CDS 101/110: Lecture 5.2
Observability & State Estimation

October 28, 2016

Goals:

• Review Observability and Observers.
• Complete and “polish” the analysis of combined feedback and observation.
• A few thoughts on observer design.
• Brief mid-term review

Reading:

• Åström and Murray, Feedback Systems-2e, Section 8.1-8.3
## Observability

**System:** \( \dot{x} = Ax + Bu; \quad y = Cx + Du \quad (*) \)

- **Definition:** The linear system (*) is said to be **Observable** if for every \( T>0 \) it is possible to determine the system state \( x(T) \) through measurements \( y(t) \) and knowledge of \( u(t) \) on the interval \( [0,T] \).

  - Note: some texts/papers are slightly different: Observable if \( x(t = 0) \) can be determined from measurements and inputs.

  - If (*) is observable, then there are no “hidden” internal states. This is a practical issue in system design—do you have the right sensors?

### Testing for Observability:

- The Matrix, \( W_0 \) must be full rank

\[
W_0 \equiv \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]
Observable Canonical Form

System: \( \dot{x} = Ax + Bu; \ y = Cx + Du \quad (*) \)

- **Definition:** The linear system \((*)\) is said to be in **Observable Canonical Form (OCF)** if

\[
\dot{x} = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{n-1} & 0 & 0 & \cdots & 1 \\
-a_n & 0 & 0 & \cdots & 0 \\
\end{bmatrix} x + \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
b_n \\
\end{bmatrix} u
\]

\[
y = [1 \ 0 \ 0 \ \cdots \ 0] x + d_0 u
\]

Where the characteristic polynomial of \(A\) is: \(\lambda_A(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0\)

- When the system \((*)\) is in OCF, the controllability matrix takes the form:

\[
\tilde{W}_o = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-a_1 & 1 & 0 & \cdots & 0 \\
-a_1 - a_2 & -a_1 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
* & \ddots & \ddots & \ddots & 1
\end{bmatrix}; \quad \tilde{W}_o^{-1} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-a_1 & 1 & 0 & \cdots & 0 \\
-a_1^2 - a_2 & -a_1 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
* & \ddots & \ddots & \ddots & 1
\end{bmatrix}
\]
State Estimation/Observer

**State Estimator:** \( \dot{x} = A\hat{x} + Bu + L(y - C\hat{x}) \)  

- The term \( L(y - C\hat{x}) \) provides “feedback” to the estimation process.
- **Analysis:** Let \( \tilde{x} = x - \hat{x} \) denote the error in the state estimate. Then
  \[
  \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + L(y - C\hat{x})] \\
  = A(x - \hat{x}) + LC(x - \hat{x}) = (A - LC)\tilde{x}
  \]

Hence, convergence of the estimation error is governed by the eigenvalues of \((A - LC)\).

- Dual to previous reachability analysis. “Design” = eigenvalues of \((A - LC)\).
- Place poles of \((A^T - C^T L^T)\). MATLAB: `place(A^T, C^T, eigenvalues)` gives \(L^T\)

**Theorem:** If \((A, C)\) is observable, then the poles of \((A - LC)\) can be set arbitrarily.

**Design:** Specify the desired poles of \((A - LC)\) by
\[
\lambda_{A-LC}(s) = s^n + p_1s^{n-1} + \cdots + p_{n-1}s + p_n = 0
\]

Then gain matrix is found as: \( L = W_0^{-1}\hat{W}_0 \begin{bmatrix} p_1 - a_1 \\ \vdots \\ p_n - a_n \end{bmatrix} \)
Feedback of Estimated State

Feedback the estimated state: \( u = -K \hat{x} + k_r r \)

- **Analysis:** Again, let \( \tilde{x} = x - \hat{x} \) denote the error in the state estimate. The dynamics of the controlled system under this feedback are:

\[
\dot{x} = Ax + Bu = Ax - BK \hat{x} - Bk_r r = Ax - BK (x - \tilde{x}) + Bk_r r
\]
\[
= (A - BK)x + BK \tilde{x} + Bk_r r
\]

- Introduce a new *augmented* state: \( q = [x \quad \tilde{x}]^T \). The dynamics of the system defined by this state is:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\tilde{x}}
\end{bmatrix} =
\begin{bmatrix}
(A - BK) & BK \\
0 & (A - LC)
\end{bmatrix}
\begin{bmatrix}
x \\
\tilde{x}
\end{bmatrix} +
\begin{bmatrix}
Bk_r \\
0
\end{bmatrix} r 
≡ Mq + B_M r
\]

The characteristic polynomial of \( M \) is:

\[
\lambda_M(s) = \det(sI - A + BK) \det(sI - A + LC)
\]

- If the system is *observable* and *reachable*, then the poles of \( (A - BK) \) and \( (A - LC) \) can be set *arbitrarily* and *independently*.
Feedback of Estimated State

Remarks:

• The controller is a dynamical system with internal state dynamics (the observer).

• *Separation principle*: The controller and observer can be designed (eigenvalues assigned) separately/independently.

• *Internal Model principle*: the control system includes and *internal model* of the system being controlled.
Reachability

For LTI system \( \dot{x} = Ax + Bu, \ y = Cx + Du, \) reachability assessed by rank of:

\[
W_r = [B \ AB \ \ldots \ \ A^{n-1}B]
\]

Definitions: recall \( x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) \, d\tau \)

- **Controllable** if state can be driven to \( x(T) = 0 \) for any \( x(0) \)
  - i.e., \( \exists u(t) \) s.t. \(-x(0) = e^{-AT} \int_0^T e^{A(T-\tau)}Bu(\tau) \, d\tau = \int_0^T e^{-\tau}Bu(\tau) \, d\tau \)

- **Reachable** if \( x(0) = 0 \) can be driven to any state \( x_f = x(T) \) in time \( T \)
  - i.e. \( \exists u(t) \) s.t. \( x(T) = \int_0^T e^{A(T-\tau)}Bu(\tau) \, d\tau \)

**General Principle:** Linear independence of \( N \) functions \( l_i(t), i = 1, \ldots, N \) over interval \([t_o, t_f]\) is determined using a Gramian:

\[
G = [G_{ij}], \quad G_{ij} = \int_{t_0}^{t_f} l_i(\tau)l_j(\tau) \, d\tau
\]

Linear independence is proven when \( G \) has full rank
Controllability

Controllability Gramian:

\[ C(t_0, t_1) = \int_{t_0}^{t_f} e^{A(t_0-\tau)}BB^T e^{A^T(t_0-\tau)} d\tau \quad \rightarrow \quad C(0, t_f) = \int_{0}^{t_f} e^{-A\tau}BB^T e^{-A^T\tau} d\tau \]

Since \( C(0, t_f) \) is symmetric, for it to be full rank over \([0, t_f]\), it must be positive definite.

**Lemma:** \( C(0, t_f) \) is positive definite if and only if there is no vector \( v \neq 0 \) such that

\[ v^T e^{-At}B = 0 \quad \forall t \epsilon [0, t_f] \]

**Proof** (by contradiction): suppose there is such a \( v \) with \( v^T e^{-At}B = 0 \quad \forall t \epsilon [0, t_f] \)

- \( v^T C(0, t_f)v = \int_{0}^{t_f} v^T e^{-A\tau}BB^T e^{-A^T\tau}Bv \ d\tau \)
- If there is such a \( v \), then \( v^T C(0, t_f)v = 0 \), which implies that \( C(0, t_f)v \) is not positive definite.

**Theorem:** The pair \((A,B)\) is controllable if and only if the \( C(0, t_f) \) is positive definite

**Proof** (sufficiency): suppose \( C(0, t_f) \) is positive definite. Let \( x_0, x_f \) be the initial/final states

- \( x(t_f) = e^{At_f}x_0 + \int_{0}^{t_f} e^{-A(t_f-\tau)}B u(\tau) \ d\tau \)
Controllability

**Proof** (sufficiency): *(continued)*

- Choose \( u(t) = B^T e^{-A^T t} C^{-1}(0, t_f) v \) for some constant vector \( v \)
- Then: \( x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B B^T e^{-A^T \tau} C^{-1}(0, t_f) v \, d\tau \)
  \[ = e^{At_f} x_0 + e^{At_f} C(0, t_f) C^{-1}(0, t_f) v \]
  \[ = e^{At_f} (x_0 + v) \]
- If \( v = -x_0 + e^{-At_f} x_f \), then \( x(t_f) = x_f \)

That is, \( u(t) = B^T e^{-A^T t} C^{-1}(0, t_f) [e^{-At_f} x_f - x_0] \) steers \( x_0 \) to \( x_f \) for any \( x_0, x_f \)

**Proof** (necessity): show that positive definiteness of \( C(0, t_f) \) is necessary

- Contradiction: suppose \( C(0, t_f) \) is not positive definite.
- Then there exists \( z \neq 0 \) such that \( z^T e^{-At_f} B = 0 \) \( \forall t \in [0, t_f] \)
- For *controllability*, let \( x_0 = z \). Suppose that \( x(t_f) = 0 \)
  - Then: \( 0 = e^{At_f} z + \int_0^{t_f} e^{A(t_f-\tau)} B u(\tau) \, d\tau \)
  - Multiply by \( z^T e^{-At_f} : 0 = z^T z + \int_0^{t_f} z^T e^{A\tau} B u(\tau) \, d\tau \)
  - But integrand is zero for all \( t \), and thus \( z = 0 \), a contradiction
Controllability/Reachability

Proof (necessity): (continued)

• For reachability, let \( x_f = e^{At_f}z \), and suppose \( u(t) \) steers \( x_0 \) to \( x(t_f) = x_f \)
  
  • Then: \( e^{At_f}z = \int_0^{t_f} e^{A(t_f-\tau)}B \ u(\tau) \ d\tau \)

  • Multiply by \( z^T e^{-At_f} \):
    \[
    z^T e^{-At_f} e^{At_f}z = \int_0^{t_f} z^T e^{-A\tau}B \ u(\tau)d\tau = z^T z
    \]

  • But, if \( C(0, t_f) \) is not positive definite, then there exists \( z \) such that
    \( z^T e^{-At_f}B = 0 \ \forall t \in [0, t_f] \), implying that \( z = 0 \), which is a contradiction.

Theorem: \( C(0, t_f) \) is positive definite only if \( \text{rank}(W_r) = n \), where

\[
W_r = [B \ AB \ldots \ A^{n-1}B]
\]

Proof: If \( C(0, t_f) \) is not positive definite, there exists \( z \neq 0 \) s. t. \( z^T e^{-At_f}B = 0, \forall t \in [0, t_f] \)

• \[
    z^T \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A^k B = 0, \forall t \in [0, t_f]
    \]

• Same as \[
    \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} z^T A^k B = 0, \forall t \in [0, t_f]
    \]

• This implies that there exists \( z \) such that \( z^T A^k B = 0 \) for all \( k = 0, 1, \ldots \)
Proof: (continued)

• \( \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} z^T A^k B = 0, \forall t \in [0, t_f] \) implies via Cayley-Hamilton that \( z^T A^k B = 0 \) for \( k = 0, \ldots, n-1 \)

• Hence, \( z^T [B \ AB \ A^2 B \cdots A^{n-1} B] = 0 \), which implies that \( W_r \) is not full rank.

• Therefore, \((A,B)\) is reachable (controllable) only if \( W_r \) is full rank

Note: in LTI case, reachability is independent of time.

Observability Gramian:

\[
O(0, t_f) = \int_0^{t_f} e^{-A^T \tau} C^T C e^{-A \tau} d\tau
\]

A nearly identical analysis shows that the \( O \) must be positive definite for observability, which in turn implies that the observability matrix \( W_O \) must be full rank.
Mid Term

Schedule: (1) Handed out in Class on Monday. (2) Due Friday at 5:00 pm.

Instructions on Front Page. Three hour limited time take-home.

Review:

• Convert control system description to 1\textsuperscript{st} order form
• Solution and characterization of o.d.e.s
  • Matrix exponential, equilibria, stability of equilibria, phase space
• Lyapunov Function and stability
• System linearization, and stability/stabilization of linearized models.
• Convolution Integral, impulse response
• Performance characterization for 1\textsuperscript{st} and 2\textsuperscript{nd} order systems:
  • Step response overshoot, rise time, settling time
• System Frequency Response
• Discrete Time System
• State Feedback, eigenvalue placement
• Reachability, reachable canonical form, test for reachability