CDS 101/110: Lecture 4.3
State Feedback

October 21, 2016

Goals:

- Clean up uncertainty from last lecture's example
- 2^{nd} –Order systems in Detail
- Introduce Control Gramian, and connect with Reachability

Reading:

- Åström and Murray, Feedback Systems 2e, Ch 7

Example #2: Predator prey

(From FBS Section 4.7)

(growth rate)

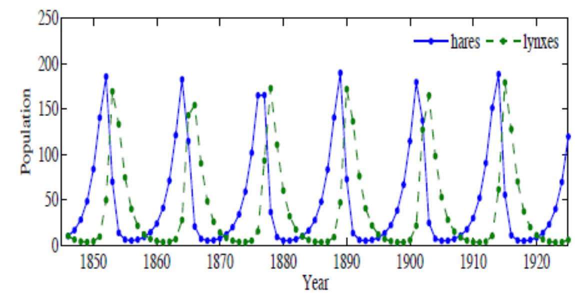
System dynamics

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k}\right) - \frac{aHL}{c+H}, \qquad H \geq 0,$$

$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0.$$

(prey consumption rate)

- Stable limit cycle with unstable equilibrium point at $H_e = 20.6$, $L_e = 29.5$
- Can we design the dynamics of the system by modulating the food supply ("u" in "$r + u$" term)

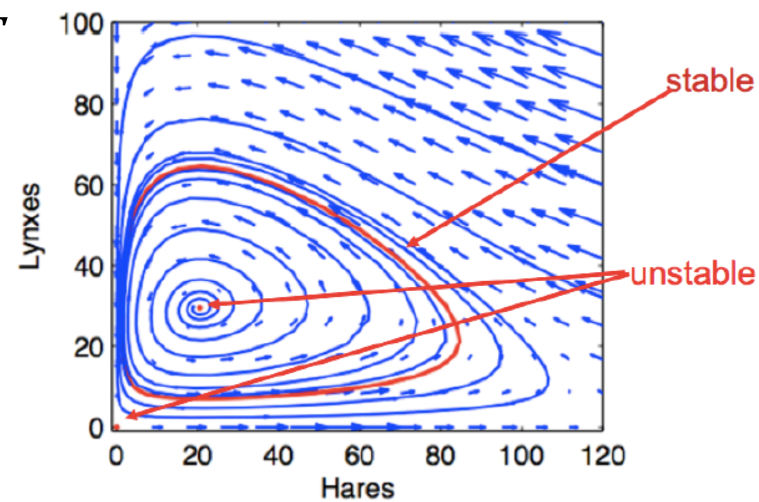


Q1: can we move from a given initial population of lynxes and rabbits to a specified one in time T by modulation of the food supply?

Q2: can we stabilize the lynx population around a desired equilibrium point (eg, L_d = ~30)?

- Try to keep lynx and hare population in check

Approach: try to stabilize using state feedback law



Example #2: Problem setup

Equilibrium point calculation

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL$$

- $x_e = (20.6, 29.5)$, $u_e = 0$, $L_e = 29.5$

```
f = inline('predprey(0, x)', 'x');
xeq = fsolve(f, [20, 30])'; He = xeq(1); Le = xeq(2);

% Generate the linearization around the eq point
App = [
  -((a*c*k*Le + (c + He)^2*(2*He - k)*r)/((c + He)^2*
  (a*b*c*Le)/(c + He)^2, -d + (a*b*He)/(c + He)
];
Bpp = [He*(1 - He/k); 0];

% Check reachability
if (det(ctrb(App, Bpp)) ~= 0) disp "reachable"; end
```

Linearization

- Compute linearization around equilibrium point, x_e:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} \qquad \frac{dx}{dt} \approx A\boxed{(x-x_e)} + B\boxed{(u-u_e)} + \text{higher order terms}$$

$$v = (u - u_e)$$
$$z = (x - x_e)$$

- Redefine local variables: $z = x - x_e$, $v = u - u_e$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{acL_e}{(c+H_e)^2} - \frac{2H_e r}{k} + r & -\frac{aH_e}{c+H_e} \\ \frac{abcL_e}{(c+H_e)^2} & \frac{abH_e}{c+H_e} - d \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_e\left(1 - \frac{H_e}{k}\right) \\ 0 \end{bmatrix} \boxed{v}$$

- Reachable? YES, if $a, b \neq 0$ (check [B AB]) ⇒ can locally steer to any point

Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{acL_e}{(c+H_e)^2} - \frac{2H_e r}{k} + r & -\frac{aH_e}{c+H_e} \\ \frac{abcL_e}{(c+H_e)^2} & \frac{abH_e}{c+H_e} - d \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_e\left(1 - \frac{H_e}{k}\right) \\ 0 \end{bmatrix} v$$

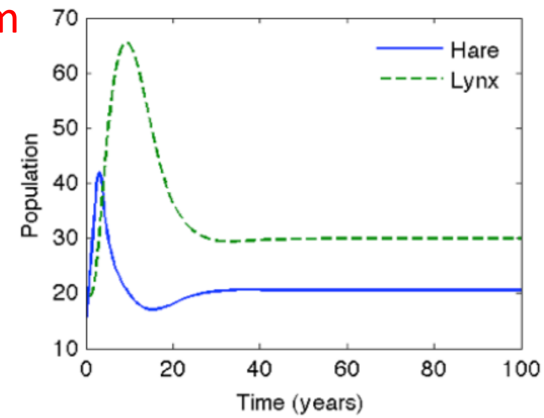
Control design: v is control input for *linearized* system

$$v = -Kz = -k_1(H - H_e) - k_2(L - L_e)$$

$$u = u_e + K(x - x_e)$$

$$v = (u - u_e)$$

$$z = (x - x_e)$$

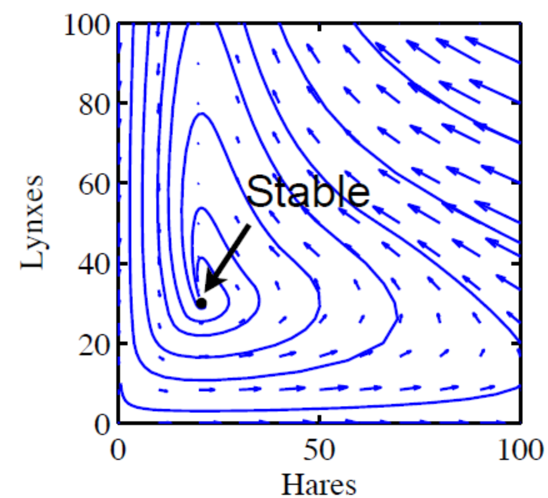


Place poles at stable values

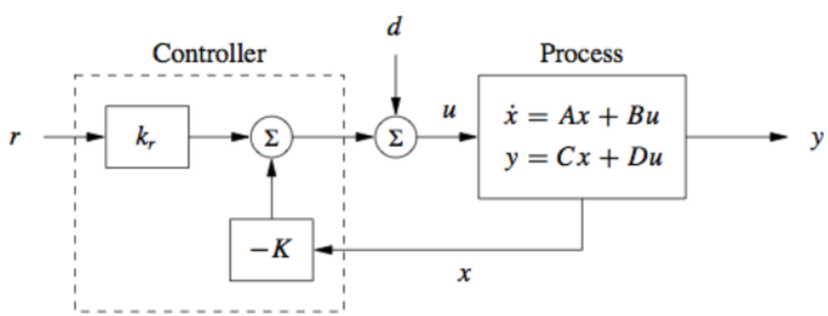
- Choose λ = -0.1, -0.2
- MATLAB: Kpp = place(App, Bpp, [-0.1; -0.2]);

Key principle: *design of dynamics*

- Use feedback to create a stable equilibrium point



More advanced: control to desired value $r = L_d$



Second Order Systems

General Form: $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \quad y = q$

- Convert to 1st-order from, with $z = [q \quad \dot{q}]^T$:

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \mathrm{z} + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u \qquad\qquad y = [1\ 0]z$$

- Roots of the characteristic polynomial are $\lambda_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$
  - Stable if $\zeta > 0$. Complex conjugates if $\zeta < 1$, real otherwise
  - Solution, and behavior, depends upon *damping ratio* ζ
  - ω_0 is the *natural frequency* of the system
  - $\omega_d = \omega_0\sqrt{\zeta^2 - 1}$ is the *damped frequency* of the system
- For convenience, introduce $\dot{x} = \begin{bmatrix} q & \frac{\dot{q}}{\omega_0} \end{bmatrix}^T$

$$\dot{x} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u \qquad\qquad y = [1\ 0]x$$

Second Order Systems

Behavior (homogeneous solution):

- $\zeta < 1$: *underdamped* (oscillatory behavior)

$$y(t) = e^{-\zeta\omega_0 t}\left(x_{10}\cos\omega_d t + \left(\frac{\zeta\omega_0}{\omega_d}x_{10} + \frac{1}{\omega_d}x_{20}\right)\sin\omega_d t\right)$$

- If $\zeta > 1$: *overdamped*

$$\alpha = \omega_0\left(\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$\beta = \omega_0(\zeta - \sqrt{\zeta^2 - 1})$$

$$y(t) = \frac{\beta x_{10} + x_{20}}{\beta - \alpha}e^{-\alpha t} - \frac{\alpha x_{10} + x_{20}}{\beta - \alpha}e^{-\beta t}$$

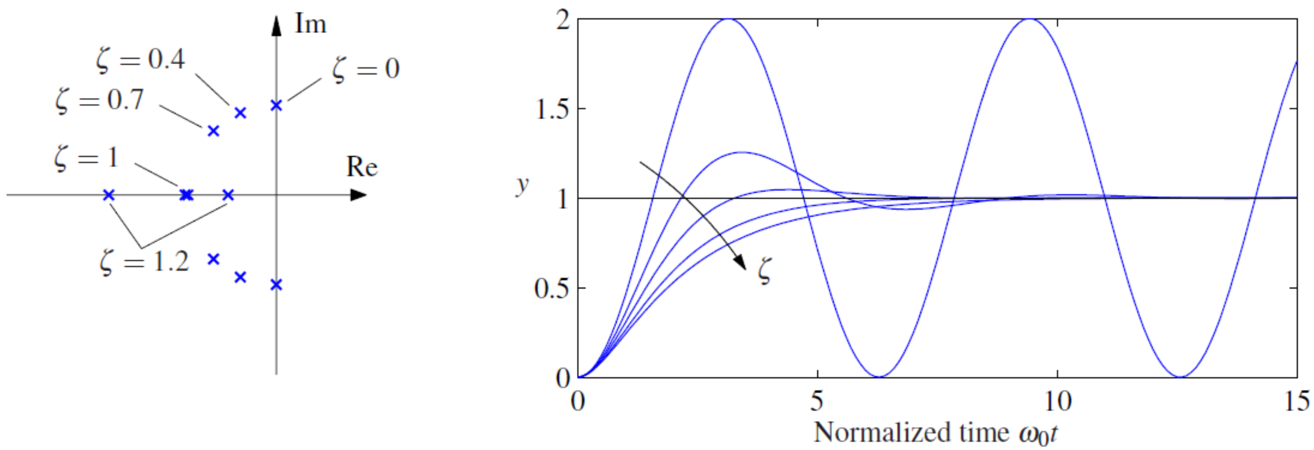
- If $\zeta = 1$: critically damped

$$y(t) = e^{-\zeta\omega_0 t}(x_{10} + (x_{20} + \zeta\omega_0 x_{10})t)$$

Second Order System Step Response

From the convolution Integral: $y(t) = \int_0^t Ce^{A(t-\tau)}Bd\tau$

$$y(t) = \begin{cases} \left(1 - e^{-\zeta\omega_0 t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin\omega_d t\right), & \zeta < 1; \\ \left(1 - e^{-\omega_0 t}(1+\omega_0 t)\right), & \zeta = 1; \\ \left(1 - \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2-1}} + 1\right) e^{-\omega_0 t(\zeta - \sqrt{\zeta^2-1})} \right. & \\ \left. \quad + \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2-1}} - 1\right) e^{-\omega_0 t(\zeta + \sqrt{\zeta^2-1})}\right), & \zeta > 1, \end{cases}$$



(a) Eigenvalues (b) Step responses

Second Order System Step Response

Maximum Overshoot:

- Find first response peak time (set $dy/dt = 0$), and then peak amplitude
- note that step response expression can be rearranged to

$$y(t) = k\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin(\omega_d t + \varphi)\right) \qquad \varphi = \cos^{-1}\zeta$$

$$\frac{dy(t)}{dt} = 0 = -\frac{e^{-\zeta\omega_o}}{\sqrt{1-\zeta^2}}[\omega_d\cos(\omega_d t^* + \varphi) - \zeta\omega_0\sin(\omega_d t^* + \varphi)]$$

Or: $\tan(\omega_d t^* + \varphi) = \frac{\omega_d}{\zeta\omega_0} = \frac{\omega_0\sqrt{1-\zeta^2}}{\zeta\omega_0} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\varphi \quad \rightarrow \quad \omega_d t^* = n\pi$

First peak at $t_{peak} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$; $y_{peak} = 1 - \frac{e^{\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\sin(\pi + \varphi)$

$$y_{peak} = 1 + e^{\pi\zeta/\sqrt{1-\zeta^2}} \qquad \rightarrow \qquad overshoot = e^{\pi\zeta/\sqrt{1-\zeta^2}}$$

Reachability

Review: For LTI control systems,

$$\dot{x} = Ax + Bu, \qquad x\epsilon\mathbb{R}^n,\ A\epsilon\mathbb{R}^{nxn},\ \ B\epsilon\mathbb{R}^{nxr}, u\epsilon\mathbb{R}^r$$
$$y = Cx + Du, \qquad y\epsilon\mathbb{R}^m,\ A\epsilon\mathbb{R}^{mxn},\ \ B\epsilon\mathbb{R}^{mxr}$$

reachability can be assessed from the rank of:

$$W_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

Some Analysis: $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

- ***Controllable*** if state can be driven to $x(T) = 0$ for any $x(0)$
  - i.e., $\exists\ u(t)$ s.t. $-e^{AT}x(0) = \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau$
  - i.e., $\exists\ u(t)$ s.t. $-x(0) = e^{-AT}\int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau = \int_0^T e^{-\tau}Bu(\tau)\,d\tau$
- ***Reachable*** if $x(0) = 0$ can be driven to any state $x_f = x(T)$ in time T
  - i.e. $\exists\ u(t)$ s.t. $x(T) = \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau$

Reachability

Discrete Approximation (for intuition)**:** For LTI control systems,

$$-x(0) = \sum_{i=1}^{N-1} L(\tau_i)u(\tau_i)\Delta$$

where $L(\tau_i) = e^{A(T-\tau_i)}B$

- Let $U = [u(\tau_1), u(\tau_2), \cdots, u(\tau_{N-1})]^T$; $\quad \mathcal{L} = [L(\tau_0)\Delta, L(\tau_1)\Delta, \cdots, L(\tau_{N-1})\Delta]$
- Then $-x(0) = \mathcal{L}U$
- A solution exists if $x(0)$ lies in the *range space* of $\mathcal{L}$. For reachability, where $x(0)$ can be arbitrary, $\mathcal{L}$ must be full rank. $\mathcal{L}$ is full rank if the following matrix is full rank:

$$\mathcal{L}\mathcal{L}^T$$

More Formally: Linear independence of N *functions* $l_i(t), i = 1, \ldots, N$ over interval $[t_o, t_f]$is determined using a *Gramian:*

$$G = [G_{ij}], \qquad G_{ij} = \int_{t_0}^{t_f} l_i(\tau)l_j(\tau)\, d\tau$$

Linear independence is proven when G has full rank

Controllability

Controllability Gramian:

$$C(t_0, t_1) = \int_{t_0}^{t_f} e^{A(t_0-\tau)} B B^T e^{A^T(t_0-\tau)} d\tau \quad \rightarrow \quad C(0, t_f) = \int_0^{t_f} e^{-A\tau} B B^T e^{-A^T\tau} d\tau$$

Since $C(0, t_f)$ is symmetric, for it to be full rank over $[0, t_f]$, it must be positive definite.

Lemma: $C(0, t_f)$ is positive definite *if and only if* there is no vector $v \neq 0$ such that

$$v^T e^{-At} B = 0 \quad \forall t \epsilon [0, t_f]$$

Proof (by contradiction)**:** suppose there is such a v with $v^T e^{-At} B = 0 \quad \forall t \epsilon [0, t_f]$

- $v^T C(0, t_f) v = \int_0^{t_f} v^T e^{-A\tau} B B^T e^{-A^T\tau} B v \; d\tau$
- If there is such a v, then $v^T C(0, t_f) v = 0$, which implies that $C(0, t_f) v$ is not positive definite.

Theorem: The pair (A,B) is controllable if and only if the $C(0, t_f)$ is positive definite

Proof (sufficiency)**:** suppose $C(0, t_f)$ is positive definite. Let x_0, x_f be the initial/final states

- $x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{-A(t_f-\tau)} B \, u(\tau) \, d\tau$

Controllability

Proof (sufficiency)**:** *(continued)*

- Choose $u(t) = B^T e^{-A^T t} C^{-1}(0, t_f) v$ for some constant vector v
- Then: $x(t_f) = e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B\, B \wedge T e^{-A^T \tau} C^{-1}(0, t_f) v\, d\tau$
  $= e^{At_f} x_0 + e^{At_f} C(0, t_f) C^{-1}(0, t_f) v$
  $= e^{At_f} (x_0 + v)$
- If $v = -x_0 + e^{-At_f} x_f$, then $x(t_f) = x_f$

That is, $u(t) = B^T e^{-A^T t} C^{-1}(0, t_f)[e^{-At_f} x_f - x_0]$ steers x_0 to x_f for any x_0, x_f

Proof (necessity)**:** show that positive definiteness of $C(0, t_f)$ is necessary

- Contradiction: suppose $C(0, t_f)$ is not positive definite.
- Then there exists $z \neq 0$ such that $z^T e^{-At_f} B = 0 \quad \forall t \epsilon [0, t_f]$
- For ***controllability***, let $\mathrm{x}_0 = z$**.** Suppose that $x(t_f) = 0$
  - Then: $0 = e^{At_f} z + \int_0^{t_f} e^{A(t_f-\tau)} B\, u(\tau)\, d\tau$
  - Multiply by $z^T e^{-At_f}$: $0 = z^T z + \int_0^{t_f} z^T e^{A\tau} B\, u(\tau)\, d\tau$
  - But integrand is zero for all t, and thus $z = 0$, a contradiction

Controllability/Reachability

Proof (necessity)**:** *(continued)*

- For ***reachability***, let $\mathrm{x_f} = e^{At_f}z$, and suppose $u(t)$ steers x_0 to $x\left(t_f\right) = x_f$
  - Then: $e^{At_f}z = \int_0^{t_f} e^{A\left(t_f-\tau\right)}B\,u(\tau)\,d\tau$
  - Multiply by $z^Te^{-At_f}$: $z^Te^{-At_f}e^{At_f}z = \int_0^{t_f} z^Te^{-A\tau}B\,u(\tau)d\tau = z^Tz$
  - But, if $C(0,t_f)$ is not positive definite, then there exists z such that $z^Te^{-At_f}B = 0 \ \ \forall t\epsilon[0,t_f]$, implying that $z = 0$, which is a is a contradiction.

Theorem: $C(0,t_f)$ is positive definite only if $rank(W_r) = n$, where

$$W_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

Proof: If $C(0,t_f)$ is not positive definite, there exists $z \neq 0$ s. t. $z^Te^{-At_f}B = 0, \forall t\epsilon[0,t_f]$

- $z^T\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}A^kB = 0, \forall t\epsilon[0,t_f]$
- Same as $\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}z^TA^kB = 0, \forall t\epsilon[0,t_f]$
- This implies that there exists z such that $z^TA^kB = 0$ for all $k = 0,1, \ldots$

Controllability/Reachability

Aside: Cayley-Hamilton Theorem

- Let A be an $n \times n$ matrix.
- Let $\lambda_A(s) = \det(sI - A) = s^n + a_1\, s^{n-1} + \cdots + a_{n-1}s + a_n$ be characteristic poly.
- A satisfies its own characteristic polynomial: $A^n + a_1\, A^{n-1} + \cdots + a_{n-1}A + a_n I = 0$
  - Hence, A^k for $k \geq n$ are linear combinations of $I, A, \cdots, A^{n-1}$

Proof: (*continued)*

- $\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} z^T A^k B = 0, \forall t \epsilon [0, t_f]$ implies via Cayley-Hamilton that

$$z^T A^k B = 0 \ \text{ for } k = 0, \ldots, n-1$$

- Hence, $z^T \left[B \ AB \ A^2 B \cdots A^{n-1} B \right] = 0$, which implies that W_r is not full rank.
- Therefore, (A,B) is reachable (controllable) only if W_r is full rank n

Note: in LTI case, reachability is independent of time.