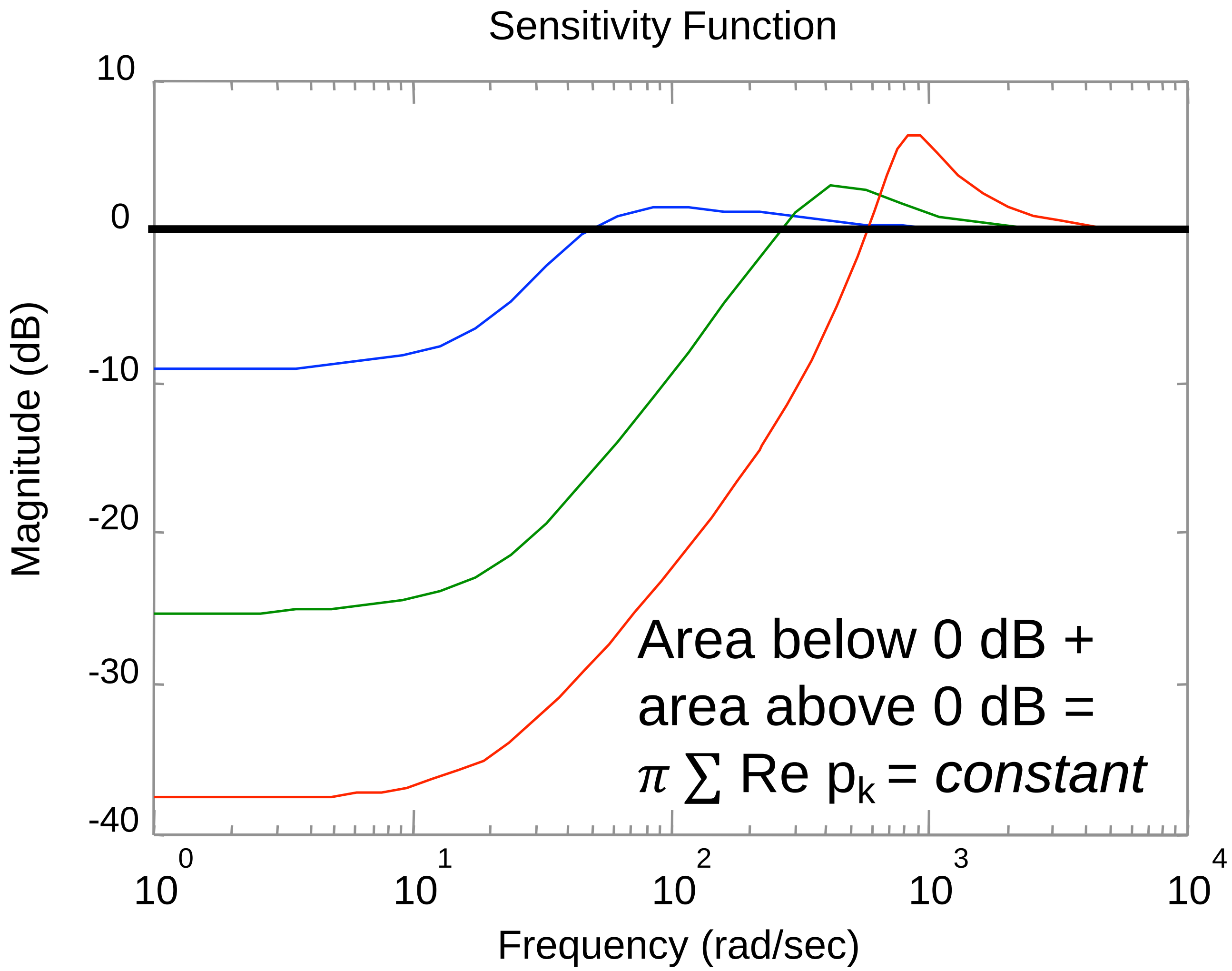
- Transition between large gain and small gain complicated by stability (phase margin)

Bode's Integral Formula and the Waterbed Effect

Bode's integral formula for $S(s) = \frac{1}{1+L(s)} = G_{er} = G_{yn} = G_{vd} = -G_{en}$

- Let p_k be the unstable poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem:** the area under the sensitivity function is a conserved quantity:

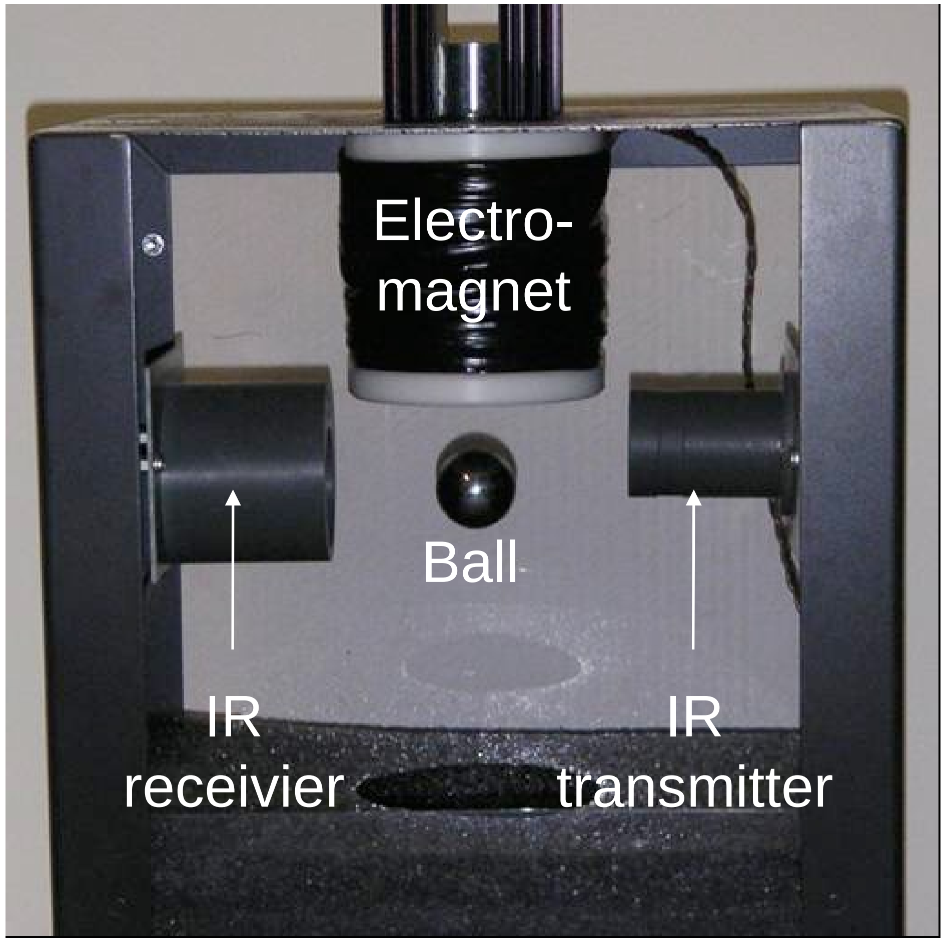
$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1+L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$



Waterbed effect:

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in ω; Bode plots are logarithmic

Example: Magnetic Levitation



System description

- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball, z, (from IR sensor)
- States: $z, \dot{z}$
- Dynamics: $F = ma$, F = magnetic force generated by wire coil

System Dynamics

$$m\ddot{z} = mg - k_m (k_A u)^2 / z^2$$

$$v_{ir} = k_T z + v_0$$

where:

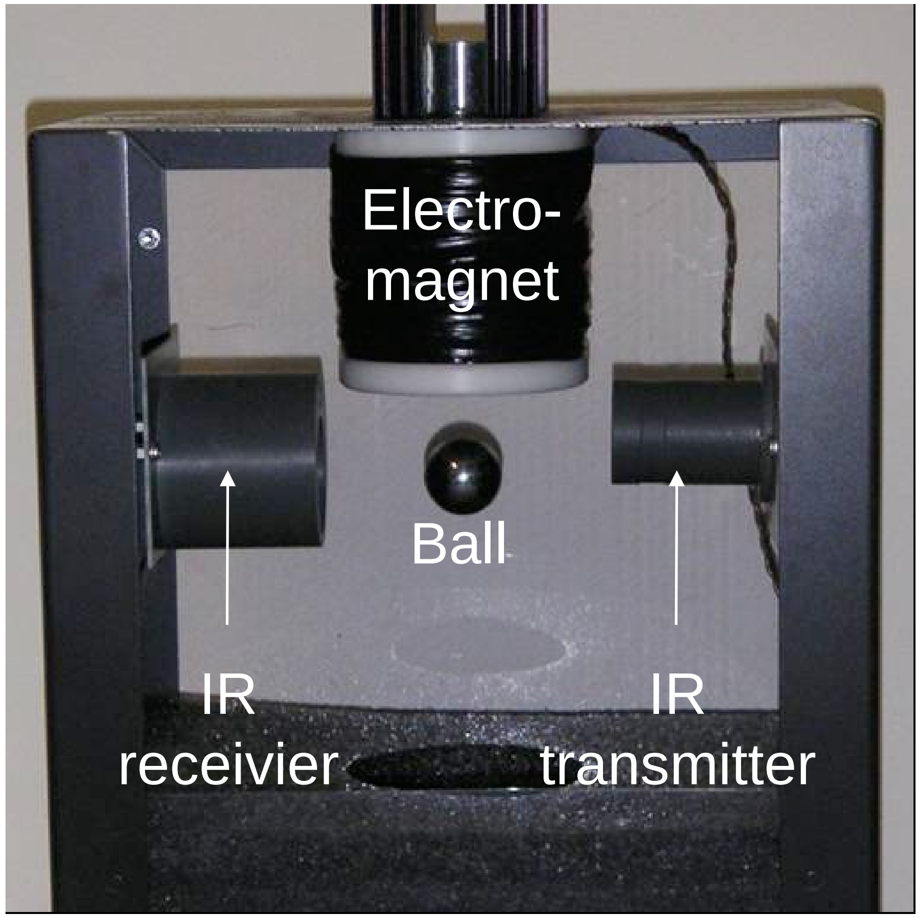
- u = current to electromagnet
- v_{ir} = voltage from IR sensor

Linearization:

$$P(s) = \frac{-k}{s^2 - r^2}$$

- Poles at $s = \pm r \quad \Rightarrow$ open loop unstable

Bode Plot of Open Loop System



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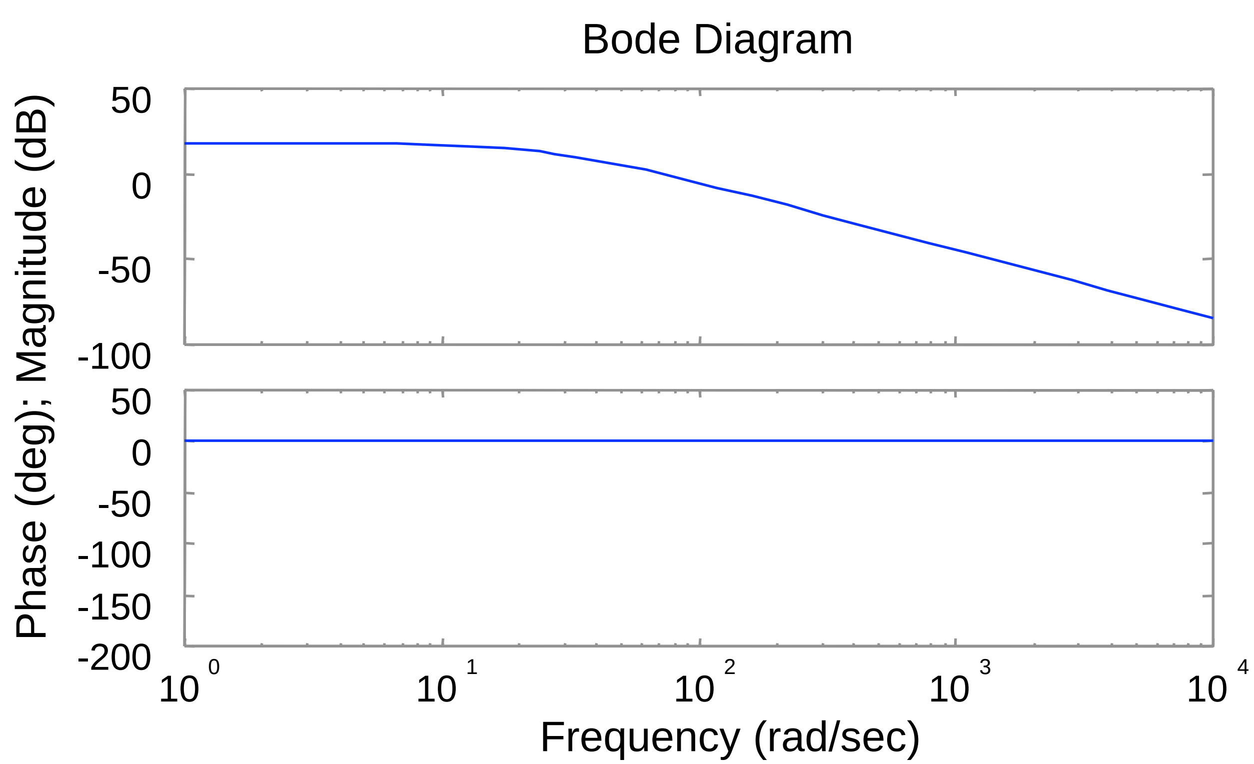
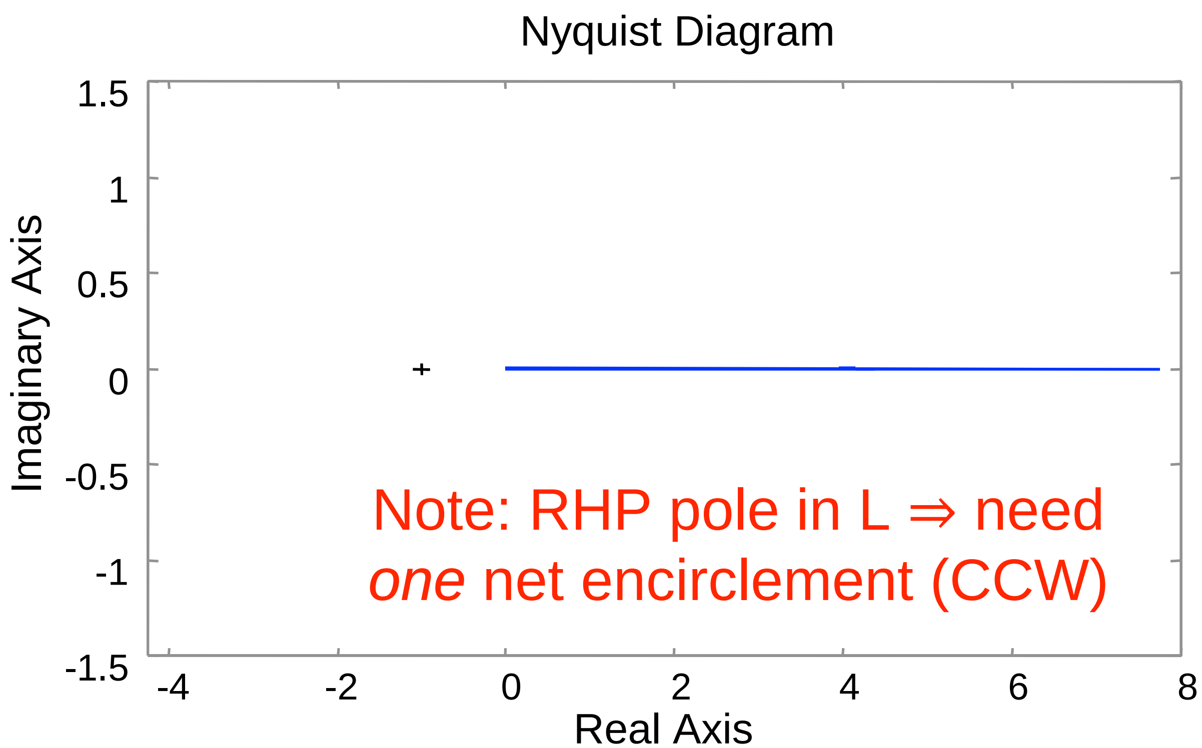




Control Design

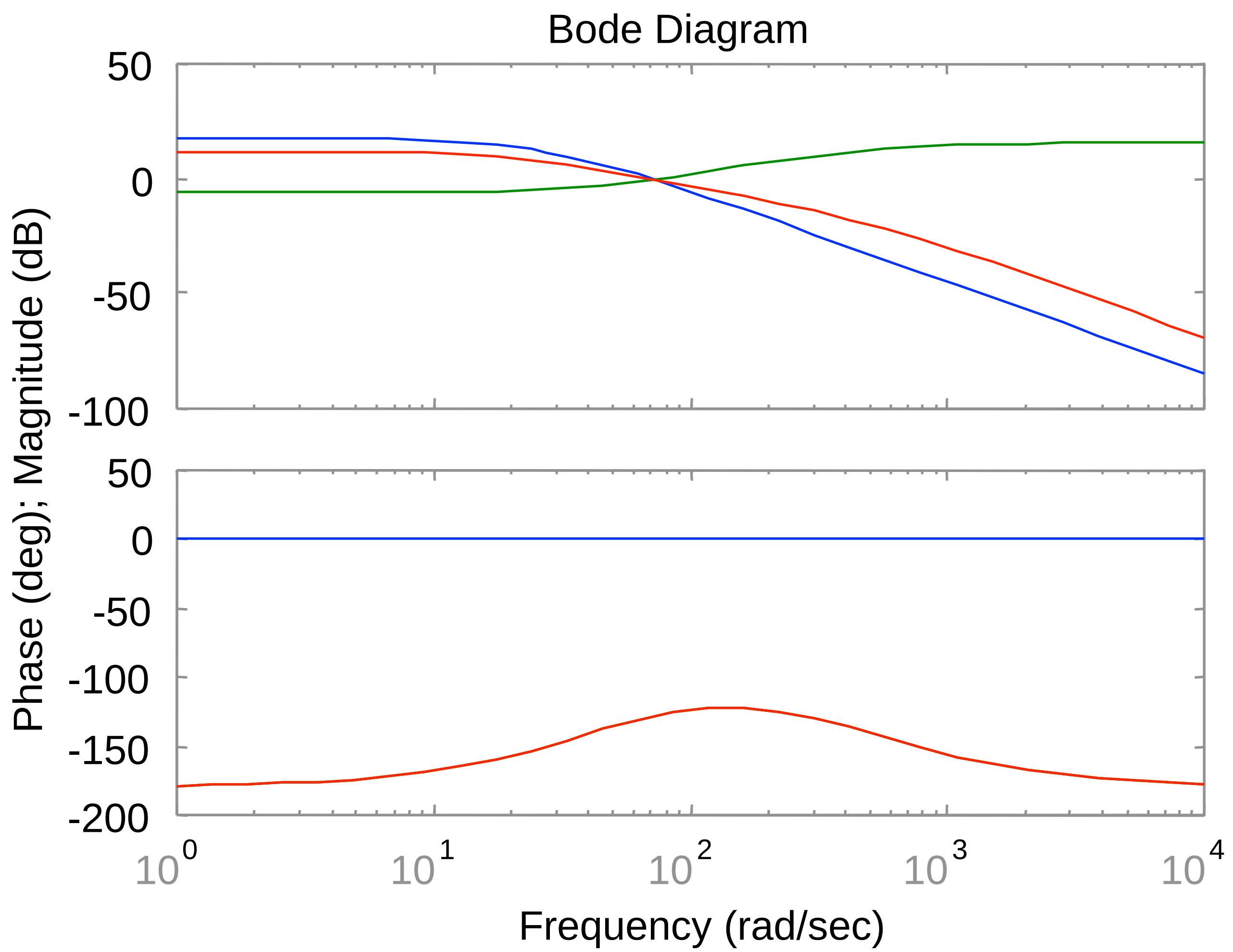
Need to create encirclement

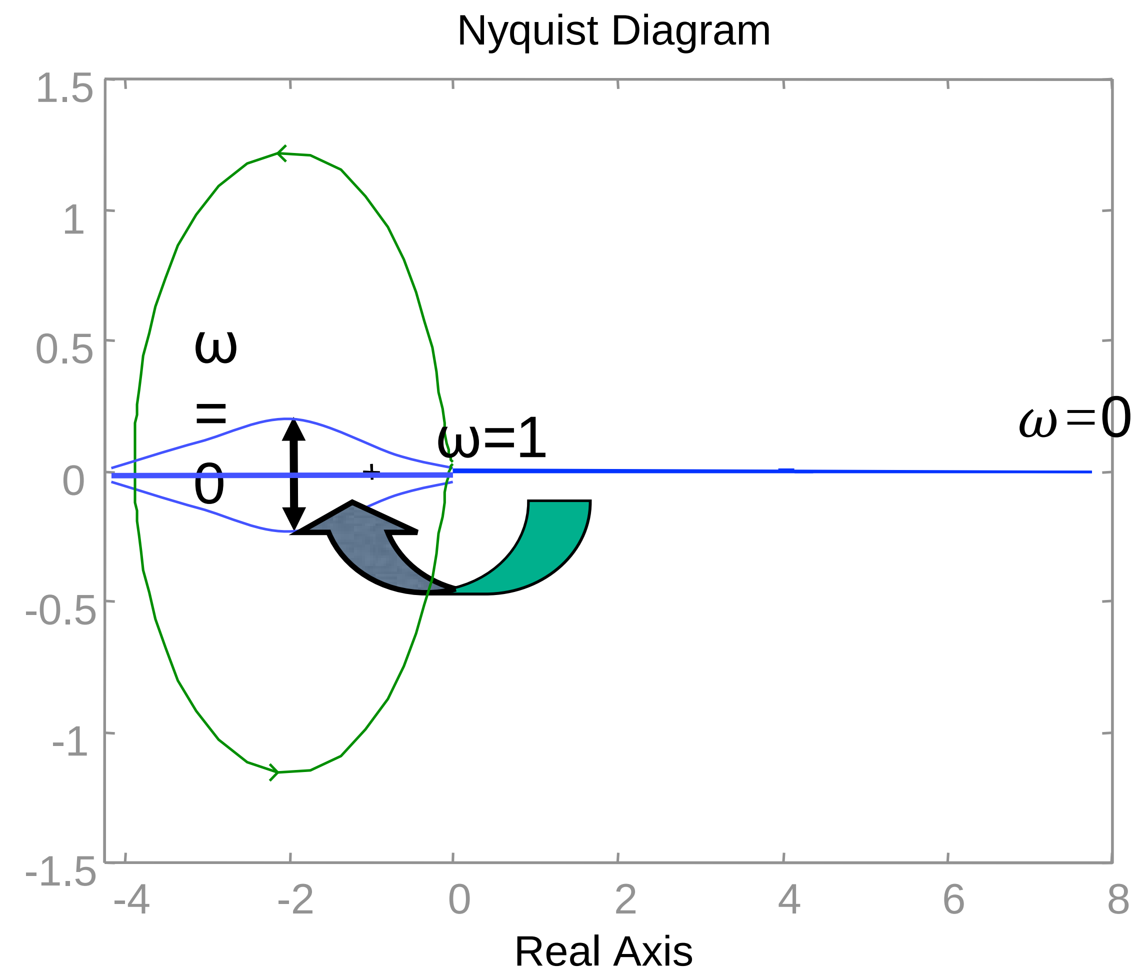
- To offset RHP pole
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator

- Produce phase lead at crossover
- Generates loop in Nyquist plot

$$C(s) = -k\frac{s+a}{s+b}$$





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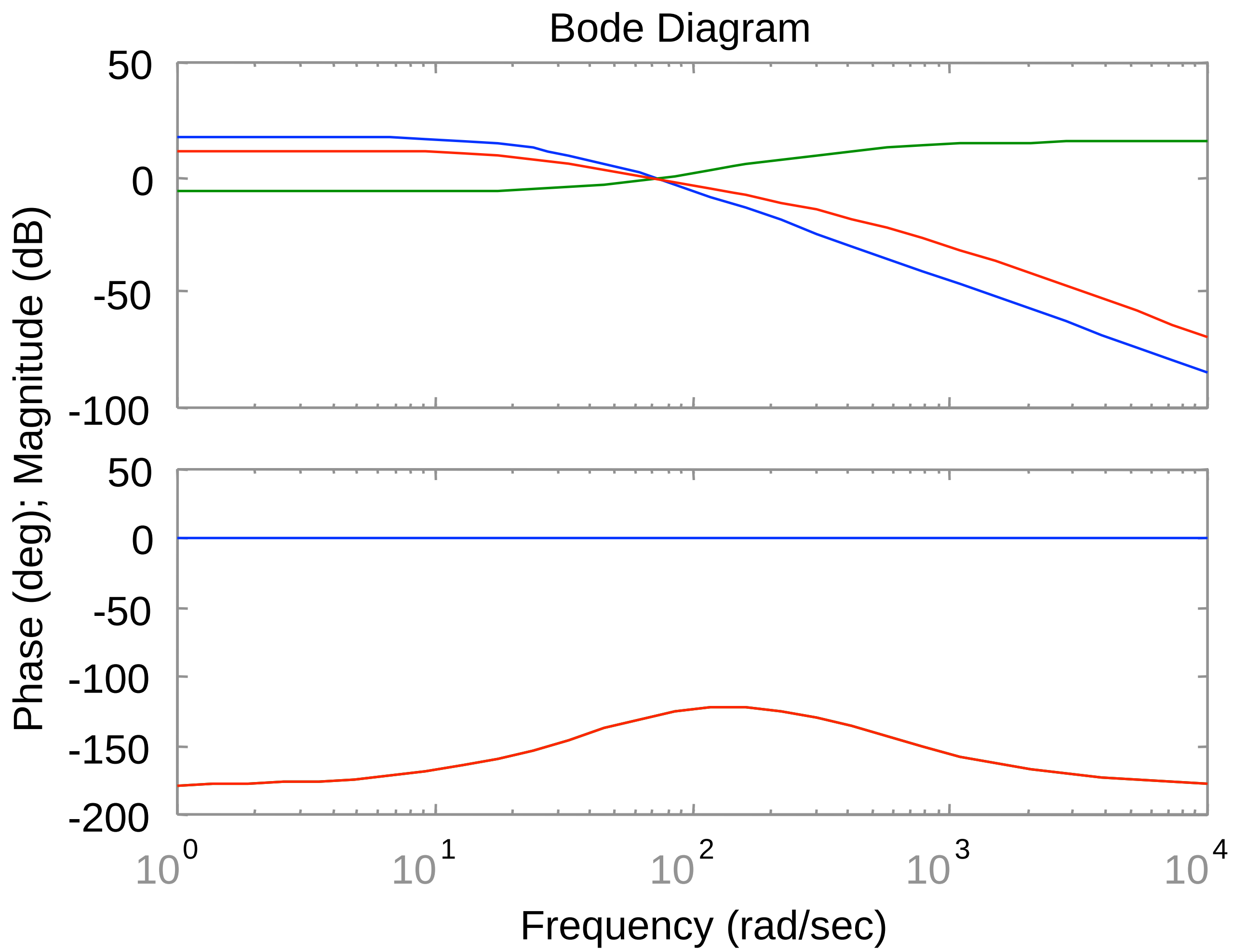
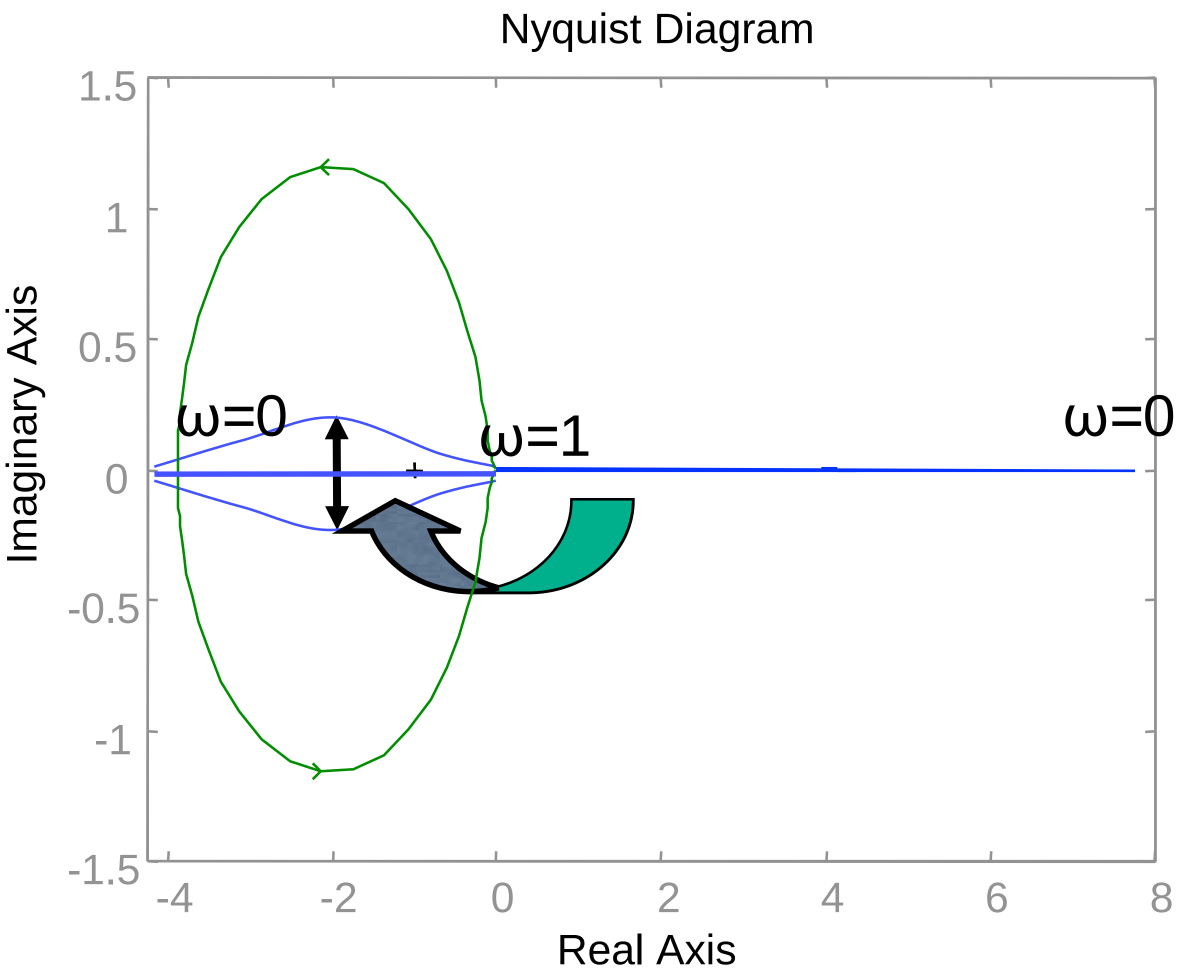




Performance Limits

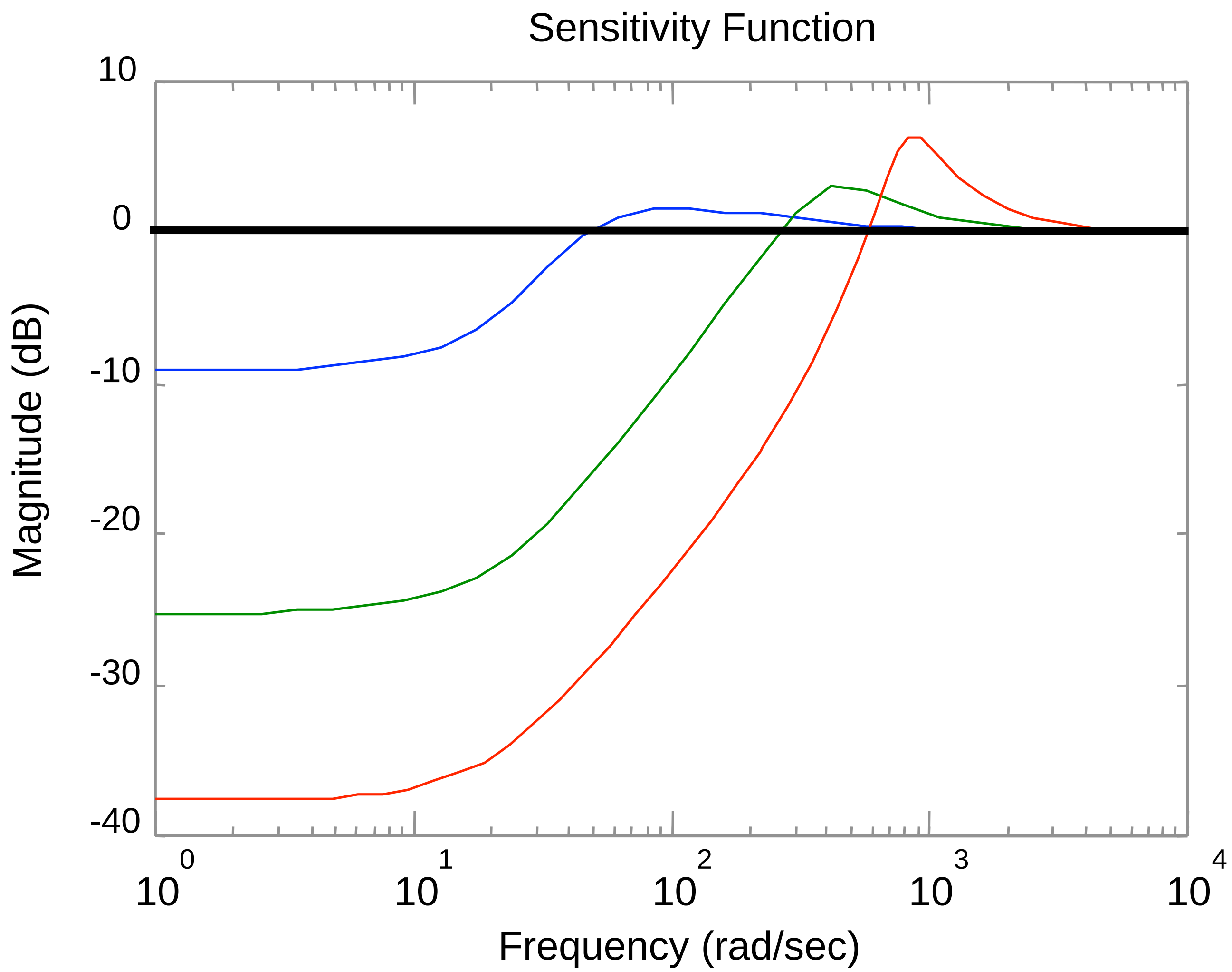
Nominal design gives low perf

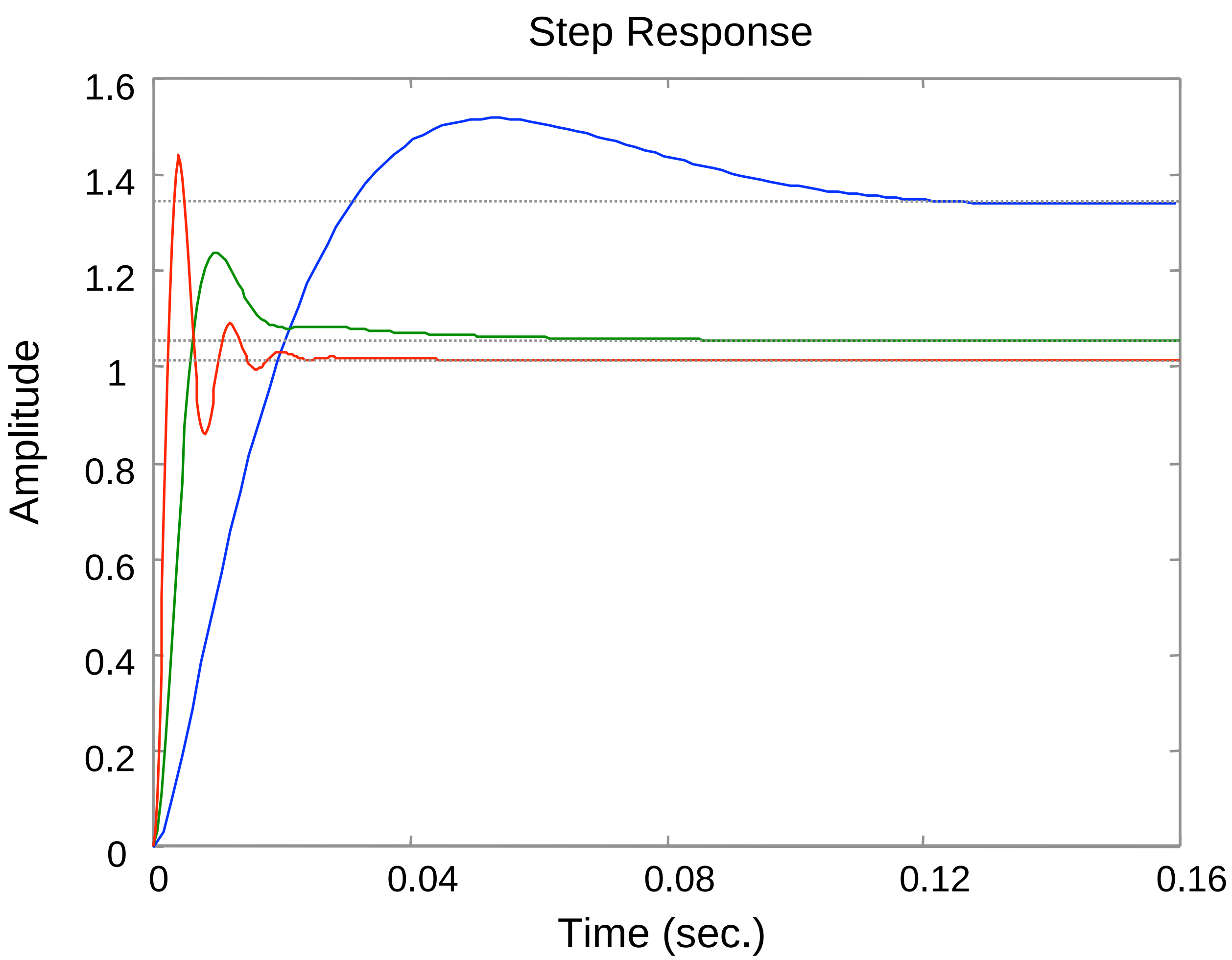
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi r$$

- Must increase sensitivity at some point



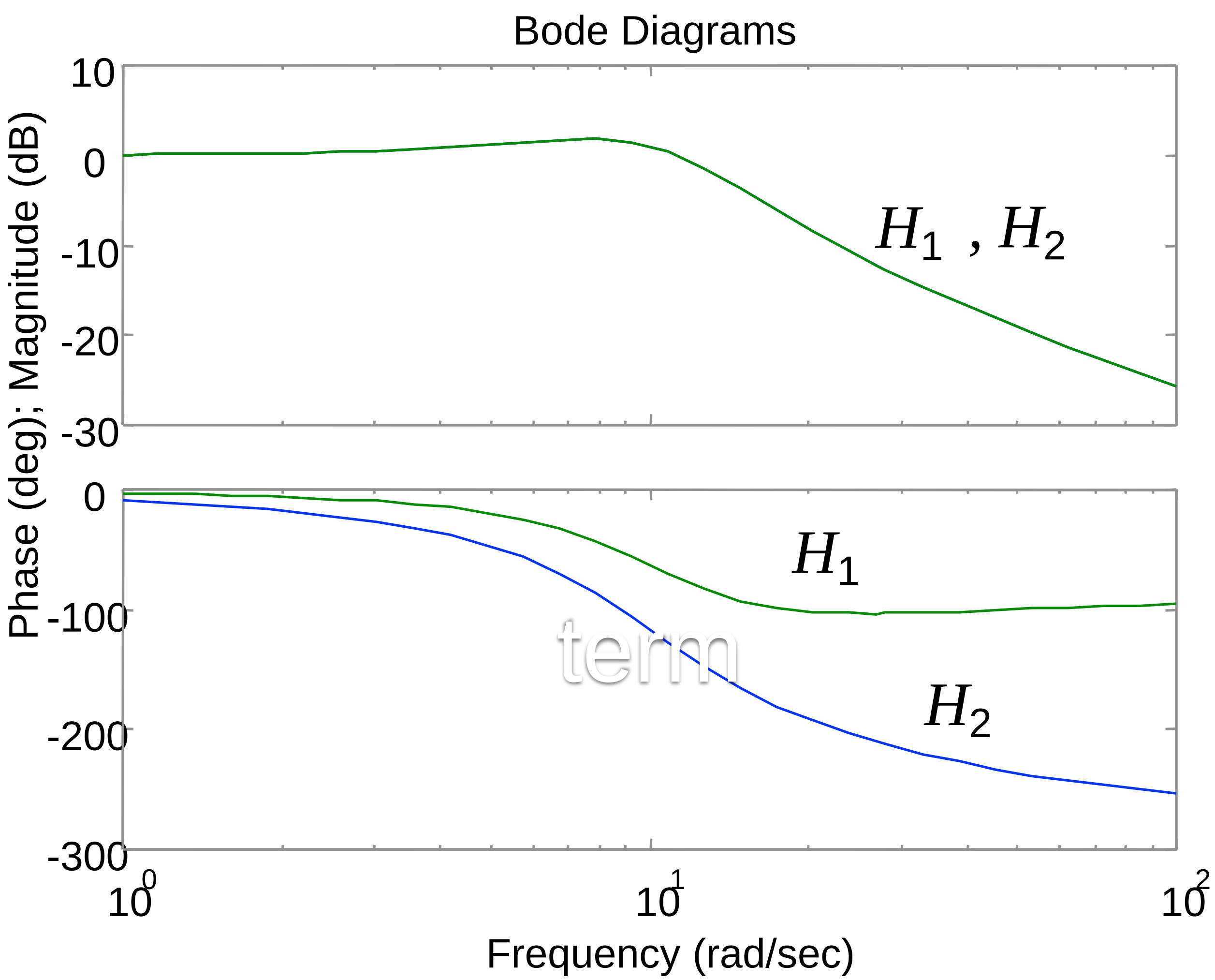


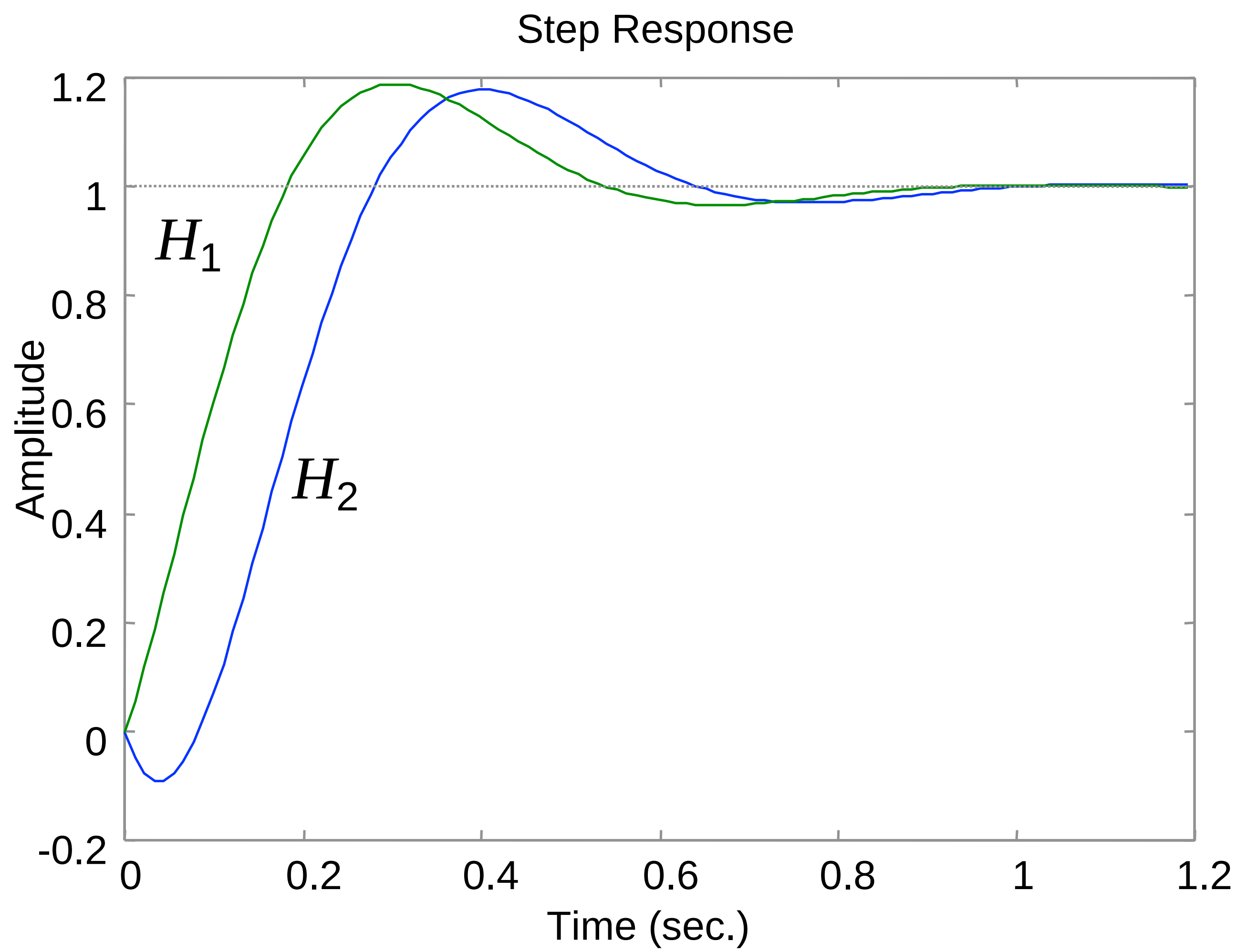
Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior

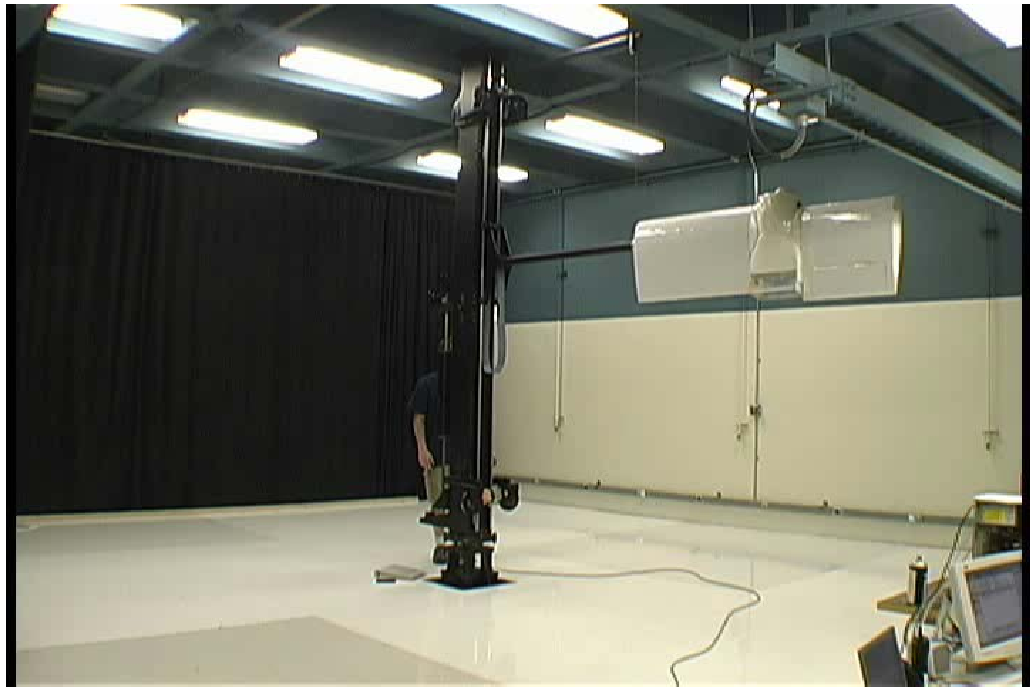
- Phase vs. frequency has additional lag (not “minimum”) for a given magnitude
- Can cause output to move opposite from input for a short period of time

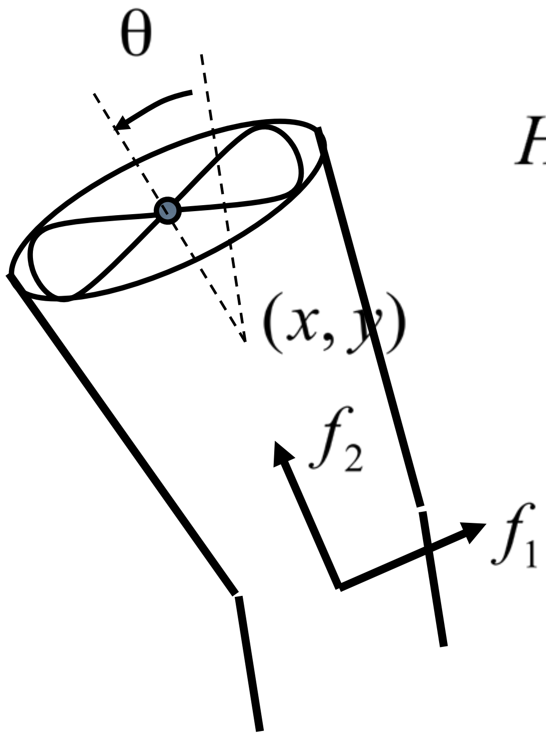
Example: $H_1(s) = \dfrac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ **vs** $H_2(s) = \dfrac{s-a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$





Example: Lateral Control of the Ducted Fan



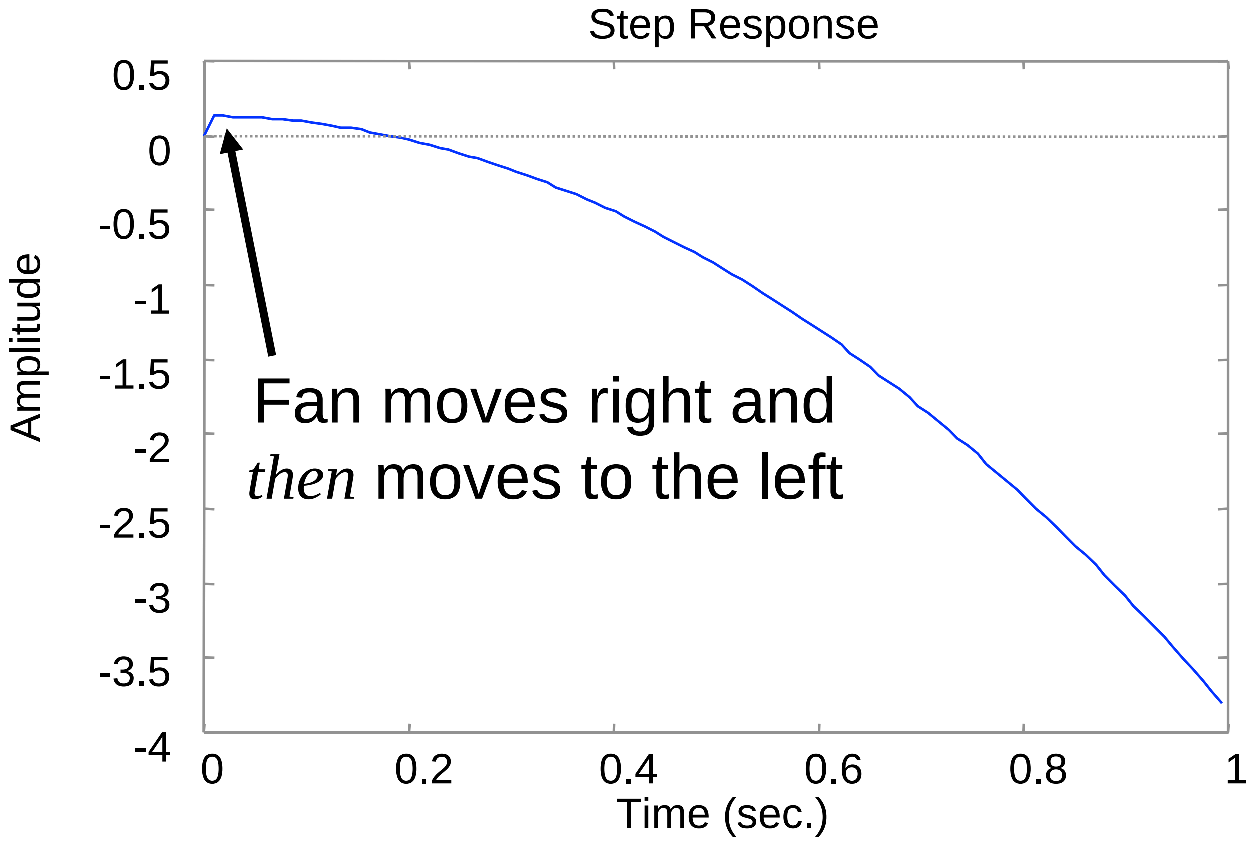


$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: $0, 0, -\sigma \pm i\,\omega_d$
- Zeros: $\pm\sqrt{mgl}$

Source of non-minimum phase behavior

- To move left, need to make θ > 0
- To generate positive θ, need $f_1 > 0$
- Positive f_1 causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)



Stability in the Presence of (RHP) Zeros

Loop gain limitations

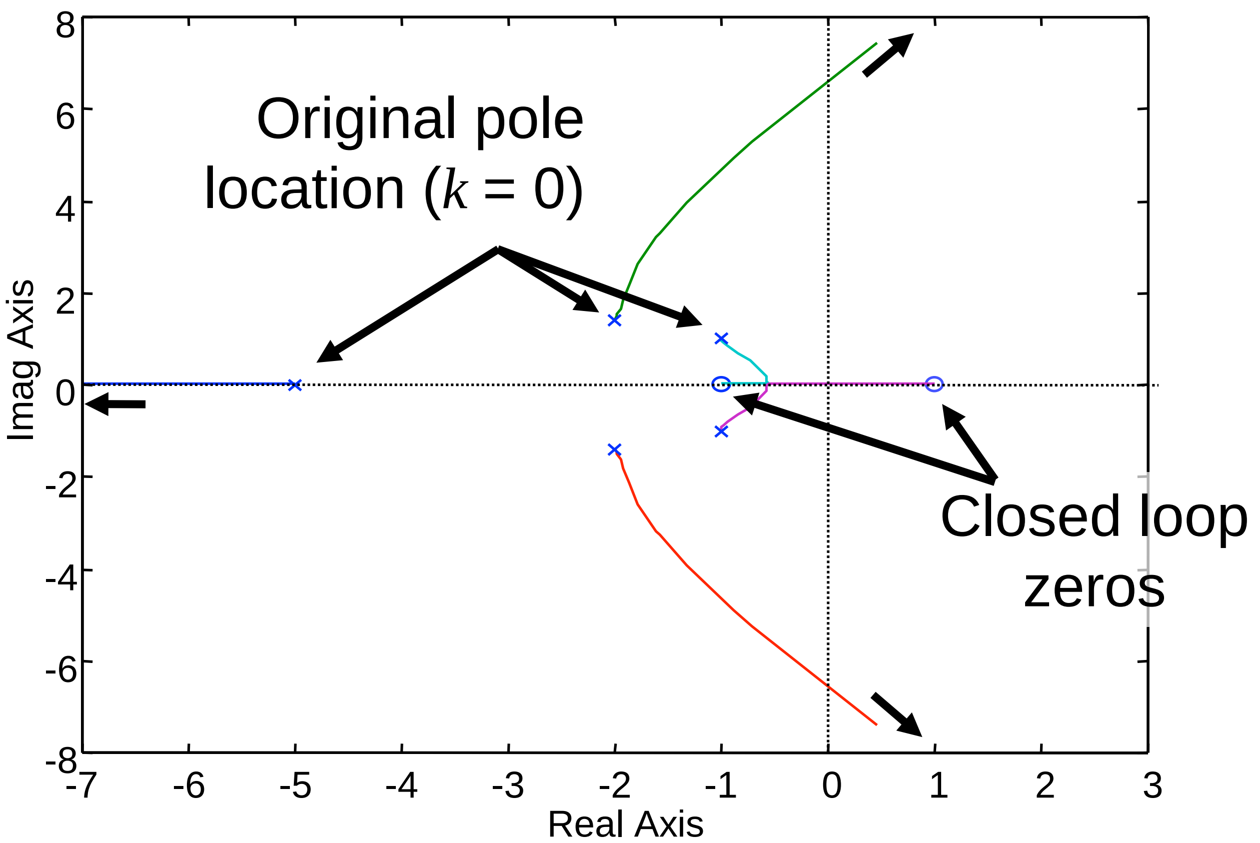
- Poles of closed loop = poles of 1 + L. Suppose $C(s) = k\, n_c(s)/d_c(s)$, where k is the controller gain

$$1 + L = 1 + k\frac{n_c n_p}{d_c d_p} = \frac{d_c d_p + k n_c n_p}{d_c d_p}$$

- For large k, closed loop poles approach open loop zeros
- RHP zeros limit maximum gain ⇒ serious design constraint!

Root locus interpretation

- Plot location of eigenvalues as a function of the loop gain k
- Can show that closed loop poles go from open loop poles (k = 0) to open loop zeros ($k = \infty$)



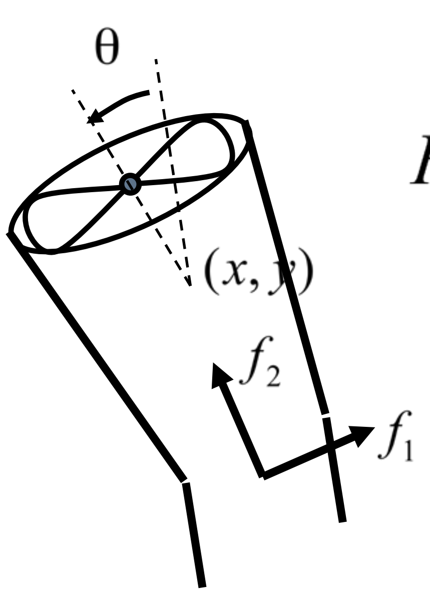
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of H_{er} over frequency range:

$$M_1 = \max_{\omega_1 \le \omega \le \omega_2} |H_{er}(j\omega)| \qquad M_2 = \max_{0 \le \omega \le \infty} |H_{er}(j\omega)|$$

Theorem: Suppose that $P(s)$ has a RHP zero at z. Then there exist constants c_1 and c_2 (depending on ω_1, ω_2, z) such that $c_1 \log M_1 + c_2 M_2 \geq 0$.

- M_1 typically << 1 ⇒ M_2 must be larger than 1 (since sum is positive)
- If we increase performance in active range (make M_1 and H_{er} smaller), we must lose performance (H_{er} increases) some place else
- Note that this affects peaks not integrals (different from RHP poles)



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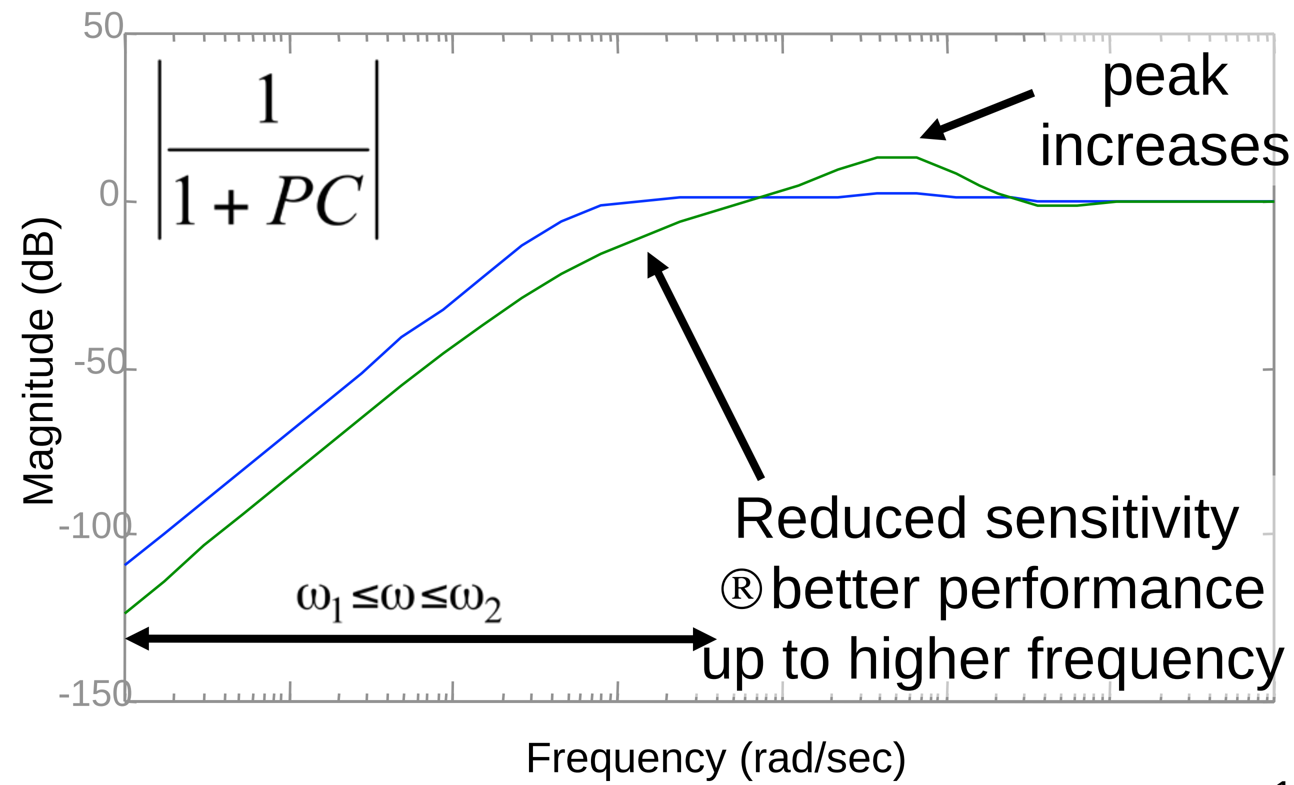


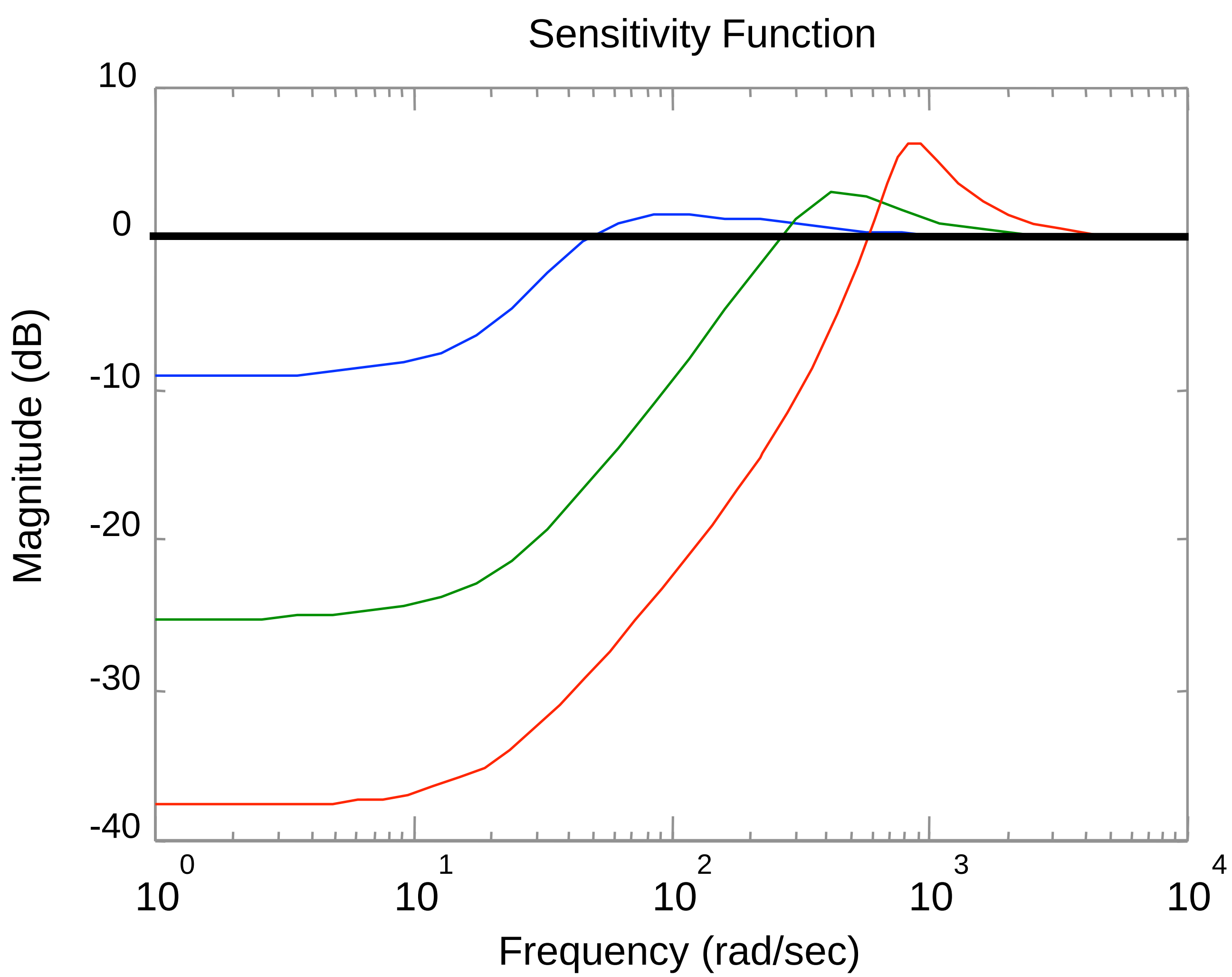
Summary: Limits of Performance

Many limits to performance

- Algebraic: S + T = 1
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of S

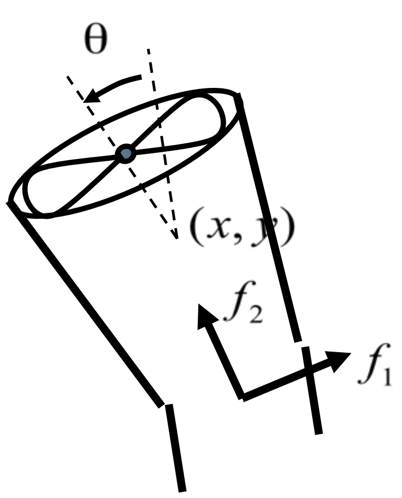
Main message: try to avoid RHP poles and zeros when-ever possible (eg, re-design)

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \mathrm{Re}\, p_k$$



$$\int_0^\infty \frac{\log |T(i\omega)|}{\omega^2} d\omega = \pi \sum \frac{1}{z_i}$$

RHP poles



Announcements

Homework #8 is due on Friday, 2 pm

- In class or HW slot (102 STL)

Final exam

- Out on 5 Dec (Mon.)
- Due on Fri. December 9, by 5 pm:
  - turn in to Sonya Lincoln
  - 250 Gates-Thomas
- Final exam review: December 2 from 2-3 pm, 105 Annenberg
- Office hours during study period
  - 5 Dec (Mon), 3-5 pm
  - 6 Dec (Tue), 3-5 pm