CDS 101/110: Lecture 9.1
Frequency DomainLoop Shaping

November 21, 2016

Goals:

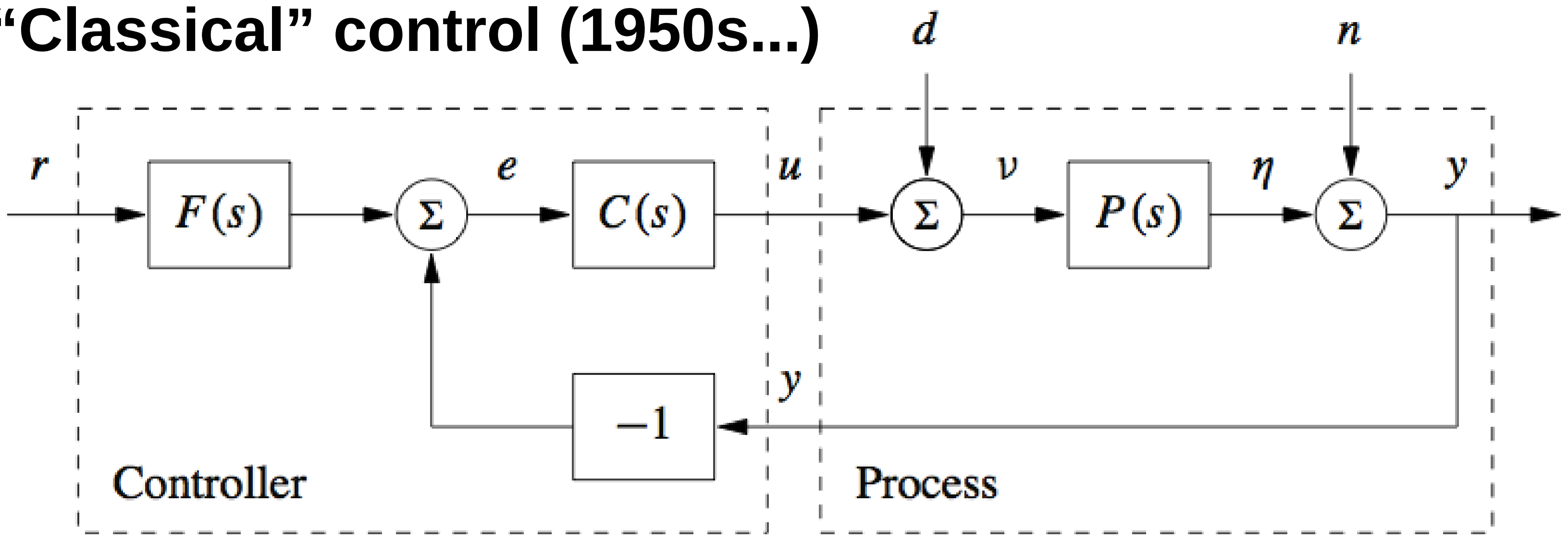
- Review:
  - canonical control design problem
  - performance measures
- Loop Transfer Functions (Gang of Four, Seven)
- Show how to use “loop shaping” to achieve a performance specification
- Work through example(s)

Reading:

- Åström and Murray, Feedback Systems 2-e, Section 12.1, 12.2-12.4

Design Patterns for Control Systems

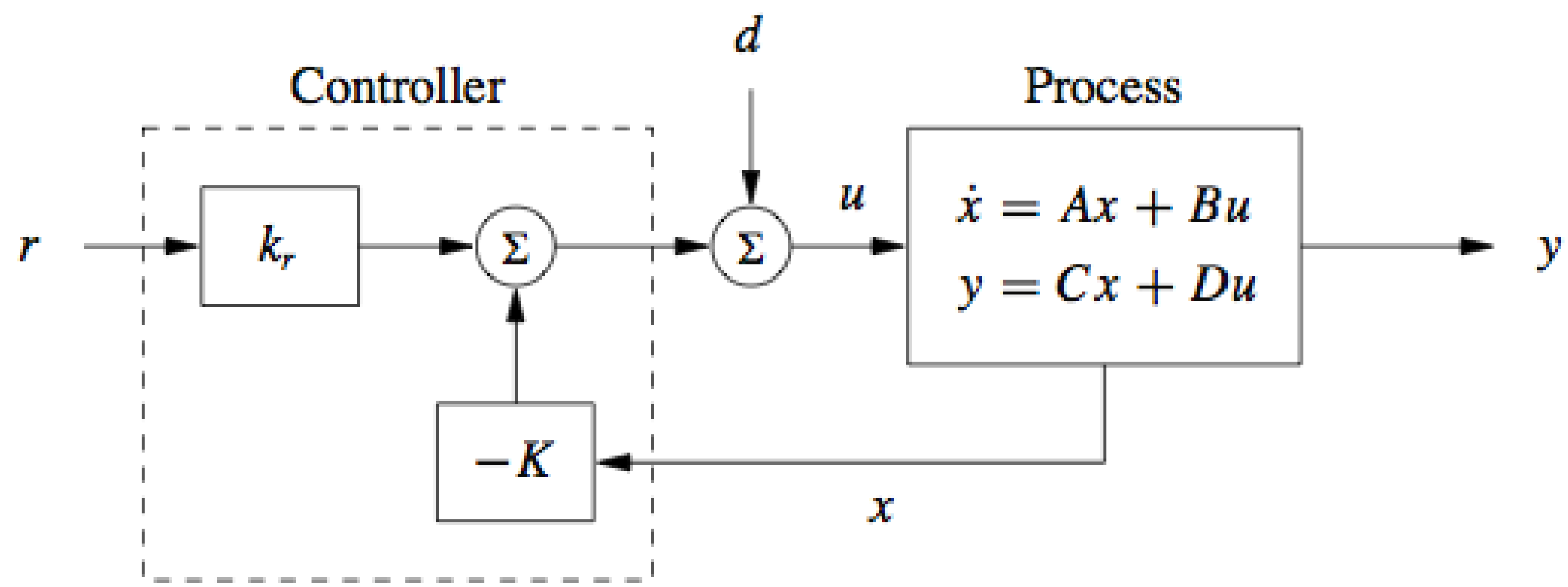
"Classical" control (1950s...)



- Reference input shaping
- Feedback on output error
- Compensator dynamics shape closed loop response
- *Uncertainty* in process dynamics $P(s)$ + external disturbances (d) & noise (n)

- **Goal:** output y(t) should track reference trajectory r(t)
- Design typically done in "frequency domain" (second half of CDS 101/110)

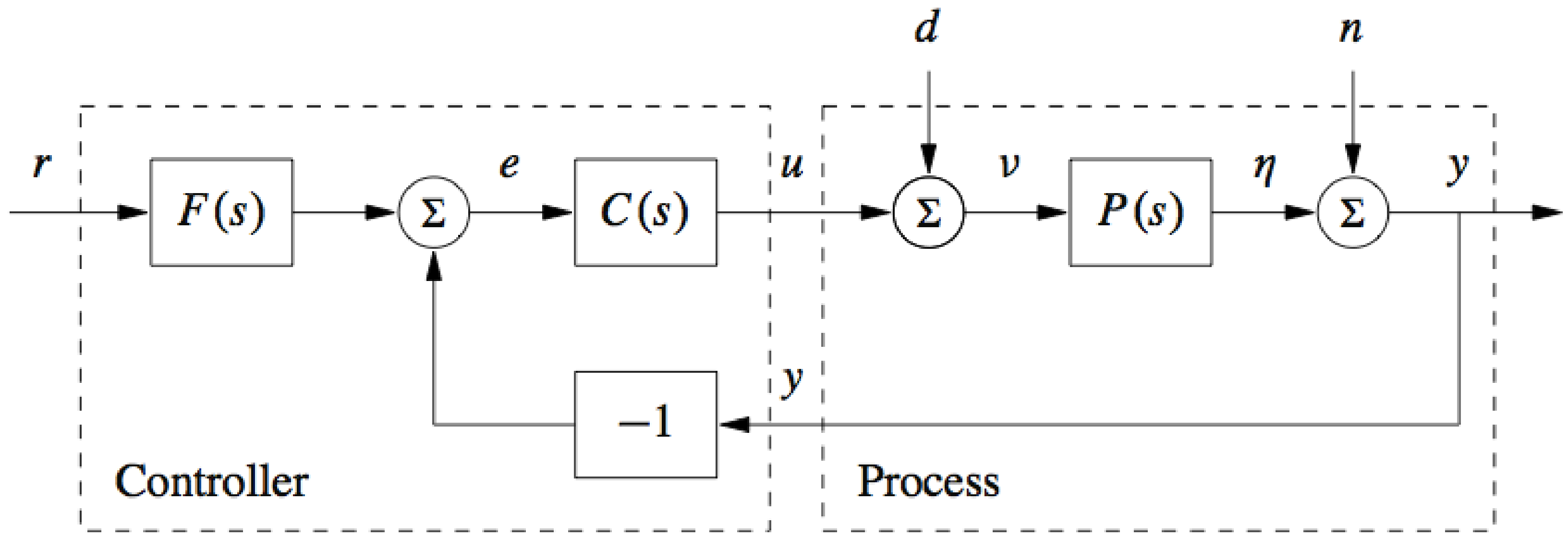
Modern" (state space) control (1970s...)



- Assume dynamics are given by linear system, with known A, B, C, D matrices
- Measure the state of the system and use this to modify the input
- $u = -K\,x + k_r\,r$

- Goal unchanged: output y(t) should track reference trajectory r(t) [often constant]

Input/Output Control Design Specifications



Common Sense Goals:

- Keep error small for reference signals r
- Attenuate effect of sensor noise n and *load disturbances* d
- Avoid large input commands u

Designs represent tradeoffs, e.g.:

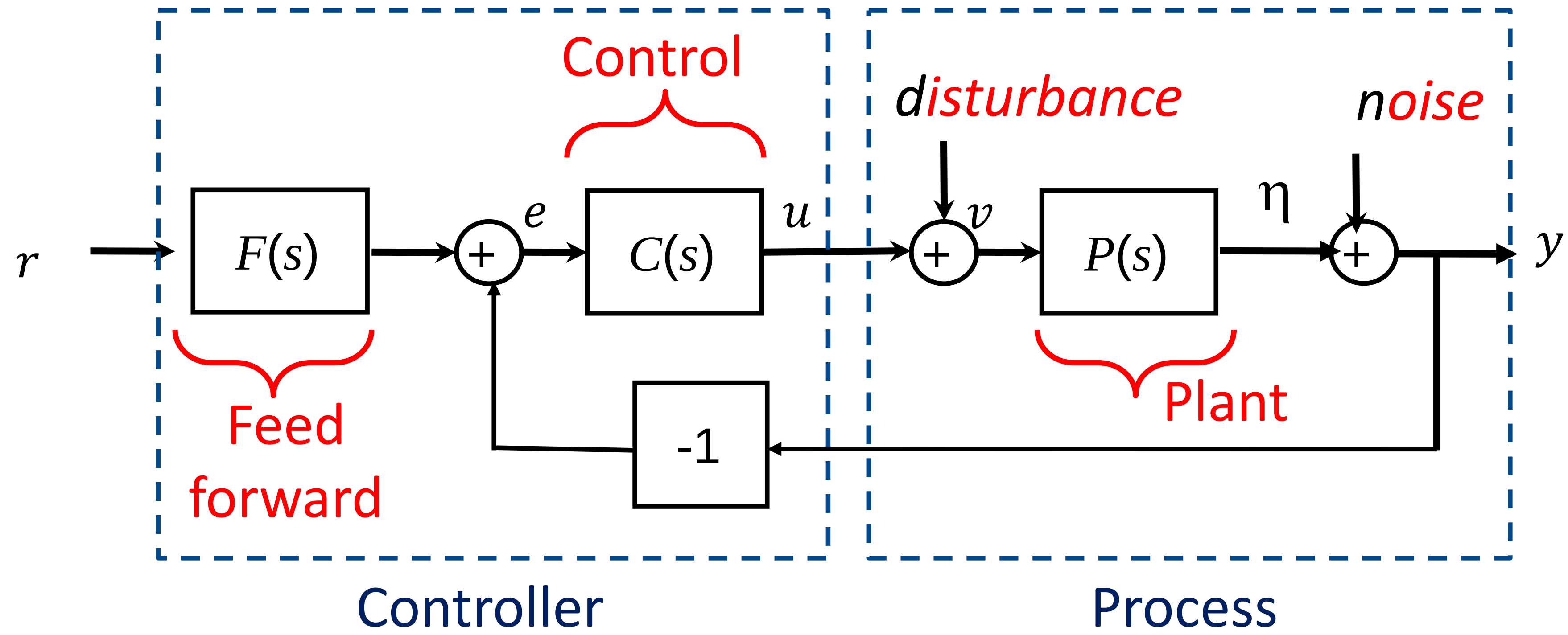
- Keep $L = PC$ large for good performance (G_{er} << 1)
- Keep $L = PC$ small for good noise rejection ($G_{\eta n}$ < 1)

- Load disturbances (d) typically low frequency, while noise (v) is high frequency
- Stability always determined by 1/(1+*PC*) assuming stable process & controller
- Numerator determined by forward path between input and output

More generally: 7 primary transfer functions; simultaneous design of each

- Controller *C*(*s*) enters in multiple places ⇒ possibly hard to understand tradeoffs

General Loop Transfer Functions



r = reference input
e = error
u = control
v = control + disturbance
η = true output (***what we want to control***!)
y = measured output

System "outputs" · System "inputs"

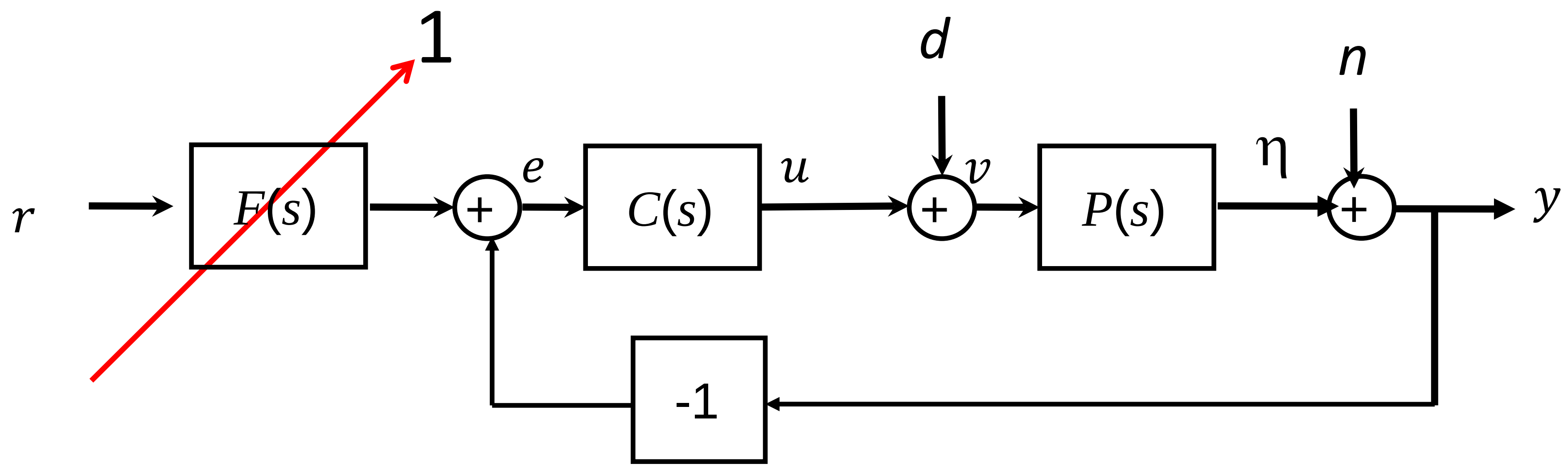
$$\begin{pmatrix} y \\ \eta \\ v \\ u \\ e \end{pmatrix} = \begin{pmatrix} \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{CF}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{CF}{1+PC} & \frac{-PC}{1+PC} & \frac{-C}{1+PC} \\ \boxed{\frac{F}{1+PC}} & \frac{-P}{1+PC} & \frac{-1}{1+PC} \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

"Gang of Seven"

"Gang of Six"

$TF = \frac{PCF}{1+PC}$	$T = \frac{PC}{1+PC}$	$PS = \frac{P}{1+PC}$
$CFS = \frac{CF}{1+PC}$	$CS = \frac{C}{1+PC}$	$S = \frac{1}{1+PC}$
Response of (y, u) to r	Response of u to (d,n)	Response of y to (d,n)

Key Loop Transfer Functions



$F(s) = 1$: Four unique transfer functions define performance ("Gang of Four")

Sensitivity: Function

$$G_{er} = S(s) = \frac{1}{1+L(s)}$$

Complementary Sensitivity Function:

$$G_{yr} = T(s) = \frac{L(s)}{1+L(s)}$$

Load Sensitivity Function:

$$G_{yd} = PS(s) = \frac{P(s)}{1+L(s)}$$

Noise Sensitivity Function:

$$G_{yn} = CS\,(s) = \frac{C(s)}{1+L(s)}$$

$$L(s) = P(s)C(s)$$

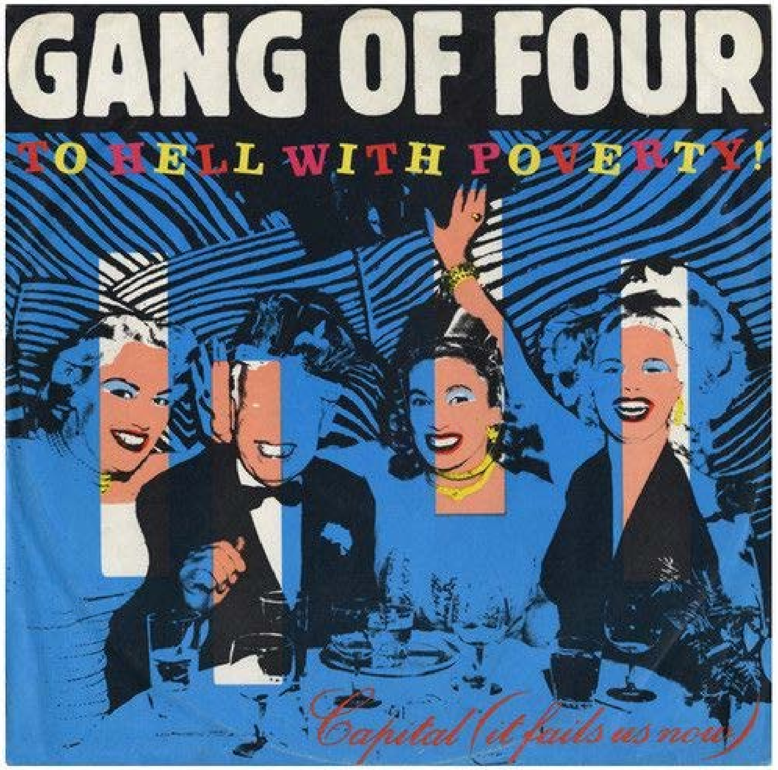
"Gang of Four"

(the "sensitivity" functions)

Characterize most performance criteria of interest

Culture, not Control

Politics

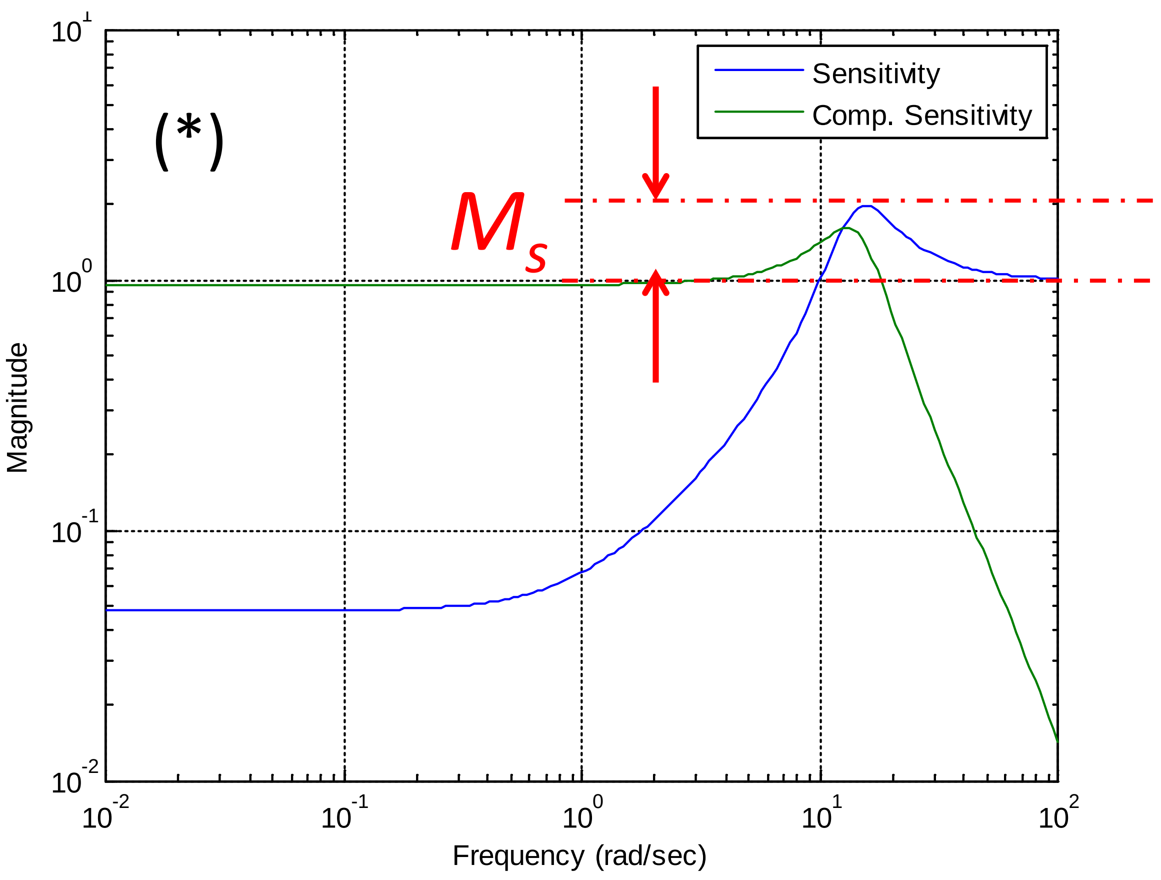


Music

Sensitivity

Note: $$H_{yd} = \frac{P(s)}{1+L(s)} = P(s)\frac{1}{1+L(s)} = P(s)S(s)$$

- I.e., Closed Loop Response to Disturbance = Open Loop Response x Sensitivity
- I.e., S(s) tells us how variations in output are influenced by feedback
  - Disturbances with $|S(i\omega)|<1$ attenuated by feedback
  - Disturbances with $|S(i\omega)|>1$ amplified
- Max Sensitivity = max $|S(i\omega)|$:= M_s
  - $M_s = 1/s_m$,
  - s_m is *stability margin* (from Nyquist)
  - Related to *robustness*



$$L(s) = \frac{20}{(s+1)(s/10+1)(s/100+1)}$$

Sensitivity (continued)

Example plotted is: $L(s) = \dfrac{20}{(s+1)(s/10+1)(s/100+1)}$ (*)

$$S(s) = \frac{1}{1+L(s)}$$

- |S(iw)|>1 Inside unit circle centered at -1
  (disturbances amplified)
- |S(iw)|<1 outside unit circle centered at -1
  (disturbances attenuated)

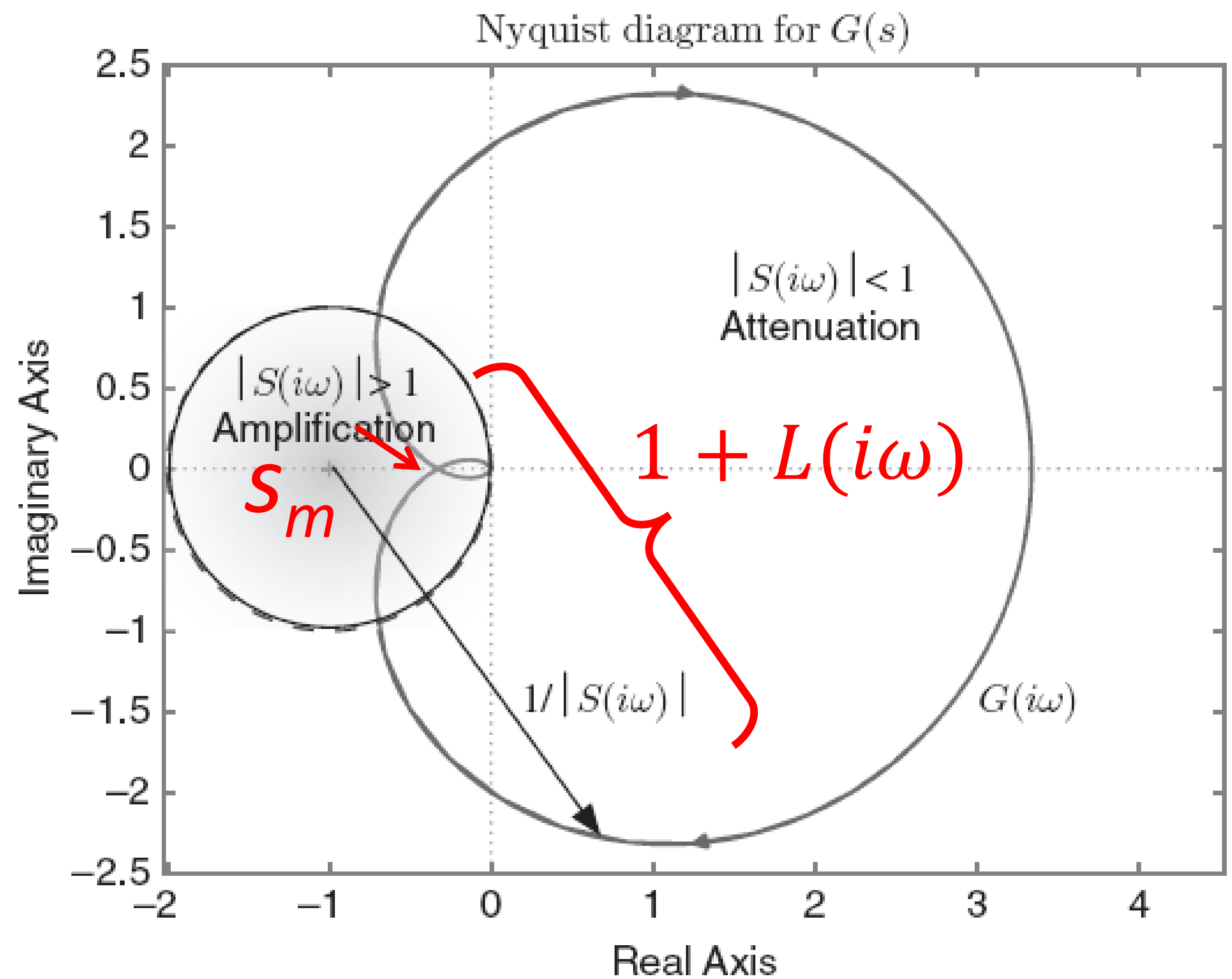
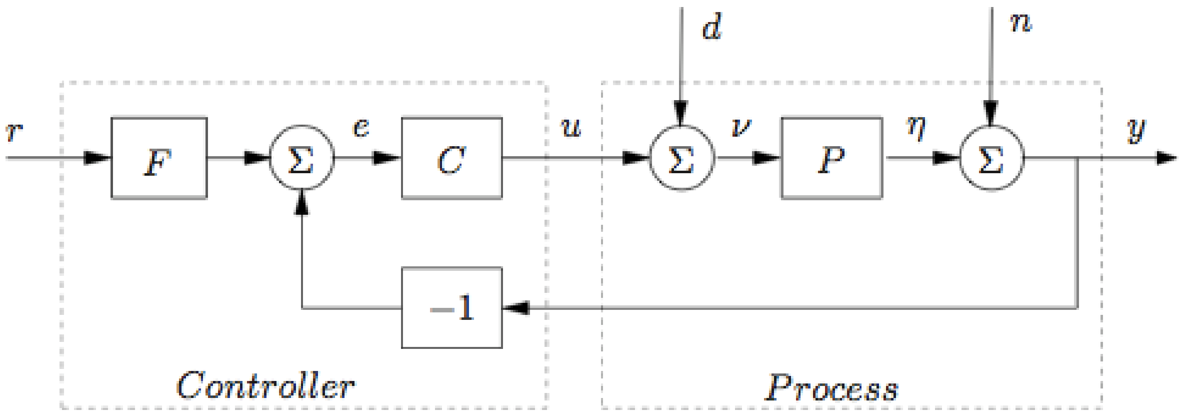
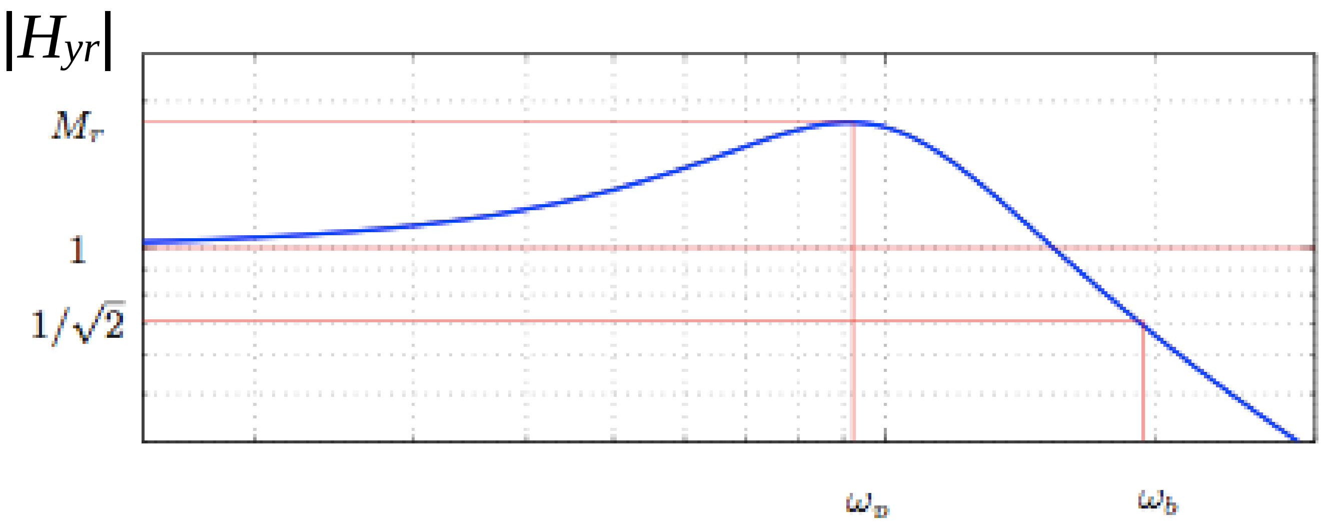


Fig. 4 Nyquist plot of the loop gain $G(s) = P(s)C(s)$ for the system (29). For frequencies for which $G(i\omega)$ enters the unit circle centered about the −1 point, disturbances are amplified and, for frequencies for which $G(s)$ lies outside this circle, disturbances are attenuated relative to open-loop.

(*) From Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5

Frequency Domain Specifications (review)





Specifications on the *open loop* transfer function (*L*)

- ***Gain crossover frequency***, ω_{gc}**:** the lowest frequency at which loop gain = 1
- ***Gain margin***, g_m**:** amount the loop gain can be increased before instability
- ***Phase margin***, φ_m**:** amount of phase lag required to generate instability

Specifications on *closed loop* frequency response (eg G_{yr}, G_{yd}, etc)

- Resonant peak, M_r, is the largest value of the frequency response
- Peak frequency, ω_p , is the frequency where the maximum occurs
- Bandwidth, ω_b, is the frequency where the gain has decreased to $1/\sqrt{2}$

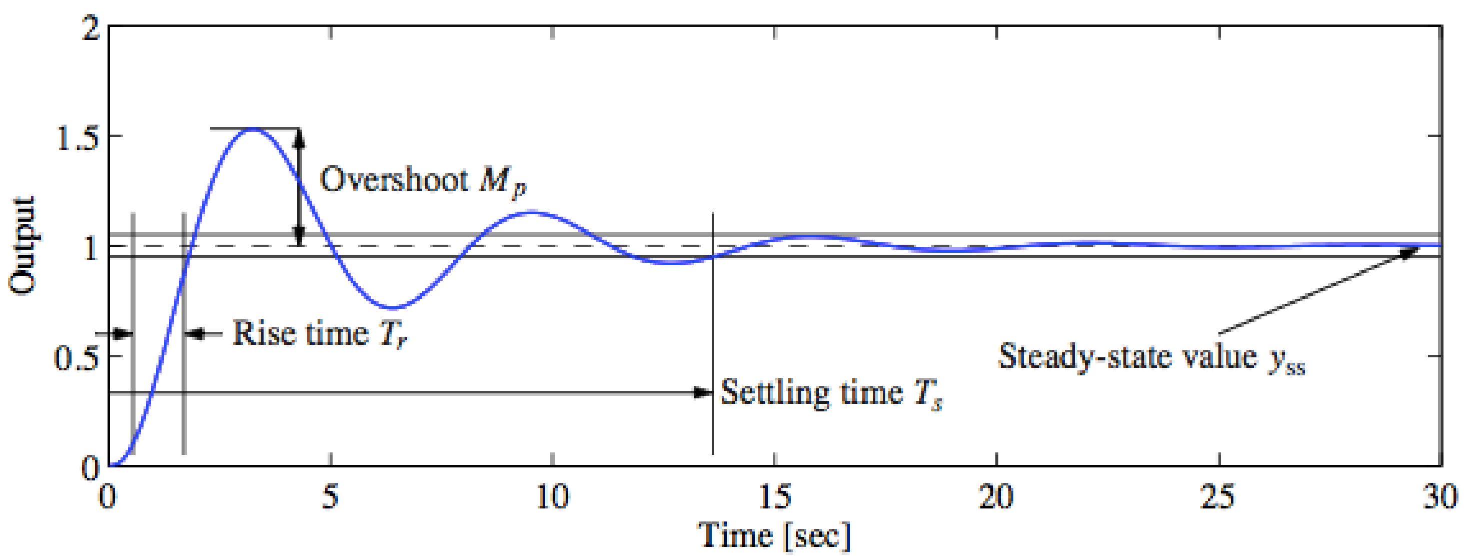
Basic idea: ***convert specs on closed loop to specs on open loop***

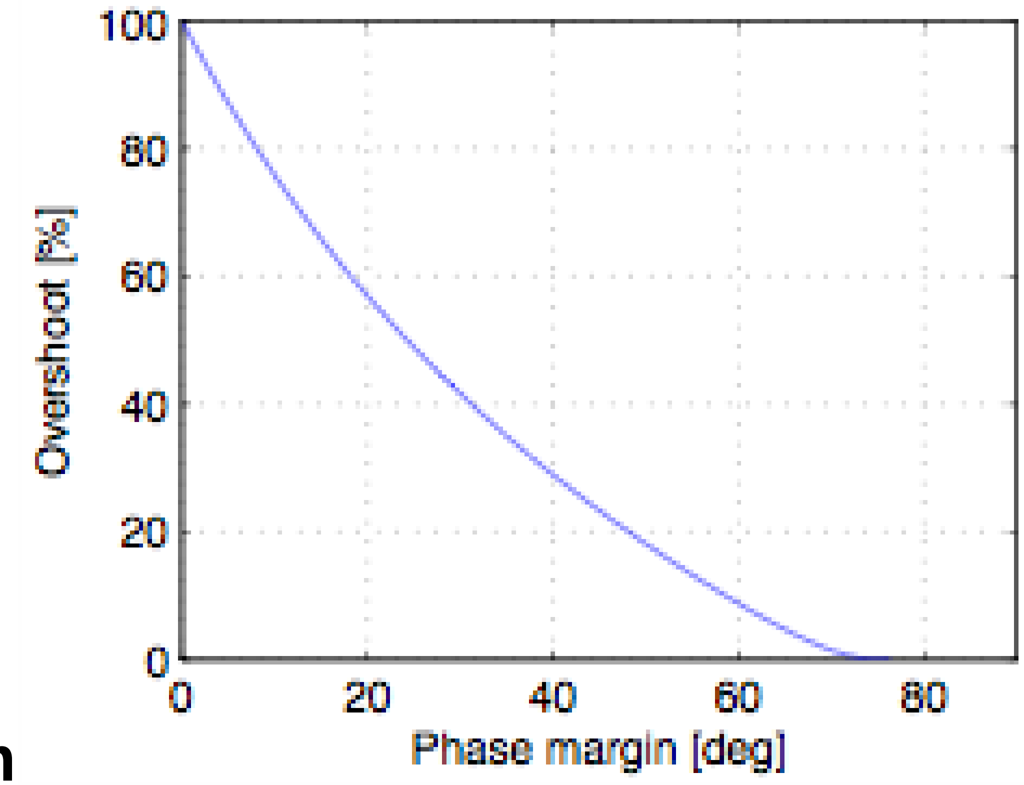
- Bandwidth ≈ value for which $|L| = 1$
- Resonant peak set by phase margin
- Keep $L(s)$ large to set $H_{yr} \approx 1$

$$H_{yr} = \frac{L}{1+L} \qquad H_{er} = \frac{1}{1+L}$$

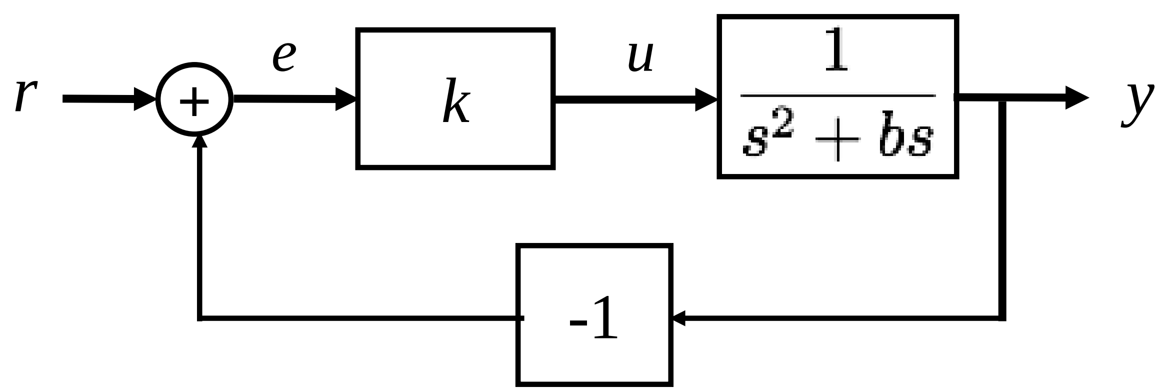
Time Domain Specs → Frequency Domain Specs

Time domain specifications



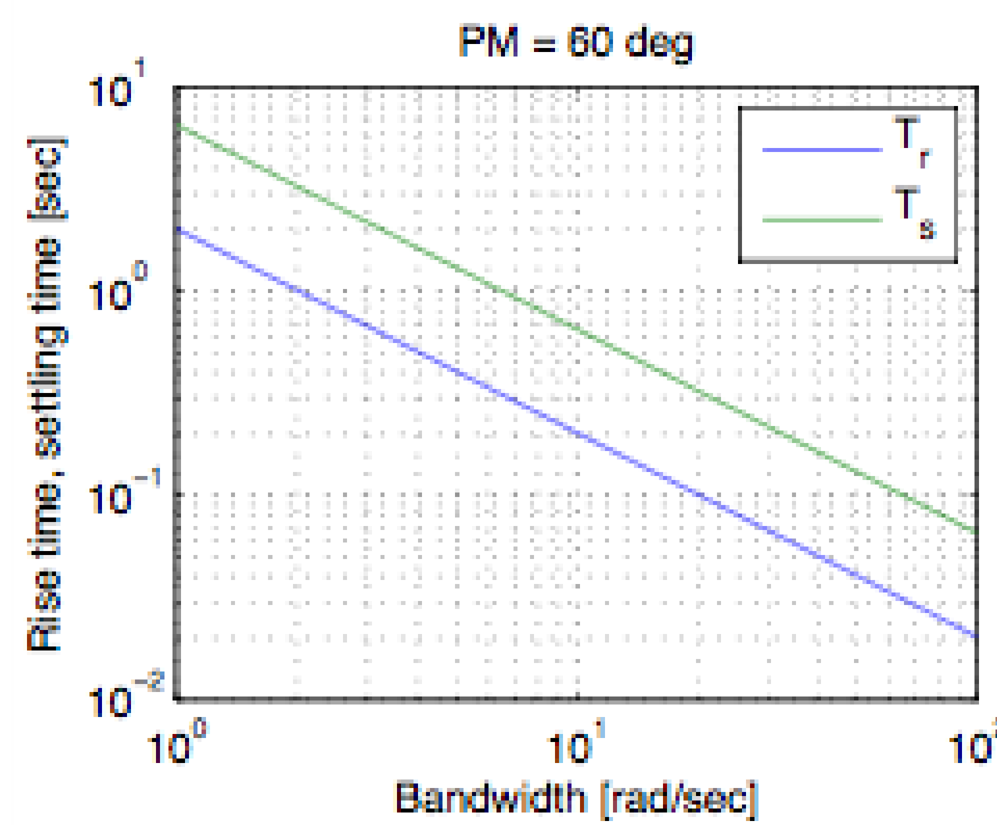


Map to frequency domain for second order system

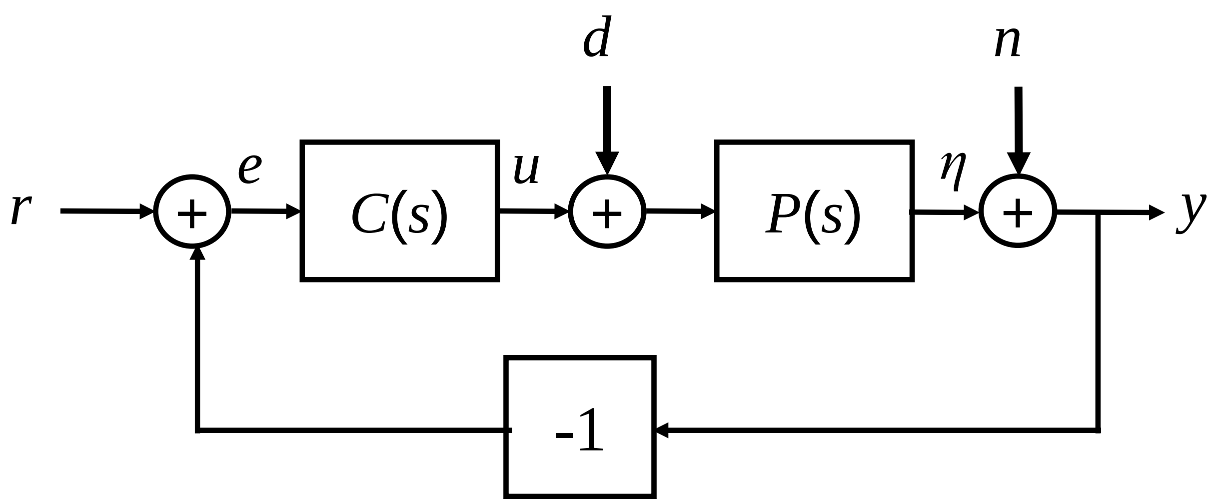


$$L(s) = \frac{k}{s^2 + bs} \qquad H_{yr} = \frac{k}{s^2 + bs + k}$$

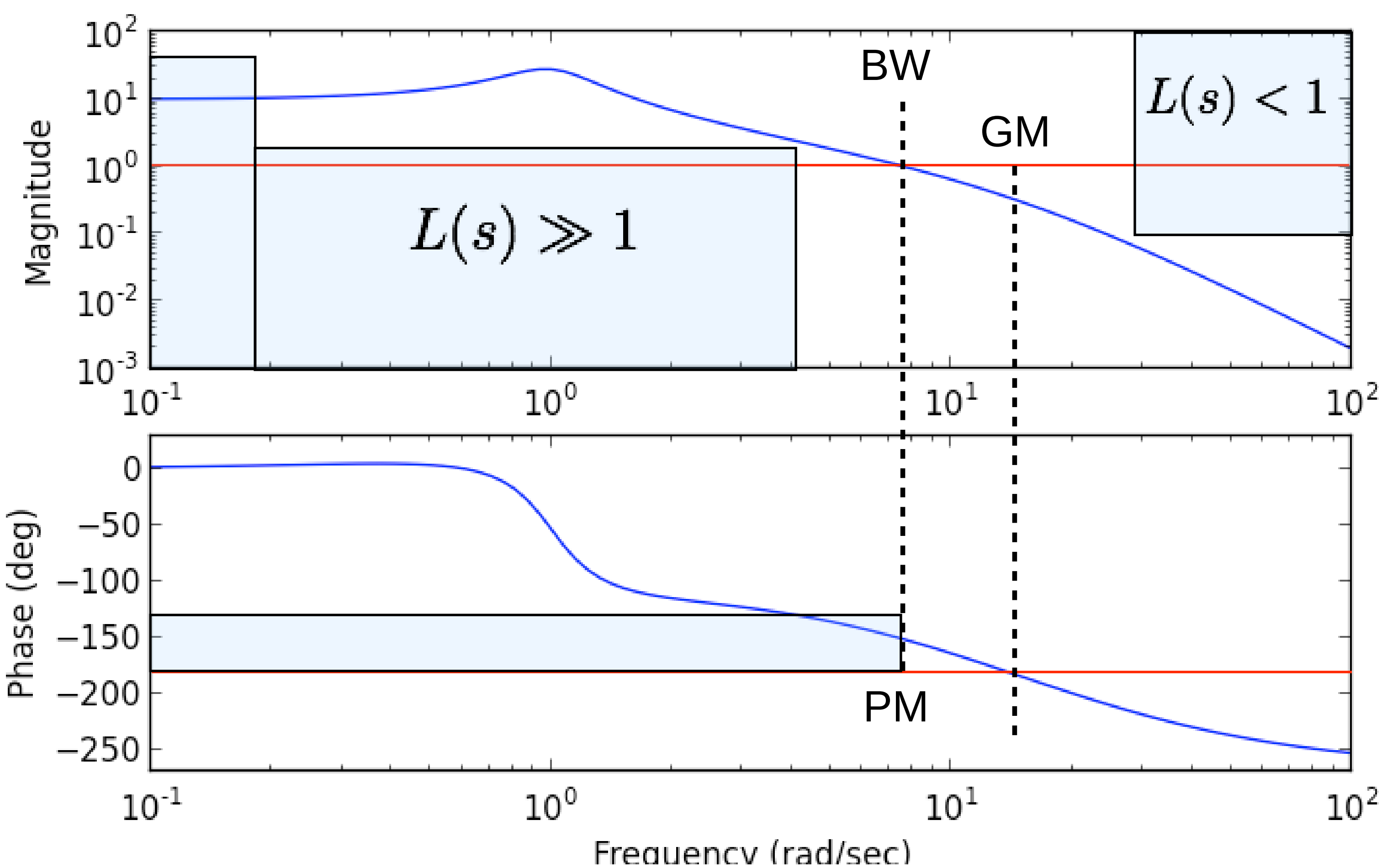
- Use properties of second order systems (Ch 7)



“Loop Shaping”: Design Loop Transfer Function



$$H_{er} = \frac{1}{1+L} \qquad H_{\eta n} = \frac{-L}{1+L}$$



Translate specs to “loop shape”

$$L(s) = P(s)C(s)$$

Naïve Idea: let $L(s)$ have desired properties, then $C(s) = \frac{L(s)}{P(s)}$

Design C(s) to obey constraints

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Loop shaping is *trial and error*

Aside: Relation of Gain to Phase in Bode Plot

If no poles/zeros in RHP (a *minimum phase system)* , then (see A&M FBS-2e,10.4) phase curve uniquely related to gain curve

$$\arg G(i\omega_0) = \frac{\pi}{2}\int_0^{\infty} f(\omega)\,\frac{d\,\log|G(i\omega)|}{d\,\log\omega}\,d\,\log\omega$$

$$\approx \frac{\pi}{2}\frac{d\,\log|G(i\omega_0)|}{d\,\log\omega_0}$$

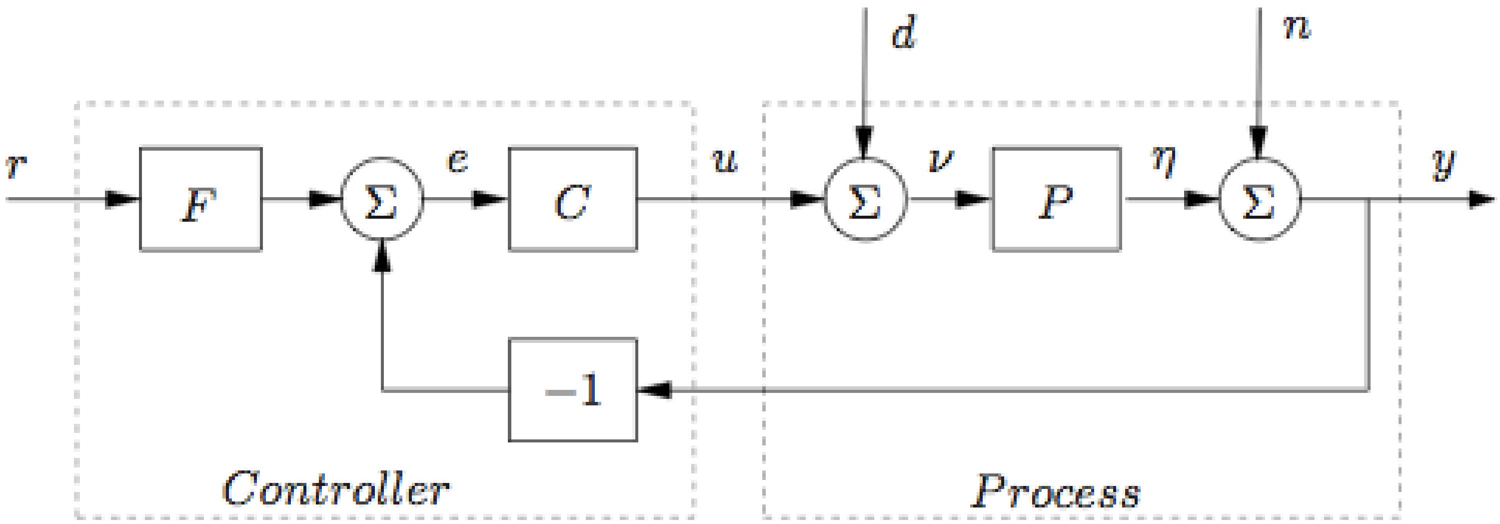
Where

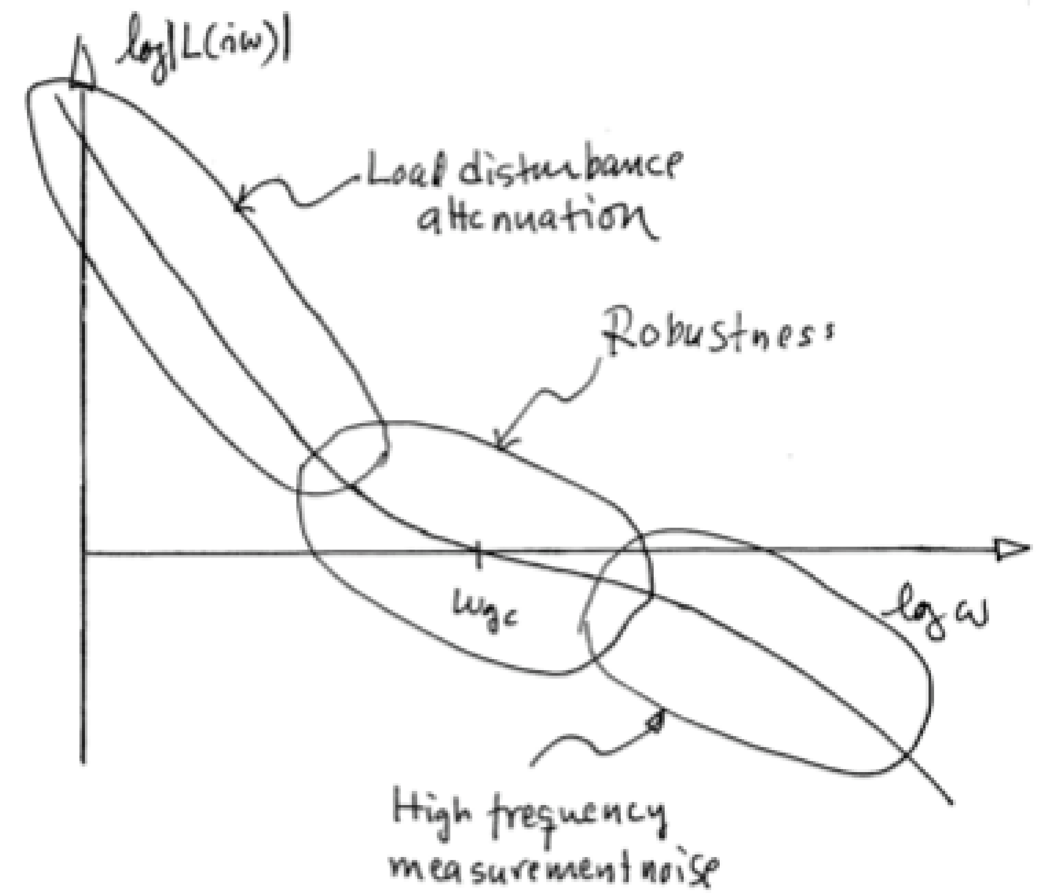
$$f(\omega) = \frac{2}{\pi}\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|$$

Consequence:

- Shape of phase curve is dictated by shape of gain curve
- Phase curve is weighted average of derivative of the gain curve
- Where gain curve is ~constant slope, phase curve is constant

Loop Shaping: Basic Approach





Disturbance rejection $$H_{ed} = \frac{-P}{1+L}$$

- Would like H_{ed} to be small make ⇒ large L(s)
- Typically require this in low frequency range

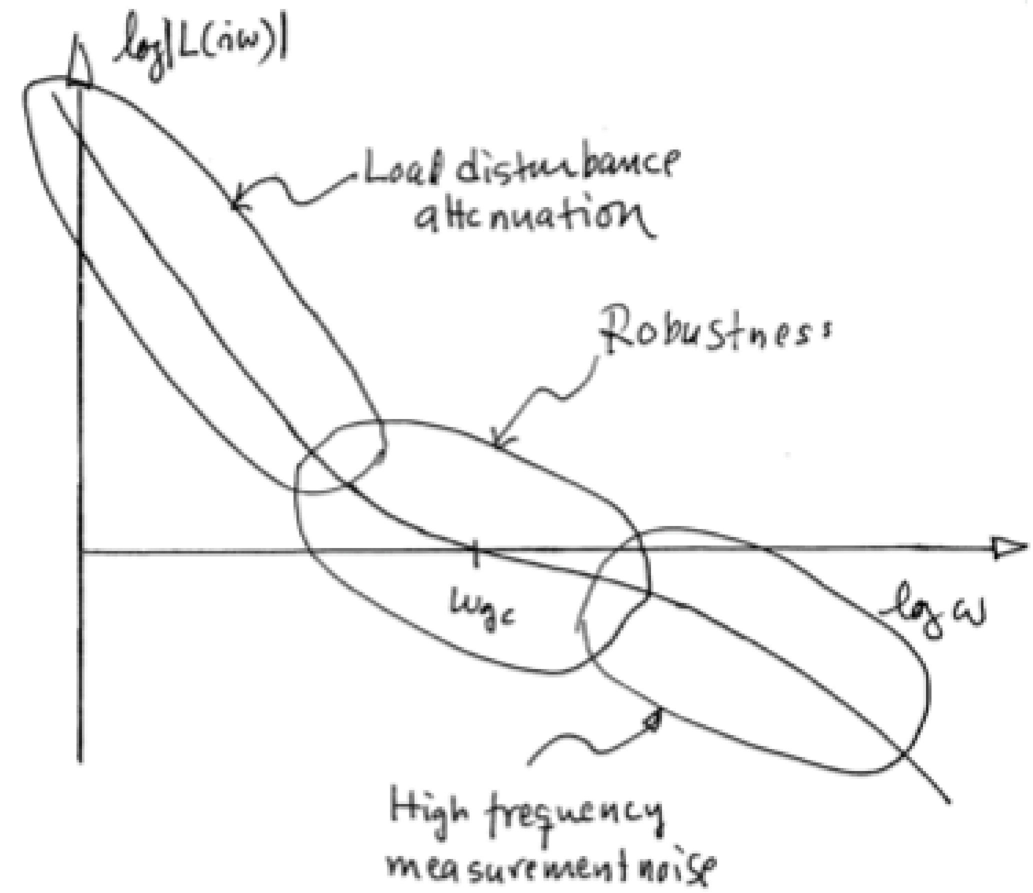
High frequency measurement noise $$H_{un} = \frac{-L}{P(1+L)}$$

- Want to make sure that H_{un} is small (avoid amplifying noise) ⇒ small $L(s)$
- Typically generates constraints in high frequency range

Robustness: gain and phase margin

- Focus on gain crossover region: make sure the slope is “gentle” at gain crossover
- Fundamental tradeoff: transition from high gain to low gain through crossover

Design Method #1: Process Inversion



Simple trick: invert out process

- Write performance specs in terms of desired loop transfer function
- Choose L(s) to satisfy specfications
- Choose controller by *inverting* P(s)

$$C(s) = L(s)/P(s)$$

Pros

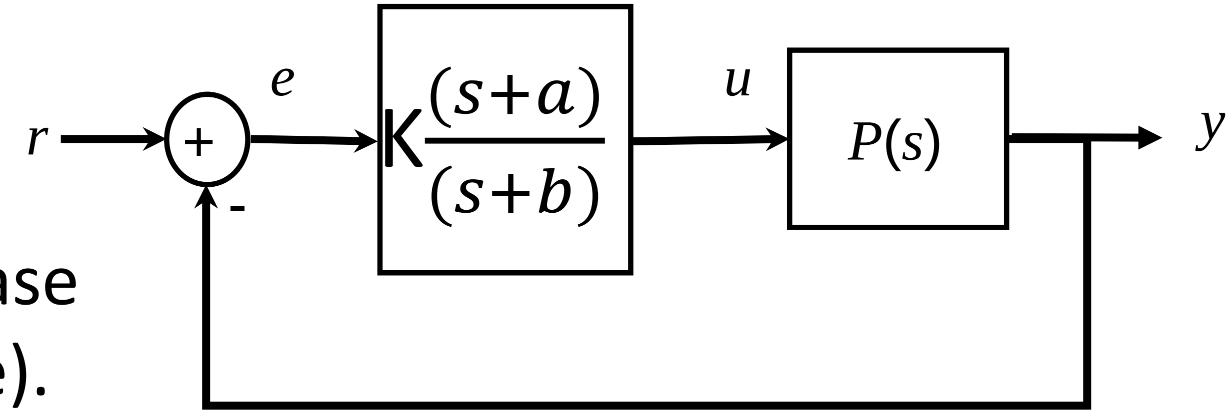
- Simple design process
- L(s) = k/s often works very well
- Can be used as a first cut, with additional tuning

Cons

- High order controllers (at least same order as plant)
- Requires "perfect" process model (due to inversion)
- Can generate non-proper controllers (order(num) > order(den))
  - Difficult to implement, plus amplifies noise at high frequency (C(∞) = ∞)
  - Fix by adding high frequency poles to roll off control response at high frequency
- Does not work if right half plane poles or zeros (*internal instability*)

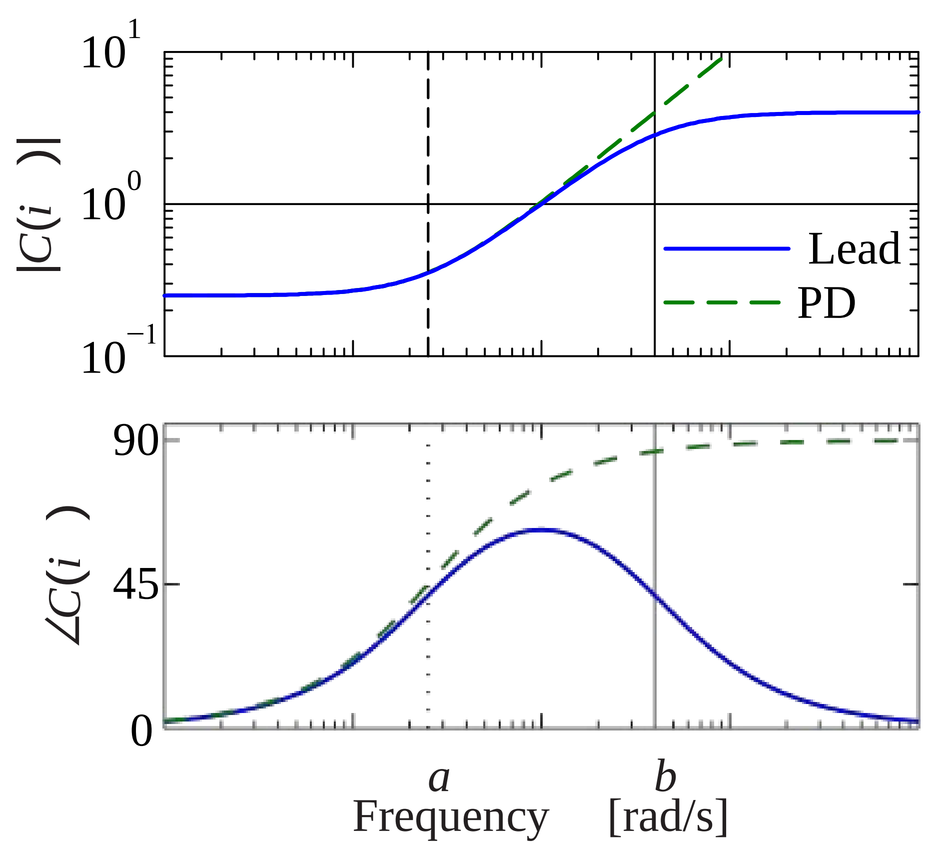
Lead & Lag Compensators

Lead: K > 0, a < b
- Add phase near crossover
- Improve gain & phase margins, increase bandwidth (better transient response).

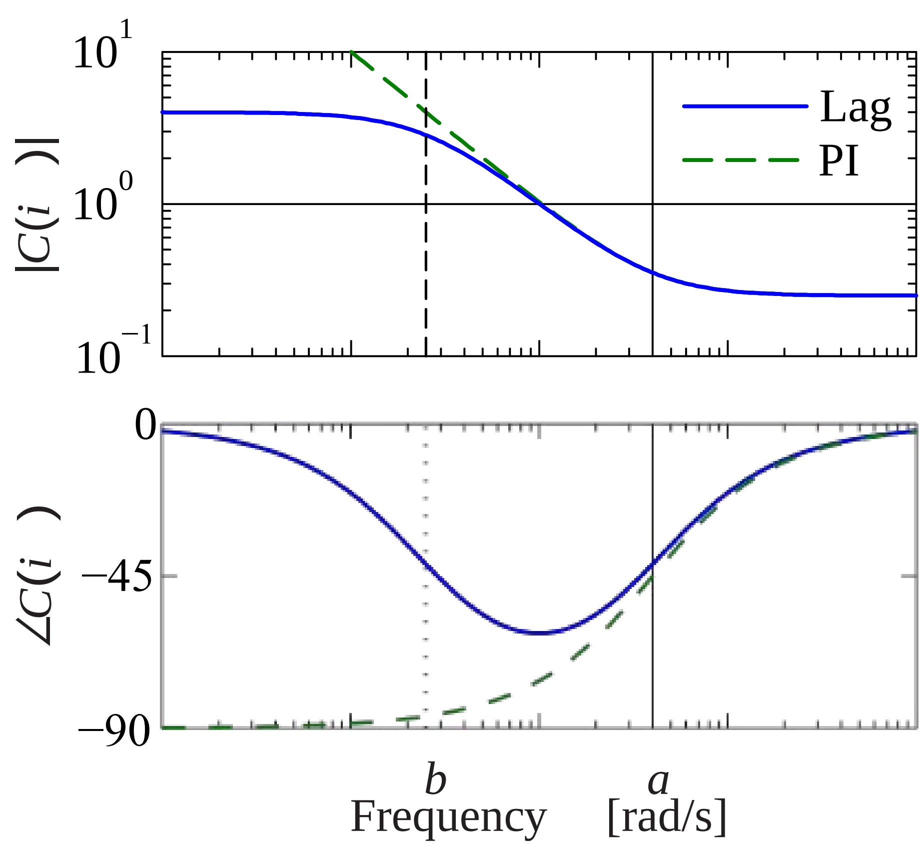


Lag: K > 0, a> b
- Add gain in low frequencies
- Improves steady state error

Lead/Lag:
- Better transient and steady state response



(a) Lead compensation, $a < b$

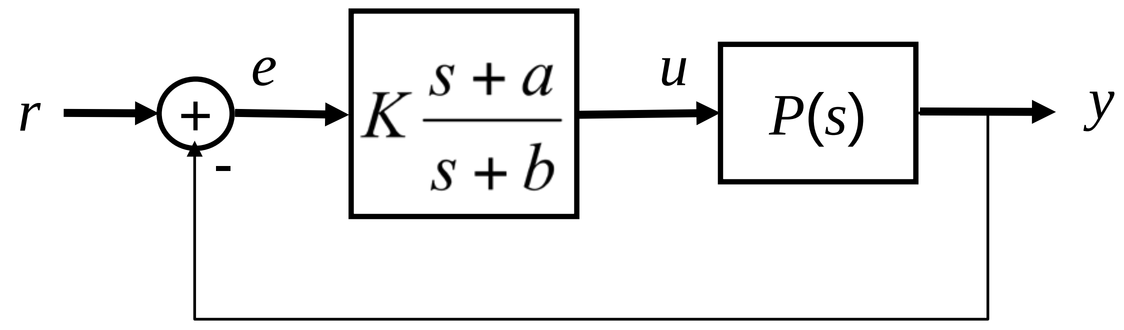


(b) Lag compensation, $b < a$

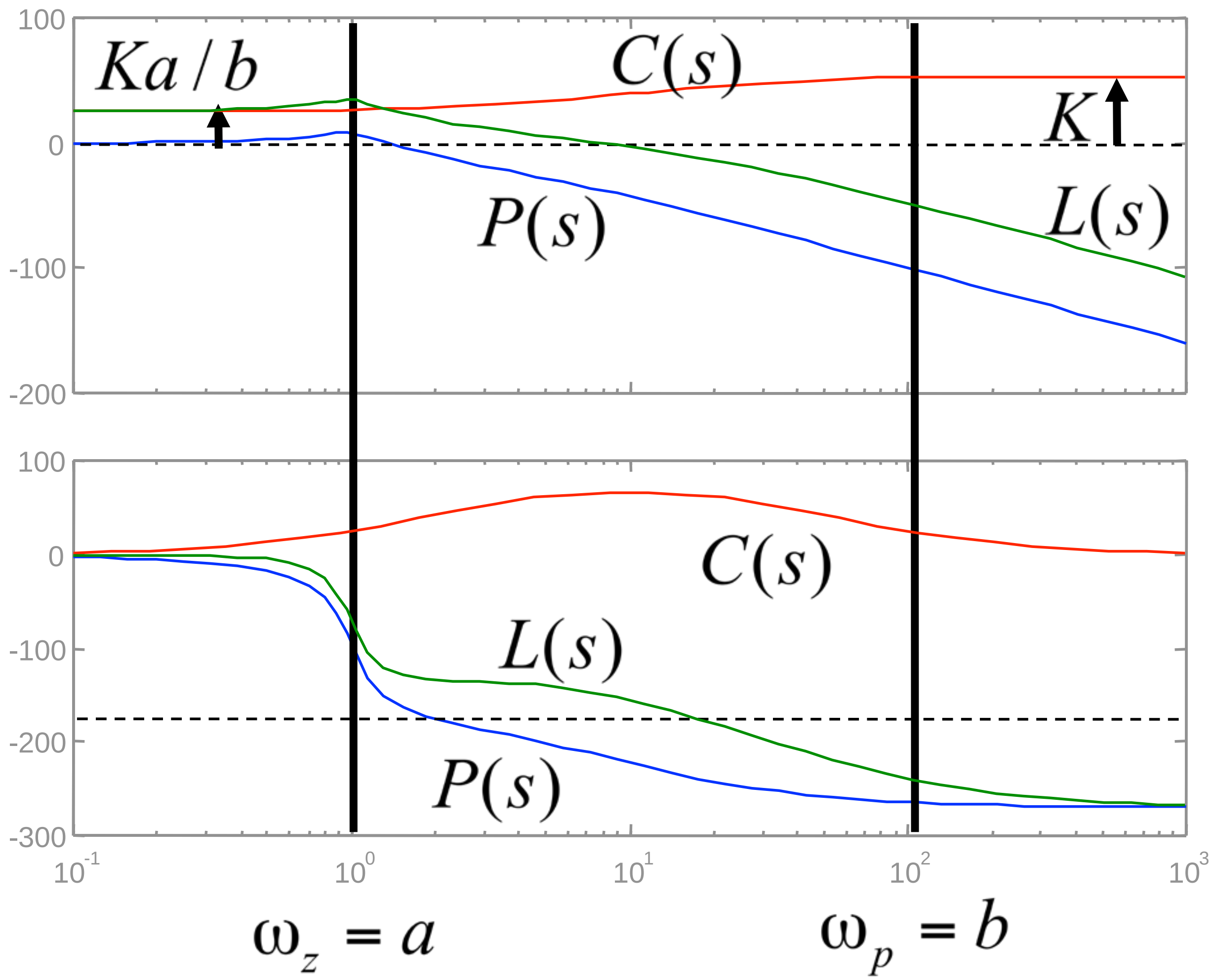
Design Method #2: Add Lead, Lag, Lead/Lag compensation

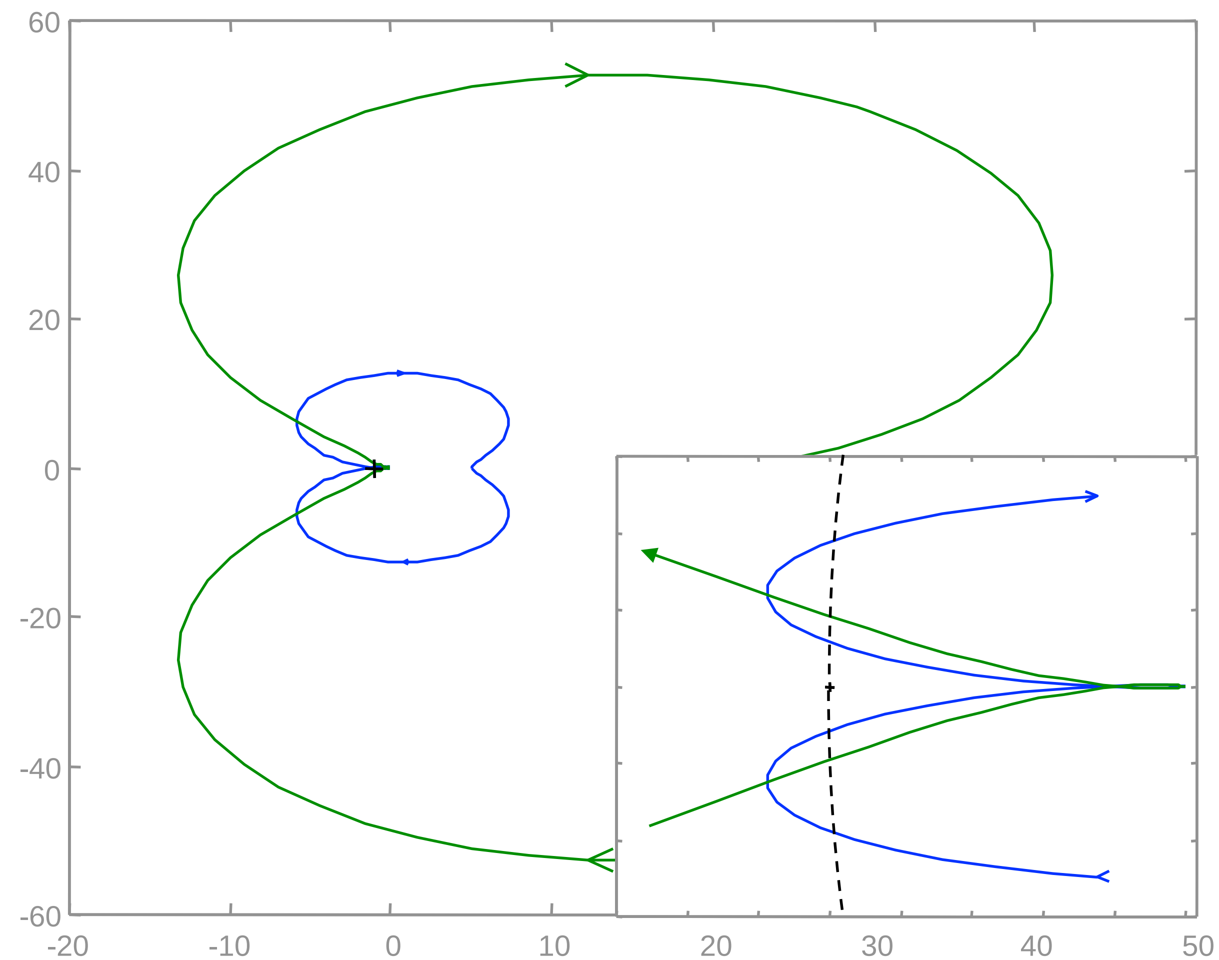
Lead: increases phase in frequency band

- Effect: lifts phase by increasing gain at high frequency
- Increases PM
- Bode: add phase between zero and pole
- Nyquist: increase phase margin



$a < b \qquad K > 0$





Example: Lead Compensation for Second Order System

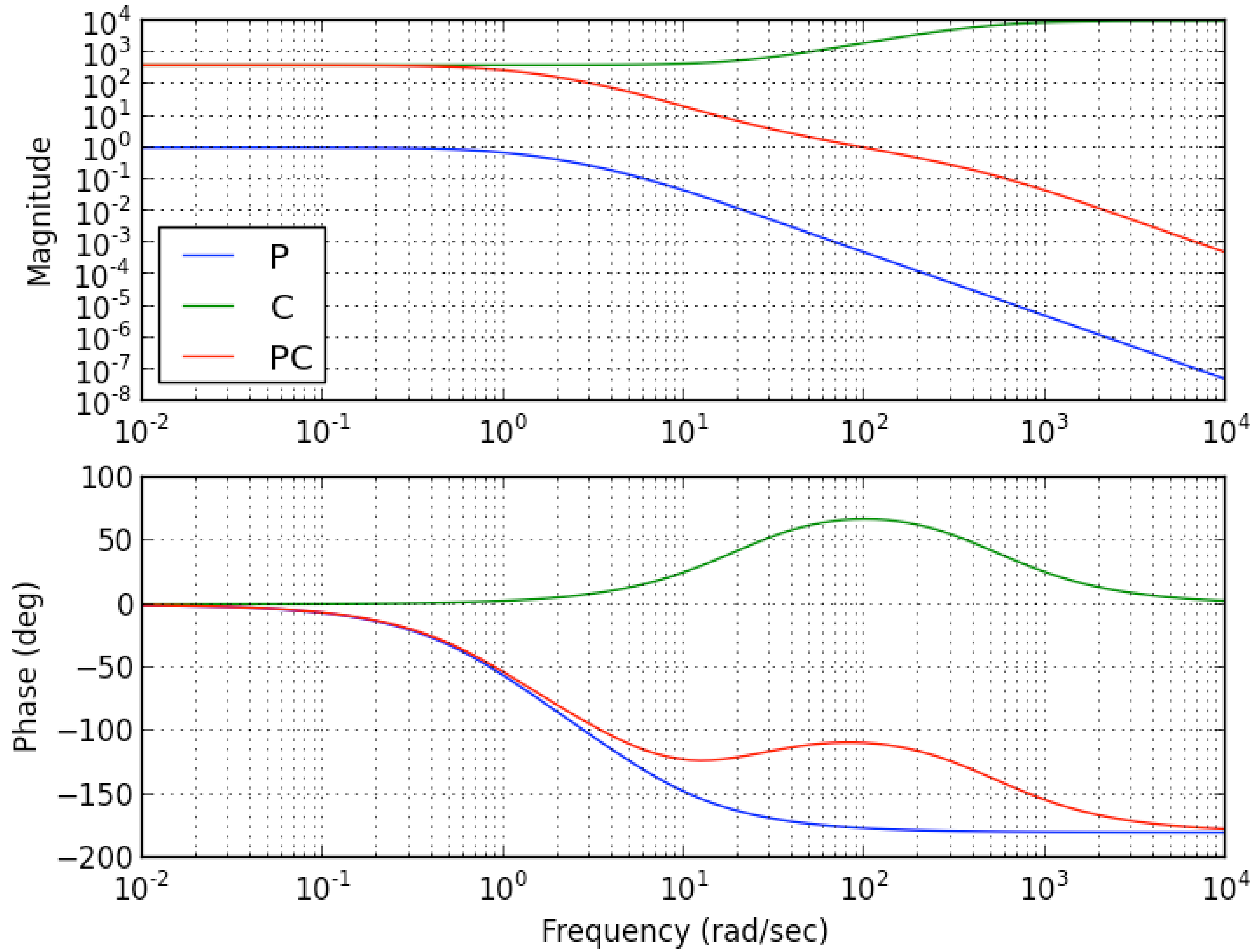
System description

$$P(s) = \frac{p_1 p_2}{(s + p_1)(s + p_2)}$$

- Poles: p_1 = 1, p_2 = 5

Control specs

- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%
  - Gives PM of ~60 deg



Try a lead compensator

$$C(s) = K\frac{s + a}{s + b}$$

- Want gain cross over at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error ⇒ $|L(0)| > 100$
- a = 20, b = 500, K = 10,000 (gives $|C(0)| = |L(0)| = 400$)

Better Loop Shaping Design Process

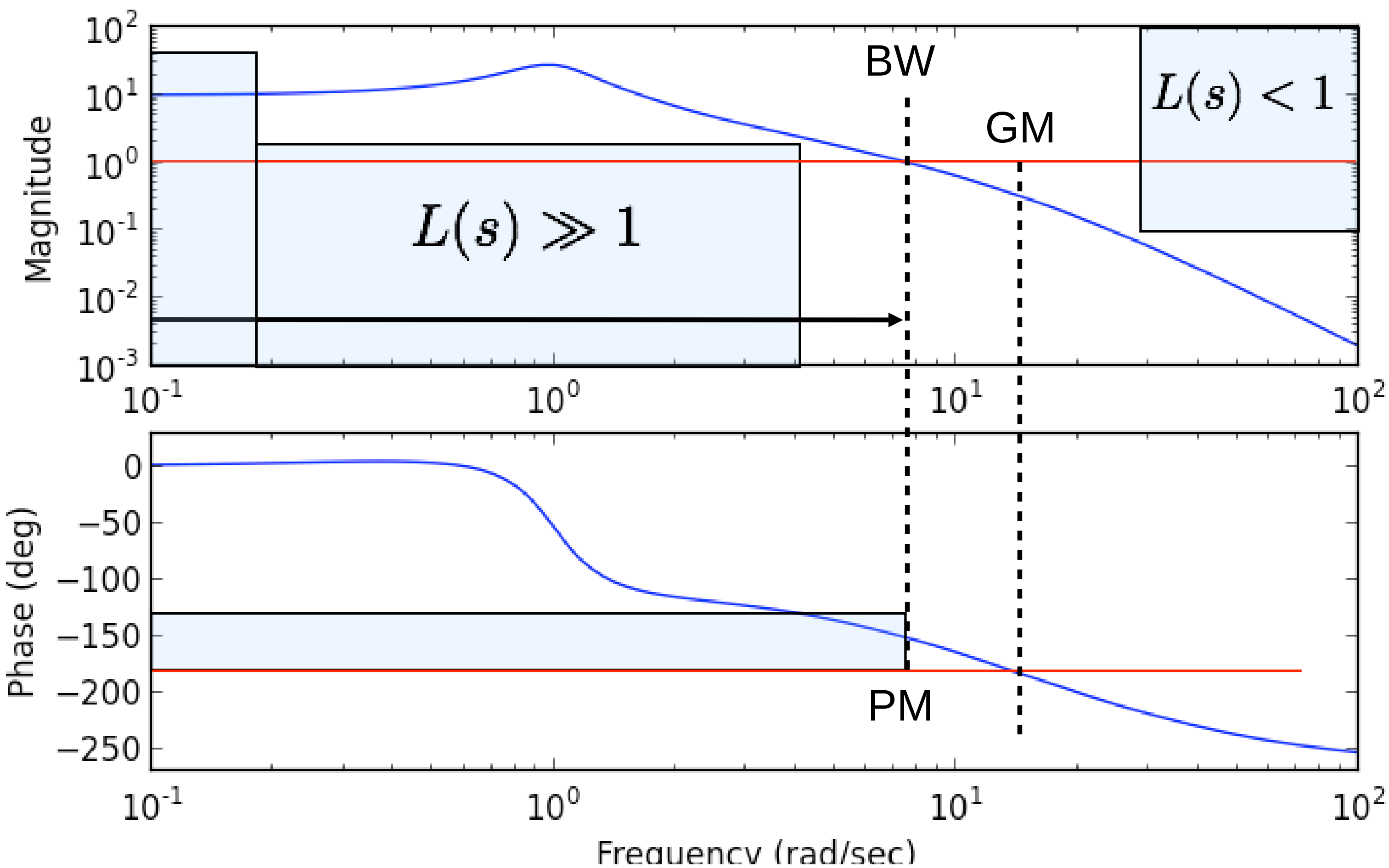
A Process**:** sequence of (nonunique) steps

1. Start with plant and performance specifications
2. If plant not stable, first stabilize it (e.g., PID)
3. Adjust/increase simple gains
   - Increase proportional gain for tracking error
   - Introduce integral term for steady-state error
   - Will derivative term improve overshoot?
4. Analyze/adjust for stability and/or phase margin
   - Adjust gains for margin
   - Introduce *Lead* or *Lag Compensators* to adjust phase margin at crossover and other critical frequencies
   - Consider PID if you haven't already

Summary: Loop Shaping

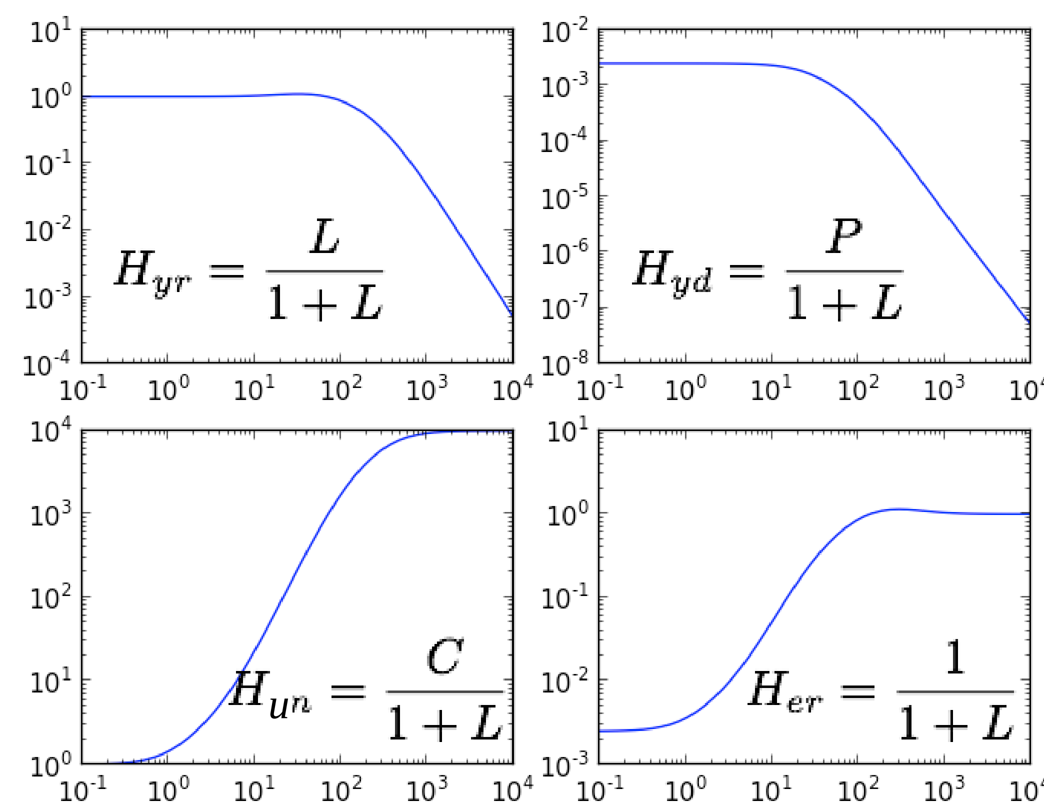
Loop Shaping for Stability & Performance

- Steady state error, bandwidth, tracking response
- Specs can be on any input/output response pair



Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI



Things to remember (for homework and exams)

- Always plot Nyquist to verify stability/robustness
- Check gang of 4 to make sure that noise and disturbance responses also look OK