Problem 1

Consider the system whose dynamics are given by:

\[ \frac{dx}{\tau} = -x + u \]
\[ y = x \]

We know the response is \( y(t) = CA^{-1}e^{At}B + D - CA^{-1}B \) where \( C = 1, B = \frac{1}{\tau}, A = -\frac{1}{\tau} \).

Plugging in and simplifying, we get \( y(t) = 1 - e^{\frac{t}{\tau}} \)

Time to get to 0.1 \( y_{ss} \rightarrow 0.1 \) \( \rightarrow t = 0.105 \tau \)

Time to get to 0.9 \( y_{ss} \rightarrow 0.9 \) \( \rightarrow t = 2.3 \tau \).

Rise time \( t_r = (2.3 - 0.105)\tau = 2.2\tau \approx 2\tau \).

1% Settling time: \( \rightarrow 0.99 = 1 - e^{\frac{t}{\tau}} \) \( \rightarrow t = 4.6\tau \)

2% Settling time: \( \rightarrow 0.98 = 1 - e^{\frac{t}{\tau}} \) \( \rightarrow t = 3.91\tau \approx 4\tau \)

5% Settling time: \( \rightarrow 0.95 = 1 - e^{\frac{t}{\tau}} \) \( \rightarrow t = 3.0\tau \)

Problem 2

Consider the system

\[ \ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = w_0^2 u \]

Part (a): Convert the dynamic system to first order form

Denote \( x_1 = x, \ x_2 = \dot{x} \).

\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_0^2 & -2\zeta w_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ w_0^2 \end{bmatrix} u \]
Part (b): Determine and plot the impulse response of this system for the case $C = [1 \ 0]$

Response $h(t) = C e^{At} B$

$A = VDV^{-1}$ where

$$V = \begin{bmatrix} \frac{-(\zeta + \sqrt{\zeta^2 - 1})}{w_0} & \frac{-(\zeta - \sqrt{\zeta^2 - 1})}{w_0} \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -w_0(\zeta - \sqrt{\zeta^2 - 1}) & 0 \\ 0 & -w_0(\zeta + \sqrt{\zeta^2 - 1}) \end{bmatrix}$$

$h(t) = C e^{At} B = CV e^{Dt} V^{-1} B$

$$h(t) = \begin{bmatrix} \frac{-(\zeta + \sqrt{\zeta^2 - 1})}{w_0} & \frac{-(\zeta - \sqrt{\zeta^2 - 1})}{w_0} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \exp(-w_0(\zeta - \sqrt{\zeta^2 - 1})) & 0 \\ 0 & \exp(-w_0(\zeta + \sqrt{\zeta^2 - 1})) \end{bmatrix} \begin{bmatrix} \frac{-w_0(\zeta - \sqrt{\zeta^2 - 1})}{2\sqrt{\zeta^2 - 1}} \\ \frac{-w_0(\zeta + \sqrt{\zeta^2 - 1})}{2\sqrt{\zeta^2 - 1}} \end{bmatrix}$$

The impulse response for $w_0 = 1$, $\zeta = 0.5$ is shown below (with code for finding the impulse response and generating the plot).

```matlab
%Find Impulse Response
syms w0 zed t
A = [0 1; -w0^2 -2*zed*w0];
B = [0; w0^2];
C = [1 0];
[V, D] = eig(A);
%Get the eigendecomposition
V = simplify(V);
D = simplify(D);
y_unit = simplify(C*A^-1*V*expm(D*t)*V^-1*B - C*A^-1*B);
%Pick w_0=1 and zed=0.5 and plot impulse response
```

![Impulse Response](image)
\[ w_0 = 1; \]
\[ \zeta = 0.5; \]
\[ A = [0 1; -w_0^2 -2*\zeta w_0]; \]
\[ B = [0; w_0^2]; \]
\[ C = [1 0]; \]
\[ \text{time} = 1500; \]
\[ t = \text{zeros}(1, \text{time}); \]
\[ \text{response} = \text{zeros}(1, \text{time}); \]
\[ \text{hline} = \text{zeros}(1, \text{time}); \]
\[ \text{for} \ i = 1: \text{time} \]
\[ \quad t(i) = i*0.01; \]
\[ \quad \text{response}(i) = C*\expm(A*t(i))*B; \]
\[ \text{end} \]
\[ \text{plot}(t, \text{response}) \]
\[ \text{hold on} \]
\[ \text{plot}(t, \text{hline}, \text{'k--'}) \]
\[ \text{hold off} \]

**Part (c):** Find the response of this system to a unit step input, assuming that \( x(0) = 0, \dot{x}(0) = 0 \).

Response \( h(t) = CA^{-1}e^{At}B - CA^{-1}B \)
\[
    h(t) = 1 - e^{-\zeta w_0 t} \cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} \sin(\omega_d t)
\]
where \( \omega_d = w_0 \sqrt{\zeta^2 - 1} \).
See Lectures notes from 10/21/2016 (Slides 5-8) for detailed derivation.

**Part (d):** Determine the time until the first peak in response. Knowing this time, derive an expression for the peak overshoot.

\[
    t_{\text{peak}} = \frac{\pi}{w_0 \sqrt{1-\zeta^2}}
\]
\[
    y_{\text{peak}} = 1 - \frac{\exp(\pi \zeta / \sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}} \sin(\pi + \theta)
\]
where \( \theta = \cos^{-1}(\zeta) \).
See Lectures notes from 10/21/2016 (Slides 5-8) for detailed derivation.

**Part (e):** Estimate the rise time, which is the time it takes from the onset of the step input until the time that the response first reaches a magnitude of one (the amplitude of the step input).

The step response can be written:
\[
    y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} \sin(\omega_d t + \zeta)
\]
When \( y = 1 \), then \( \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_0 t} \sin(w_d t + \theta) = 0 \)
Thus \( \sin(w_d t + \theta) = \sin(w_0 \sqrt{\zeta^2 - 1} t + \cos^{-1} \zeta) = 0 \)
\( t_r = \frac{\pi - \cos^{-1} \zeta}{w_0 \sqrt{\zeta^2 - 1}} \)

**Problem 3**

Show that \( P = \int_0^\infty e^{A^T \tau} Q e^{A \tau} d\tau \) defines a Lyapunov function of the form \( V = x^T Px \) given that \( Q \) is positive definite.
For fixed \( x \), \( \dot{V} = x^T P x + x^T P \dot{x} = x^T (A^T P + PA) x \)
\( A^T P + PA = \int_0^\infty (A^T e^{A \tau} x Q e^{A \tau} + e^{A \tau} x Q e^{A \tau} A) d\tau = \int_0^\infty \frac{d}{d\tau} (e^{A \tau} x Q e^{A \tau}) d\tau = -Q \)
Therefore \( \dot{V} = x^T (A^T P + PA) x = -x^T Q x \)
Since \( Q \) is positive definite, then \( \dot{V} < 0 \) and we can conclude that \( V = x^T P x \) defines a valid Lyapunov function.

**Problem 4**

Part (a):
\( x[k+1] = Ax[k] + Bu[k] \) \( y[k] = Cx[k] + Du[k] \)
Proof by induction.
Consider
\( x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} Bu[i] \)
At the initial point,
\( x[1] = Ax[0] + Bu[0] \)
Now,

\[ x[k+1] = A^{k+1}x[0] + \sum_{i=0}^{k} A^{k-i}Bu[i] \]

\[ = A(A^{k}x[0]) + \sum_{i=0}^{k-1} A^{k-i}Bu[i] + Bu[k] \]

\[ = A(A^{k}x[0]) + A \sum_{i=0}^{k-1} A^{k-1-i}Bu[i] + Bu[k] \]

\[ = Ax[k] + Bu[k] \]

So,

\[ x[k] = A^{k}x[0] + \sum_{i=0}^{k-1} A^{k-i-1}Bu[i] \]

Then,

\[ y[k] = Cx[k] + Du[k] = CA^{k}x[0] + \sum_{i=0}^{k-1} CA^{k-1-i}Bu[i] + Du[k] \]

**Part (b):**

For checking asymptotic stability, assume \( u = 0 \).

Then \( x[k] = A^{k}x[0] \).

Consider the eigendecomposition of \( A = VDV^{-1} \) where \( D \) is diagonal with eigenvalues on the diagonal and \( V \) is an orthonormal matrix (represents a change of basis).

\[ x[k] = A^{k}x[0] = VD^{k}V^{-1}x[0] \]

(if direction)

If eigenvalue of \( A \) has magnitude strictly less than 1, \( D^{k} \to 0 \) as \( k \to \infty \) because \( D \) is a diagonal matrix and its diagonal are the eigenvalues. Consequently, \( x[k] \to 0 \). Therefore, the system is asymptotically stable.

(only if direction)

If eigenvalue of \( A \) has magnitude equal to 1, \( D^{k} \to D = I \) as \( k \to \infty \). Consequently, \( x[k] = x[0] \) for all \( x[0] \neq 0 \).

If eigenvalue of \( A \) has magnitude greater than 1, \( D^{k} \to \infty \) as \( k \to \infty \). Consequently, \( x[k] \to \infty \) for all \( x[0] \neq 0 \).

Therefore, the systems are not asymptotically stable.

**Part (c):**

Consider the input \( u = e^{iwk} \)

\[ y[k] = CA^{k}x[0] + \sum_{j=0}^{k-1} CA^{k-j-1}Be^{iwj} + De^{iwk} \]
If we assume asymptotic stability, then $CA^kx[0] \rightarrow 0$.

$$y[k] = C \sum_{j=0}^{k-1} A^{k-j-1} B e^{iwj} + De^{iwk} = C \sum_{j=0}^{k-1} \frac{A^{k-j-1}}{e^{iw(k-j-1)}} (e^{iw(k-1)}) + De^{iwk}$$

$$y[k] = C(I - \frac{A}{e^{iw}})^{-1} (e^{iw})^{-1} e^{iwk} + De^{iwk} = [C(e^{iw}I - A)^{-1} + D]e^{iwk}$$

Thus the response to $e^{iwk}$ is:

$$y[k] = [C(e^{iw}I - A)^{-1} + D]e^{iwk}$$

Thus by linearity, the response to $\sin(wk) = e^{iwk} - e^{-iwk} 2i$ will be:

$$y[k] = \frac{[C(e^{iw}I - A)^{-1} + D]e^{iwk} - [C(e^{-iw}I - A)^{-1} + D]e^{-iwk}}{2i}$$

**Part (d):**

As in the continuous time case, we let $z = x - x_e$, $v = u - u_e$, and $w = y - h(x_e, u_e)$. Expanding the dynamics in a Taylor series, we have:

$$x[k+1] = f(x_e, u_e) + \frac{df}{dx}(x[k] - x_e) + \frac{df}{du}(u[k] - u_e) + h.o.t.$$  

$$y[k] = h(x_e, u_e) + \frac{dh}{dx}(x[k] - x_e) + \frac{dh}{du}(u[k] - u_e) + h.o.t.$$  

The resulting linearized system is obtained by assuming the higher order terms can be neglected and the dynamics become:

$$z[k+1] = Az[k] + Bv[k]$$  

$$w[k] = Cz[k] + Dv[k]$$

where

$$A = \left. \frac{df}{dx} \right|_{(x_e, u_e)}$$  
$$B = \left. \frac{df}{du} \right|_{(x_e, u_e)}$$  
$$C = \left. \frac{dh}{dx} \right|_{(x_e, u_e)}$$  
$$D = \left. \frac{dh}{du} \right|_{(x_e, u_e)}$$