Goals:
- Review:
  - canonical control design problem
  - performance measures
- Loop Transfer Functions (Gang of Four, Seven)
- Show how to use “loop shaping” to achieve a performance specification
- Work through example(s)

Reading:
- Åström and Murray, Feedback Systems 2-e, Section 12.1-12.4
Design Patterns for Control Systems

“Classical” control (1950s...)

- **Goal:** output $y(t)$ should track reference trajectory $r(t)$
- Design typically done in “frequency domain” (second half of CDS 101/110)

- Reference input shaping
- Feedback on output error
- Compensator dynamics shape closed loop response

- Uncertainty in process dynamics $P(s) +$ external disturbances ($d$) & noise ($n$)

Modern” (state space) control (1970s...)

- Assume dynamics are given by linear system, with known $A$, $B$, $C$, $D$ matrices
- Measure the state of the system and use this to modify the input
- **Goal unchanged:** output $y(t)$ should track reference trajectory $r(t)$ [often constant]

$$u = -Kx + k_r r$$
Input/Output Control Design Specifications

Common Sense Goals:
• Keep error small for reference signals $r$
• Attenuate effect of sensor noise $n$ and load disturbances $d$
• Avoid large input commands $u$

Designs represent tradeoffs, e.g.:
• Keep $L = PC$ large for good performance ($G_{er} << 1$)
• Keep $L = PC$ small for good noise rejection ($G_{nn} < 1$)

• Load disturbances ($d$) typically low frequency, while noise ($v$) is high frequency
• Stability always determined by $1/(1+PC)$ assuming stable process & controller
• Numerator determined by forward path between input and output

More generally: 7 primary transfer functions; simultaneous design of each
• Controller $C(s)$ enters in multiple places ⇒ possibly hard to understand tradeoffs
General Loop Transfer Functions

- \( r \) = reference input
- \( e \) = error
- \( u \) = control
- \( v \) = control + disturbance
- \( \eta \) = true output (\textit{what we want to control}!)
- \( y \) = measured output

System “outputs”

\[
\begin{pmatrix}
y \\
\eta \\
v \\
u \\
e
\end{pmatrix} = \begin{pmatrix}
F(s) & P & 1 & 1+PC & 1+PC \\
1+PC & 1+PC & 1+PC & 1 & 1+PC \\
1+PC & 1+PC & 1 & 1+PC & 1+PC \\
1+PC & CF & 1 & 1+PC & 1+PC \\
1+PC & CF & 1+PC & 1+PC & 1+PC \\
1+PC & F & 1+PC & 1+PC & 1+PC
\end{pmatrix} \begin{pmatrix}
r \\
d \\
n
\end{pmatrix}
\]

System “inputs”

- TF = \( \frac{PCF}{1+PC} \)
- CFS = \( \frac{CF}{1+PC} \)
- T = \( \frac{PC}{1+PC} \)
- CS = \( \frac{C}{1+PC} \)
- PS = \( \frac{P}{1+PC} \)
- S = \( \frac{1}{1+PC} \)

“Gang of Six”

- Response of \((y, u)\) to \(r\)
- Response of \(u\) to \((d, n)\)
- Response of \(y\) to \((d, n)\)

“Gang of Seven”

- \( F(s) \) = Feed forward
- \( C(s) \) = Controller
- \( P(s) \) = Process
- \( \eta \) = true output (\textit{what we want to control}!)
- \( y \) = measured output

- \( r \) = reference input
- \( e \) = error
- \( u \) = control
- \( v \) = control + disturbance
- \( \eta \) = true output (\textit{what we want to control}!)
- \( y \) = measured output
Key Loop Transfer Functions

\[ F(s) = 1: \text{Four unique transfer functions define performance ("Gang of Four")} \]

**Sensitivity: Function**

\[ G_{er} = S(s) = \frac{1}{1 + L(s)} \]

**Complementary Sensitivity Function:**

\[ G_{yr} = T(s) = \frac{L(s)}{1 + L(s)} \]

**Load Sensitivity Function:**

\[ G_{yd} = PS(s) = \frac{P(s)}{1 + L(s)} \]

**Noise Sensitivity Function:**

\[ G_{yn} = CS(s) = \frac{C(s)}{1 + L(s)} \]

\[ L(s) = P(s)C(s) \]

"Gang of Four" (the "sensitivity" functions)

Characterize most performance criteria of interest
Specifications on the open loop transfer function \((L)\)
- **Gain crossover frequency**, \(\omega_{gc}\): the lowest frequency at which loop gain = 1
- **Gain margin**, \(g_m\): amount the loop gain can be increased before instability
- **Phase margin**, \(\varphi_m\): amount of phase lag required to generate instability

Specifications on closed loop frequency response (e.g. \(G_{yr}\), \(G_{yd}\), etc)
- Resonant peak, \(M_r\), is the largest value of the frequency response
- Peak frequency, \(\omega_p\), is the frequency where the maximum occurs
- Bandwidth, \(\omega_b\), is the frequency where the gain has decreased to \(1/\sqrt{2}\)

**Basic idea:** convert specs on closed loop to specs on open loop
- Bandwidth \(\approx\) value for which \(|L| = 1\)
- Resonant peak set by phase margin
- Keep \(L(s)\) large to set \(H_{yr} \approx 1\)

\[
H_{yr} = \frac{L}{1 + L} \quad H_{er} = \frac{1}{1 + L}
\]
Time domain specifications

Map to frequency domain for second order system

\[ L(s) = \frac{k}{s^2 + bs} \quad H_{yr} = \frac{k}{s^2 + bs + k} \]

- Use properties of second order systems (Ch 7)
**Sensitivity**

**Note:**

\[ H_{yd} = \frac{P(s)}{1 + L(s)} = P(s) \frac{1}{1 + L(s)} = P(s)S(s) \]

- I.e., Closed Loop Response to Disturbance = Open Loop Response x Sensitivity
- I.e., \( S(s) \) tells us how variations in output are influenced by feedback
  - Disturbances with \(|S(i\omega)| < 1\) attenuated by feedback
  - Disturbances with \(|S(i\omega)| > 1\) amplified
- Max Sensitivity = \( \max |S(i\omega)| := M_s \)
  - \( M_s = 1/s_m \),
  - \( s_m \) is stability margin (from Nyquist)
  - Related to robustness

\[ L(s) = \frac{20}{(s + 1)(s/10 + 1)(s/100 + 1)} \]
Example plotted is:

\[ L(s) = \frac{20}{(s + 1)(s/10 + 1)(s/100 + 1)} \] 

\[ S(s) = \frac{1}{1 + L(s)} \]

- \(|S(i\omega)| > 1\) Inside unit circle centered at -1 (disturbances amplified)
- \(|S(i\omega)| < 1\) outside unit circle centered at -1 (disturbances attenuated)

Fig. 4  Nyquist plot of the loop gain \( G(s) = P(s)C(s) \) for the system (29). For frequencies for which \( G(i\omega) \) enters the unit circle centered about the \(-1\) point, disturbances are amplified and, for frequencies for which \( G(s) \) lies outside this circle, disturbances are attenuated relative to open-loop.

(*) From Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5
“Loop Shaping”: Design Loop Transfer Function

Translate specs to “loop shape”

\[ L(s) = P(s)C(s) \]

Naïve Idea: let \( L(s) \) have desired properties, then \( C(s) = \frac{L(s)}{P(s)} \)

Design \( C(s) \) to obey constraints

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Loop shaping is trial and error
Culture, not Control

Gang of Four at trial, 1981.

Yao Wenyuan  Jiang Qing  Zhang Chunqiao  Wang Hongwen

Gang of Four
To Hell With Poverty!
Aside: Relation of Gain to Phase in Bode Plot

If no poles/zeros in RHP (a minimum phase system), then (see A&M 10.4) phase curve uniquely related to gain curve

$$\arg G(i\omega_0) = \frac{\pi}{2} \int_{0}^{\infty} f(\omega) \frac{d \log|G(i\omega)|}{d \log \omega} d \log \omega$$

$$\approx \frac{\pi}{2} \frac{d \log|G(i\omega_0)|}{d \log \omega_0}$$

Where

$$f(\omega) = \frac{2}{\pi} \log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right|$$

Consequence:

• Shape of phase curve is dictated by shape of gain curve
• Phase curve is weighted average of derivative of the gain curve
• Where gain curve is ~constant slope, phase curve is constant
Loop Shaping: Basic Approach

Disturbance rejection

\[ H_{ed} = \frac{-P}{1 + L} \]

- Would like \( H_{ed} \) to be small make \( \Rightarrow \) large \( L(s) \)
- Typically require this in low frequency range

High frequency measurement noise

\[ H_{un} = \frac{-L}{P(1 + L)} \]

- Want to make sure that \( H_{un} \) is small (avoid amplifying noise) \( \Rightarrow \) small \( L(s) \)
- Typically generates constraints in high frequency range

Robustness: gain and phase margin

- Focus on gain crossover region: make sure the slope is “gentle” at gain crossover
- Fundamental tradeoff: transition from high gain to low gain through crossover
Design Method #1: Process Inversion

Simple trick: invert out process
- Write performance specs in terms of desired loop transfer function
- Choose \( L(s) \) to satisfy specifications
- Choose controller by *inverting* \( P(s) \)

\[
C(s) = \frac{L(s)}{P(s)}
\]

Pros
- Simple design process
- \( L(s) = k/s \) often works very well
- Can be used as a first cut, with additional tuning

Cons
- High order controllers (at least same order as plant)
- Requires “perfect” process model (due to inversion)
- Can generate non-proper controllers (order(num) > order(den))
  - Difficult to implement, plus amplifies noise at high frequency (\( C(\infty) = \infty \))
  - Fix by adding high frequency poles to roll off control response at high frequency
- Does not work if right half plane poles or zeros (*internal instability*)
**Lead & Lag Compensators**

**Lead:**  $K > 0$, $a < b$
- Add phase near crossover
- Improve gain & phase margins, increase bandwidth (better transient response).

**Lag:**  $K > 0$, $a > b$
- Add gain in low frequencies
- Improves steady state error

**Lead/Lag:**
- Better transient and steady state response

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**Graphs:**
- **Frequency Response**
  - Lead compensation, $a < b$ (a)
  - Lag compensation, $b > a$ (b)

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**Diagram:**
- Control system with lead and lag compensators.
Design Method #2: Add Lead, Lag, Lead/Lag compensation

Lead: increases phase in frequency band
- Effect: lifts phase by increasing gain at high frequency
- Increases PM
- Bode: add phase between zero and pole
- Nyquist: increase phase margin

\[
K \frac{s + a}{s + b} \quad a < b \quad K > 0
\]
System description

\[ P(s) = \frac{p_1 p_2}{(s + p_1)(s + p_2)} \]

- Poles: \( p_1 = 1, \ p_2 = 5 \)

Control specs

- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%
  - Gives PM of ~60 deg

Try a lead compensator

\[ C(s) = K \frac{s + a}{s + b} \]

- Want gain cross over at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error \( \Rightarrow |L(0)| > 100 \)
- \( a = 20, \ b = 500, \ K = 10,000 \) (gives \( |C(0)| = |L(0)| = 400 \))
**Better Loop Shaping Design Process**

**A Process:** sequence of (nonunique) steps

1. Start with plant and performance specifications
2. If plant not stable, first stabilize it (e.g., PID)
3. Adjust/increase simple gains
   - Increase proportional gain for tracking error
   - Introduce integral term for steady-state error
   - Will derivative term improve overshoot?
4. Analyze/adjust for stability and/or phase margin
   - Adjust gains for margin
   - Introduce *Lead or Lag Compensators* to adjust phase margin at crossover and other critical frequencies
   - Consider PID if you haven’t already
Summary: Loop Shaping

Loop Shaping for Stability & Performance

- Steady state error, bandwidth, tracking response
- Specs can be on any input/output response pair

Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI

Things to remember (for homework and exams)

- Always plot Nyquist to verify stability/robustness
- Check gang of 4 to make sure that noise and disturbance responses also look OK