1 Background

We have already looked at the problem of how to symbolically describe a portion (or “patch”) of a c-obstacle boundary corresponding to a EV contact between a planar polygonal robot, \( \mathcal{A} \), and a planar polygonal obstacle, \( \mathcal{O} \). The goal of this handout is to choose a parametrization of the robot and obstacle geometries which can then be used to derive a concrete formula that describes the boundary of a c-obstacle due to an EV contact.

2 Parametrization

Figure 1: Parametrization of polygonal obstacle and polygonal robot

Figure 1 describes a parametrization of the robot and an obstacle. Note that one must choose a fixed observing reference frame, whose basis vectors are subscripted by \( R \), and a reference frame fixed to the body of the moving robot, whose basis vectors are subscripted by \( A \). We choose a parametrization with the following variables

- \( \vec{r}_i \) is a vector from the origin of \( \mathcal{A} \)'s body fixed frame to the \( i^{th} \) vertex of \( \mathcal{A} \), \( a_i \).
• $||\vec{r}_i||$ is the Euclidean length of $\vec{r}_i$.

• By abuse of notation, let $\vec{o}_j$ be a vector from the origin of the fixed observing frame to the $j^{th}$ vertex of $O$, $o_j$.

• $||\vec{o}_i||$ is the Euclidean length of $\vec{o}_i$.

• $\alpha_i$ is the angle between $\vec{x}_A$, the $x$-axis of the robot’s body fixed frame and the vector $\vec{r}_i$.

• $\phi_i$ is the angle from $\vec{x}_A$ to $\vec{n}_A^i$, the normal to the $i^{th}$ edge of $A$, $E_A^i$.

• $\beta_j$ is the angle between $\vec{x}_R$ (the $x$-axis of the fixed observing reference frame) and $\vec{o}_j$.

• $\xi_j$ is the angle between $\vec{x}_R$ and $\vec{n}_O^j$, the normal to the $j^{th}$ edge of $O$, $E_O^j$.

With these definitions, the basic vectors that are involved in the constraint equations are:

$$\vec{o}_j = ||\vec{o}_j|| \begin{bmatrix} \cos(\beta_j) \\ \sin(\beta_j) \end{bmatrix} \quad \vec{r}_i = ||\vec{r}_i|| \begin{bmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{bmatrix}$$

(1)

$$\vec{n}_A^i(q) = \begin{bmatrix} \cos(\phi_i + \theta) \\ \sin(\phi_i + \theta) \end{bmatrix} \quad \vec{n}_O^j = \begin{bmatrix} \cos(\xi_j) \\ \sin(\xi_j) \end{bmatrix}$$

(2)

### 3 The Constraint Equations in Parametrized Form

$$a_i(q) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{r}_i = \begin{bmatrix} x + ||\vec{r}_i|| \cos(\alpha_i + \theta) \\ y + ||\vec{r}_i|| \sin(\alpha_i + \theta) \end{bmatrix}$$

(3)

**First constraint.** First consider the constraint which ensures that vertex $o_j$ lies on the line underlying the $i^{th}$ edge of $A$:

$$\vec{n}_A^i(q) \cdot (o_j - a_i(a)) = 0.$$  

(4)

Substituting in the variables from above, and performing some algebra results in the equation:

$$0 = -x \cos(\phi_i + \theta) - y \sin(\phi_i + \theta) + ||\vec{o}_j|| \cos(\phi_i + \theta - \beta_j) - ||\vec{r}_i|| \cos(\phi_i - \alpha_i).$$

(5)

This equation has the form:

$$A(\theta) \ x + B(\theta) \ y + C(\theta) = 0.$$  

(6)

For a constant orientation (i.e., when the value of $\theta$ is fixed), Equation (6) represents a straight line in the $x$-$y$ plane (i.e., a straight line in the constant orientation slice of c-space at level $\theta$). Thus, the local “patch” of the configuration-space obstacle boundary is a *ruled
surface, since this equation shows that the surface is bounded by a line whose orientation changes as a function of $\theta$.

**Second Pair of Constraints.** Next we consider the pair of inequality constraints that insure that the robot and obstacle don’t overlap:

\[
\vec{n}_i^A(q) \cdot (\vec{\partial}_{j-1} - \vec{\partial}_j) \geq 0 \tag{7}
\]
\[
\vec{n}_i^A(q) \cdot (\vec{\partial}_{j+1} - \vec{\partial}_j) \geq 0 \tag{8}
\]

Using the observation that:

\[
(\vec{\partial}_{j-1} - \vec{\partial}_j) = ||E_{j-1}^O|| \begin{bmatrix}
\cos(\xi_{j-1} - \pi/2) \\
\sin(\xi_{j-1} - \pi/2)
\end{bmatrix}
\tag{9}
\]

Substituting the parametrized terms into Equation (7), and simplifying yields the equivalent constraint:

\[
\cos(\phi_i + \theta - \xi_{j-1} + \pi/2) \geq 0 \tag{10}
\]

In general, for an angle $\gamma$ to satisfy the equation $\cos \gamma \geq 0$, we require that $-\pi/2 \leq \gamma (mod 2\pi) \leq \pi/2$. Hence, Equation (10) is equivalent to

\[
-\pi \leq \phi_i + \theta - \xi_{j-1} \leq 0 \ (mod \ 2\pi). \tag{11}
\]

Note, for this equation, only the lower bound is physically meaningful for the geometry shown in Figure 1. Thus, constraint Equation (7) reduces to:

\[
\xi_{j-1} - \phi_i - \pi \leq \theta \tag{12}
\]

Similarly, using the observation that

\[
(\vec{\partial}_{j+1} - \vec{\partial}_j) = ||E_j^O|| \begin{bmatrix}
\cos(\xi_j + \pi/2) \\
\sin(\xi_j + \pi/2)
\end{bmatrix}
\tag{13}
\]

Equation (8) can be rewritten as

\[
\cos(\phi_i + \theta - \xi_j - \pi/2) \geq 0 \tag{14}
\]

which is equivalent to

\[
0 \leq \phi_i + \theta - \xi_j \leq \pi \ (mod \ 2\pi). \tag{15}
\]

For this equation, only the upper bound is physically meaningful, and thus constraint Equation (8) reduces to:

\[
\theta \leq \pi + \xi_j - \phi_i \tag{16}
\]

These two constraints can then be summarized as:

\[
\theta \in [(\xi_{j-1} - \phi_i - \pi), (\xi_j - \phi_i + \pi)] \ (mod \ 2\pi) \tag{17}
\]
Thus, these constraints bound the range of $\theta$ over which the local “patch” is defined. Note that the “mod $2\pi$” modification applies to each of the upper and lower bounds.

**Third pair of constraints.** The final pair of inequality constraints bounds the vertex $o_j$ to lie within the line segment $E_i^A$:

$$0 \leq (o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \leq ||E_i^A||^2.$$  

(18)

Substituting the parametrized expressions for $o_j$, $a_i(q)$, and $a_{i+1}(q)$ into this equation yields:

$$0 \leq x [||\vec{r}_i|| \cos(\alpha_i + \theta) - ||\vec{r}_{i+1}|| \cos(\alpha_{i+1} + \theta)]$$

$$+ y [||\vec{r}_i|| \sin(\alpha_i + \theta) - ||\vec{r}_{i+1}|| \sin(\alpha_{i+1} + \theta)]$$

$$+ ||\vec{d}_j|| ||\vec{r}_{i+1}|| \cos(\theta + \alpha_{i+1} - \beta_j) - ||\vec{d}_j|| ||\vec{r}_i|| \cos(\theta + \alpha_i - \beta_j)$$

$$- ||\vec{r}_{i+1}|| ||\vec{r}_i|| \cos(\alpha_i - \alpha_{i+1}) + ||\vec{r}_i||^2 \leq ||E_i^A||^2.$$  

(19)

These equations have the form:

$$0 \leq D(\theta) x + E(\theta) y + F(\theta) \leq ||E_i^A||^2$$  

(23)

where:

$$D(\theta) = [||\vec{r}_i|| \cos(\alpha_i + \theta) - ||\vec{r}_{i+1}|| \cos(\alpha_{i+1} + \theta)]$$

$$E(\theta) = [||\vec{r}_i|| \sin(\alpha_i + \theta) - ||\vec{r}_{i+1}|| \sin(\alpha_{i+1} + \theta)]$$

$$F(\theta) = + ||\vec{d}_j|| ||\vec{r}_{i+1}|| \cos(\theta + \alpha_{i+1} - \beta_j) - ||\vec{d}_j|| ||\vec{r}_i|| \cos(\theta + \alpha_i - \beta_j)$$

$$- ||\vec{r}_{i+1}|| ||\vec{r}_i|| \cos(\alpha_i - \alpha_{i+1}) + ||\vec{r}_i||^2$$

**3.1 Summary**

The c-obstacle boundary patch defined by these constraint equations can thus be viewed as a ruled surface formed by sweeping a line segment (whose underlying line is given by Equation (6)) through the $\theta$-range defined by Equation (5). The end points of the line segment can be determined as follows. One end of the line segment (for a given $\theta$) occurs at the lower equality of Equation (23). Thus, this point can be found as the solution of the two linear equations

$$0 = Ax + By + C$$

$$0 = Dx + Ey + F$$

Similarly, the other end-point of the line segment (again, for a given $\theta$) can be found from the upper inequality of Equation (23). That is, the other point (for fixed $\theta$) is found by solving the linear equations:

$$0 = Ax + By + C$$

$$||E_i^A||^2 = Dx + Ey + F$$