CDS 101/110: Lecture 7.1
Loop Analysis of Feedback Systems

November 7 2016

Goals:

- Introduce concept of “loop analysis”
- Show how to compute closed loop stability from open loop properties
- Describe the Nyquist stability criterion for stability of feedback systems
- Introduce Nyquist Diagram
- First look at *gain margin* and *phase margin*

Reading:

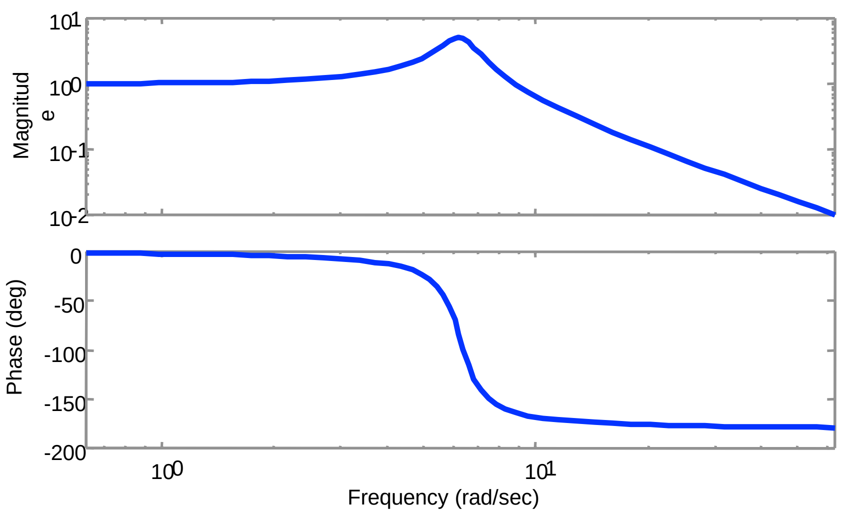
- Åström and Murray, Feedback Systems, Chapter 10, Sections 10.1, 10.2

Review From Last Week

$$u = A\sin(\omega t)$$

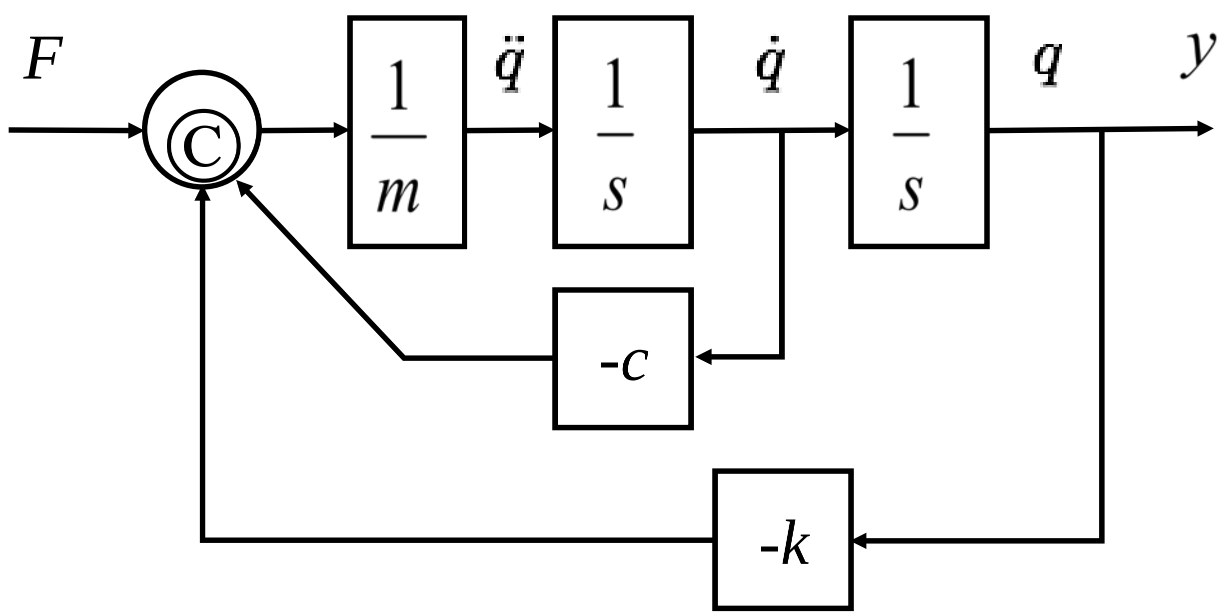
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

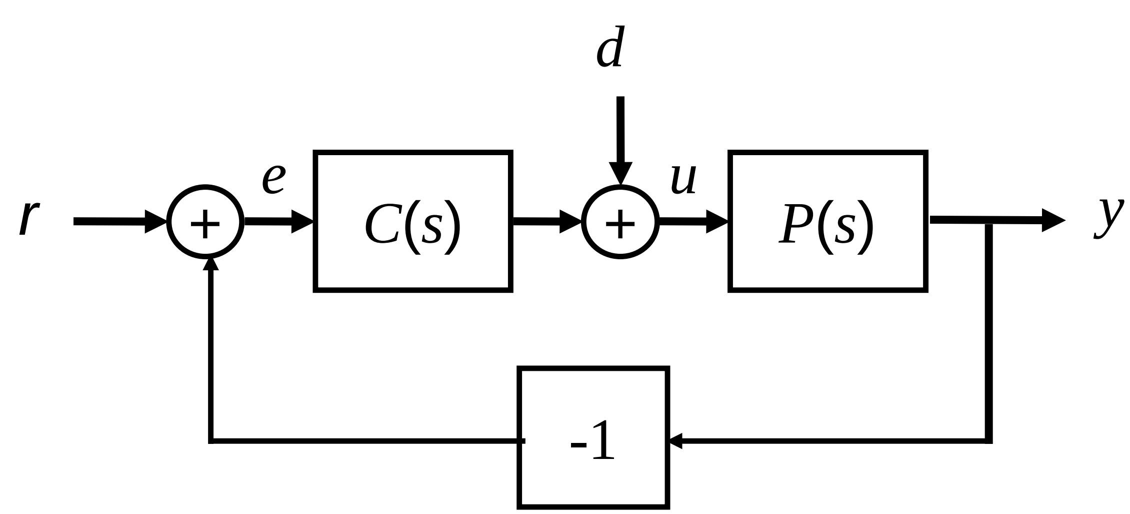
$$y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$$



$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2 u_1} = G_{y_2 u_2} G_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$





Bode Plot Units

What are the units of a Bode Plot?

- **Magnitude:** The ordinate (or "y-axis") of magnitude plot is determined by $20\log_{10}|G(i\omega)|$
  - Decibels," names after A.G. Bell
- **Phase:** Ordinate has units of degrees (of phase shift)
- The abscissa (or "x-axis") is $\log_{10}(\text{frequency})$ (usually, rad/sec)

Example: simple first order system: $G(s) = \frac{1}{1+\tau s}$

- Single pole at $s = -1/\tau$
- $|G(i\omega)| = \left|\frac{1}{1+i\tau\omega}\right| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$
- In decibels:

$$20\log_{10}|G(i\omega)| = 20\log_{10}1 - 20\log_{10}\left(1+(\omega\tau)^2\right)^{\frac{1}{2}}$$
$$= -10\log_{10}(1+(\omega\tau)^2)$$

Bode Plot Units (continued)

Example (*continued*)**:** simple first order system: $G(s) = \frac{1}{1+\tau s}$

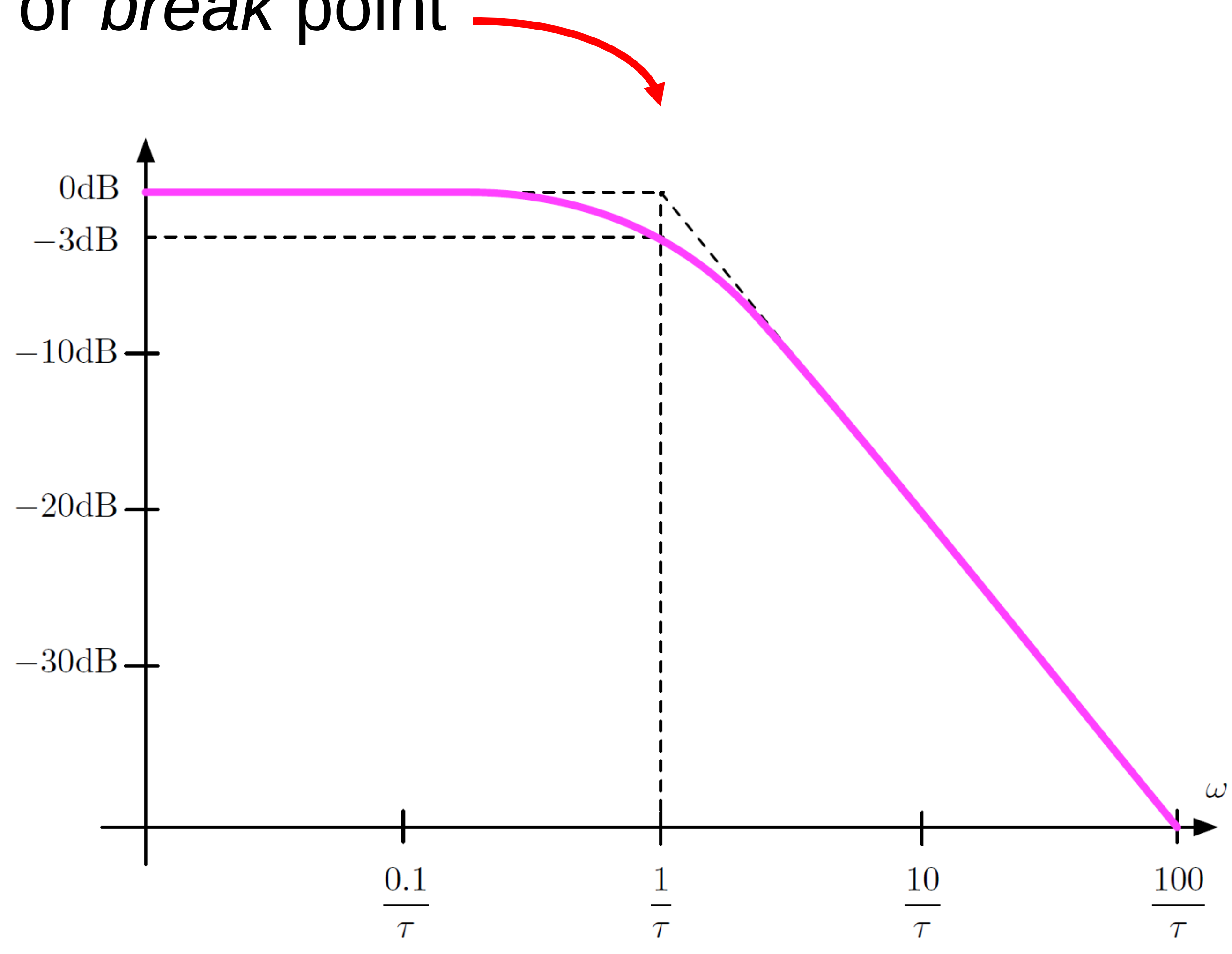
- Behavior of magnitude in decibels:

$$20\log_{10}|G(i\omega)| \approx \begin{cases} 0 & \omega \ll 1/\tau \\ -10\log_{10} 2 & \omega = 1/\tau \\ -20(\log_{10}\omega + \log_{10}\tau) & \omega \gg 1/\tau \end{cases}$$

- $\omega_{3\text{dB}} = 1/\tau$ is the -3dB *half-power* or *break* point
  - Precisely:

$$-10\log_{10}(2) = -3.0103\ \text{dB}$$

- Unit DC gain (0dB)
- Magnitude decreases at 20 dB/decade for $\omega \gg 1/\tau$



Bode Plot Units(continued)

Example (*continued*)**:** simple first order system: $G(s) = \frac{1}{1+\tau s}$

- Phase (argument) of transfer function:

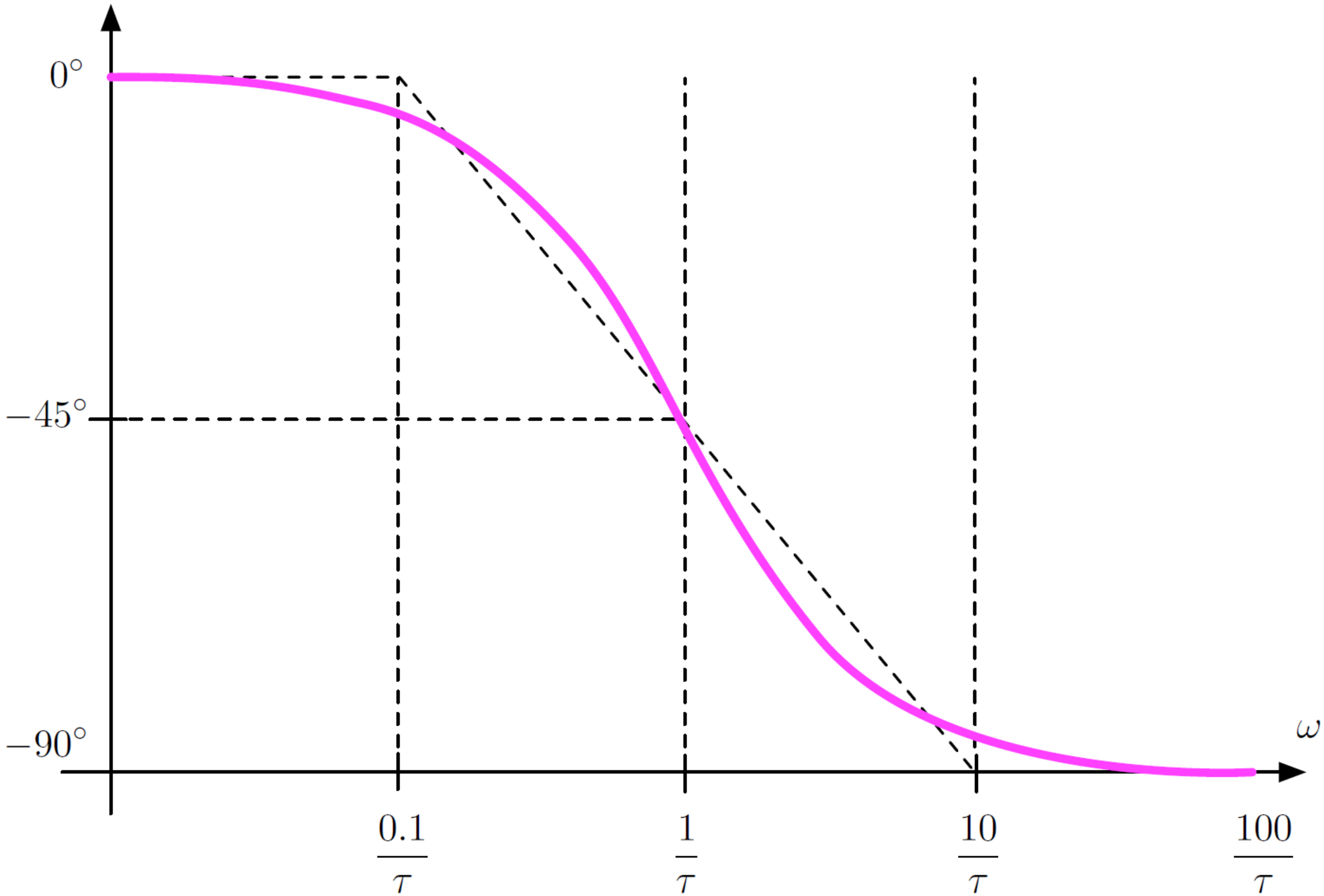
$$\angle G(i\omega) = \angle\left(\frac{1}{1+i\omega\tau}\right)$$

$$= \angle 1 - \angle(1 + i\omega\tau) = -\arctan(\omega\tau)$$

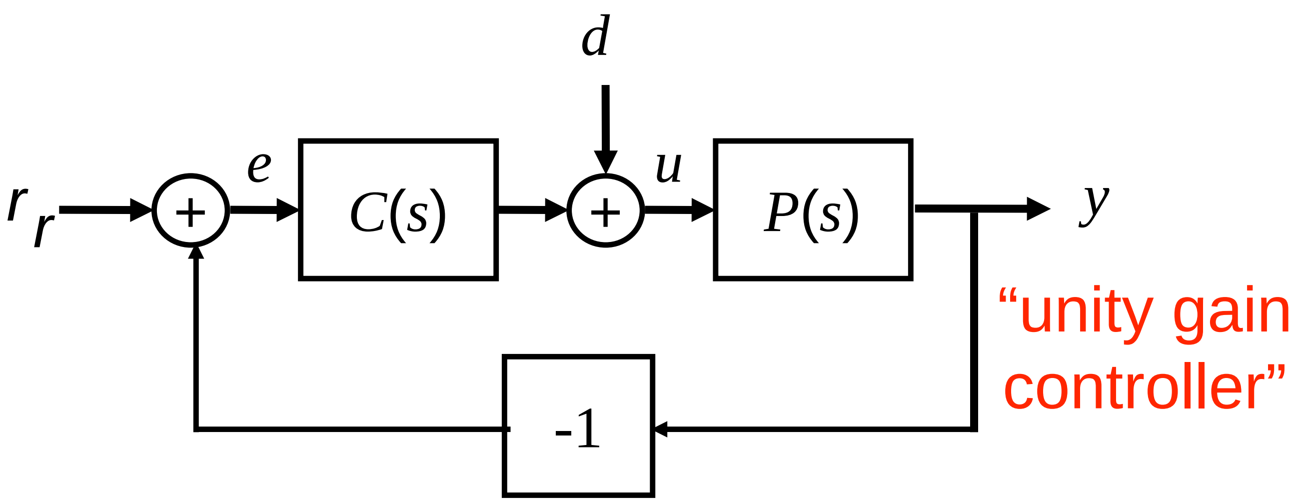
- Asymptotic Approximation

$$-\arctan(\omega\tau) \approx \begin{cases} 0 & \omega < 0.1/\tau \\ -\frac{\pi}{4}(1 + \log_{10}\omega + \log_{10}\tau) & 0.1\tau \leq \omega \leq 10/\tau \\ -\frac{\pi}{2} & \omega > 10/\tau \end{cases}$$

Bode Plot (continued)



Loop Analysis



First consider "simple" unity feedback

Performance? Trace how sinusoidal signals propagate around the closed loop system.

Does signal grow or decay?

- Can be determined from frequency response.

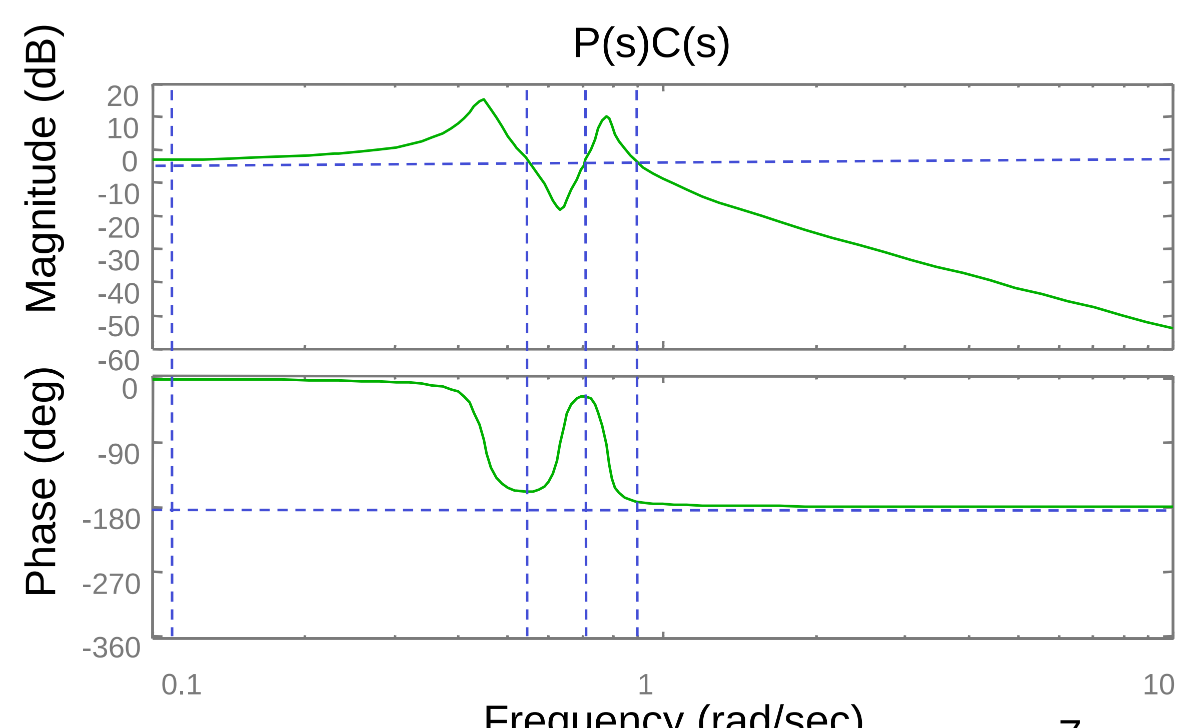
How do open loop dynamics effect closed loop dynamics?

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of 1 + PC

Alternative: look for conditions on *PC* that lead to instability

- E.g. : if $PC(s) = -1$ for some $s = i\omega$, then system is *not* asymptotically stable
- Condition on *PC* is useful because we can *design PC*(*s*) by choice of *C*(*s*)
- However, checking $PC(s) = -1$ is not enough; need more sophisticated check

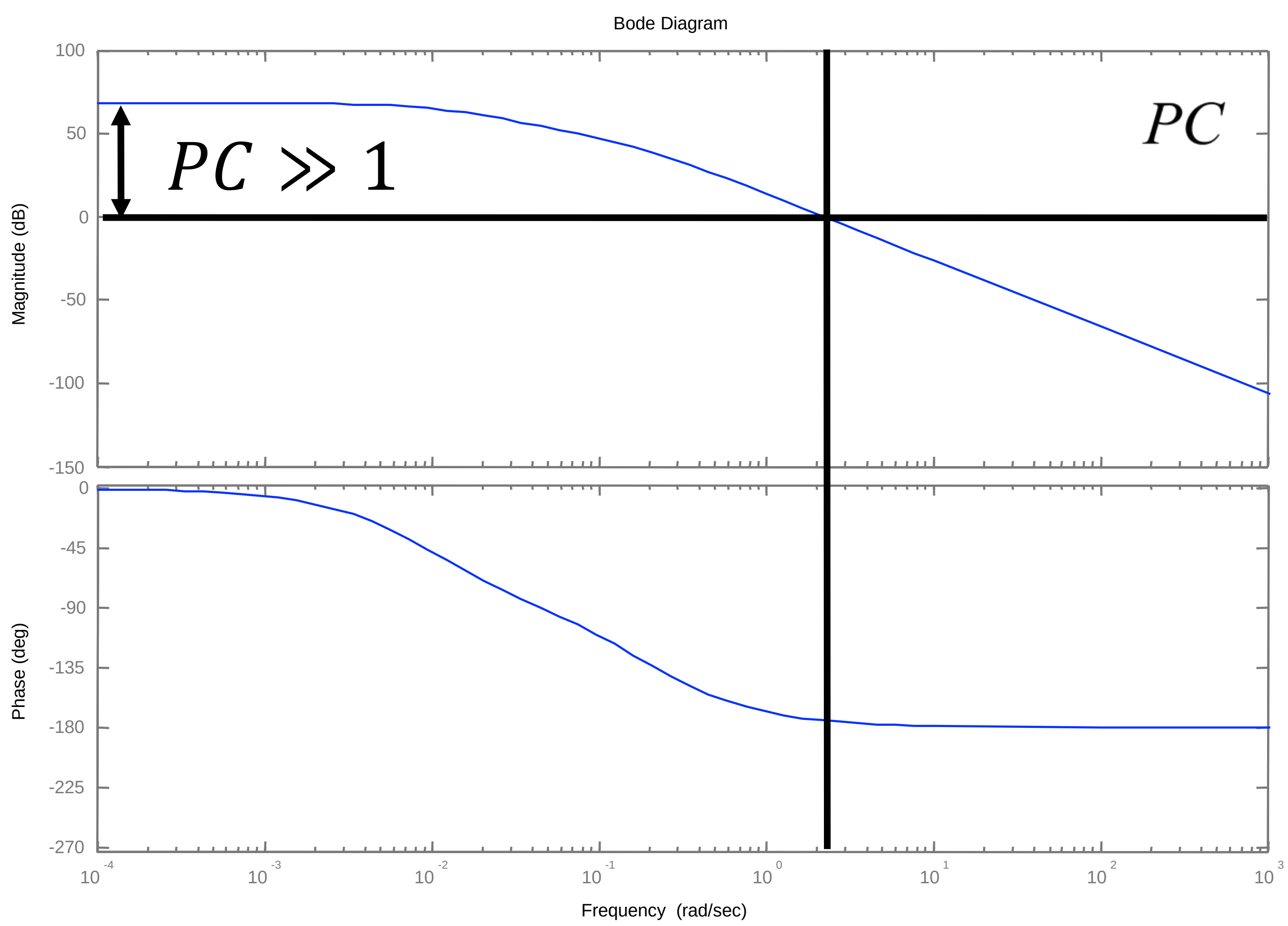


Game Plan: Frequency Domain Design

Goal: figure out how to *design* *C*(*s*) so that 1+*C*(*s*)*P*(*s*) is stable *and* we get good performance+

$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of 1 + *PC*
- Would also like to “shape” H_{yr} to specify performance at differenct frequencies



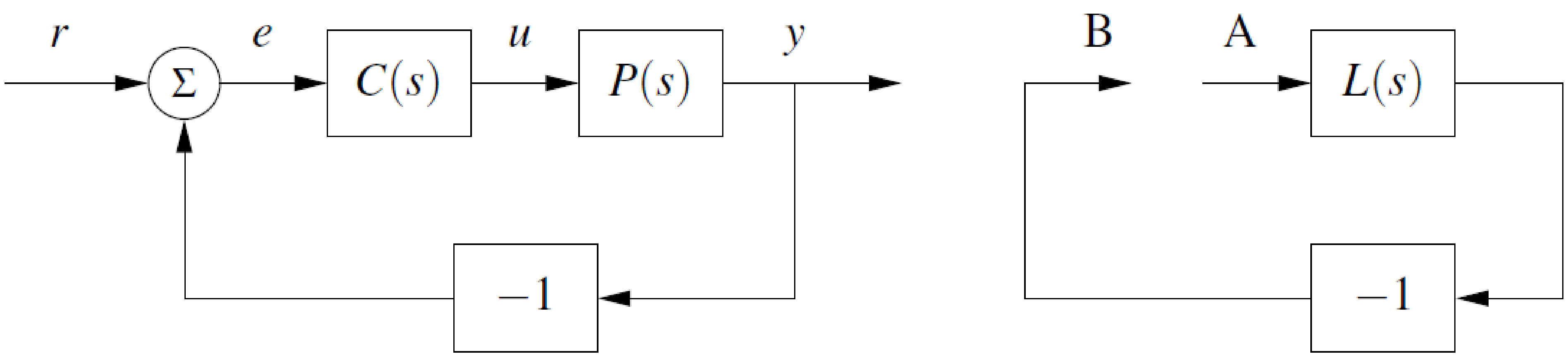
- Low frequency range:

$$PC \gg 1 \quad \Rightarrow \quad \frac{PC}{1+PC} \approx 1$$

(good tracking)

- **Bandwidth:** frequency at which closed loop gain = $\frac{1}{\sqrt{2}}$
  ⇒ open loop gain ≈ 1
- **Idea:** use *C*(*s*) to *shape PC* (under certain constraints)
- **Need** tools to analyze stability and performance for closed loop given *PC*

Nyquist Criterion: Warm up



Let the "loop transfer function" be

$$L(s) = P(s)C(s)$$

- Inject sinusoid of frequency ω_0 at pt. A.
- Signal at pt. B has frequency ω_0
- Oscillatory signal is self-maintaining if signal at B is same as signal at A.
- This can occur if there is a frequency ω_0 such

$$L(i\omega_0) = -1.$$

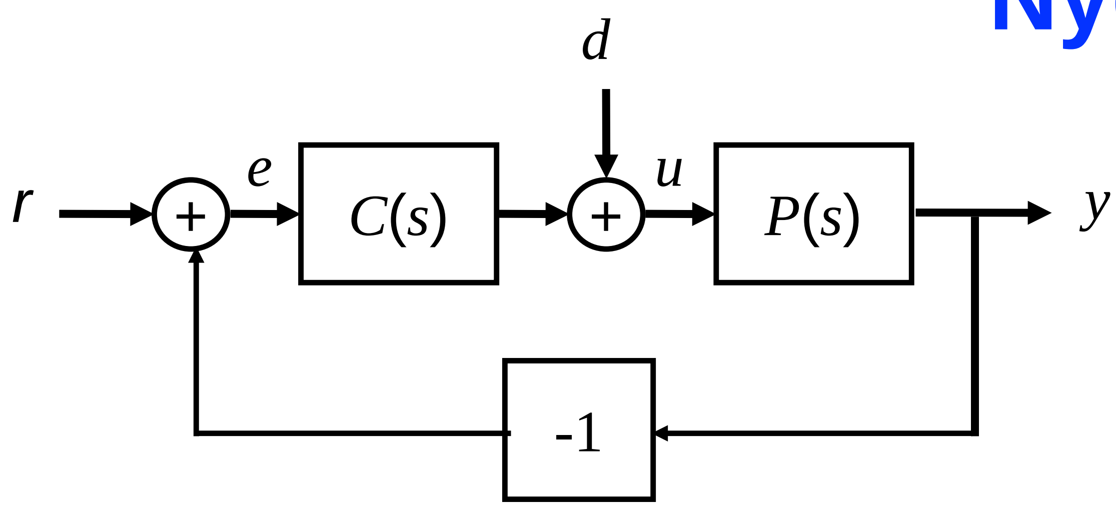
- **"critical point":** when loop transfer function =-1

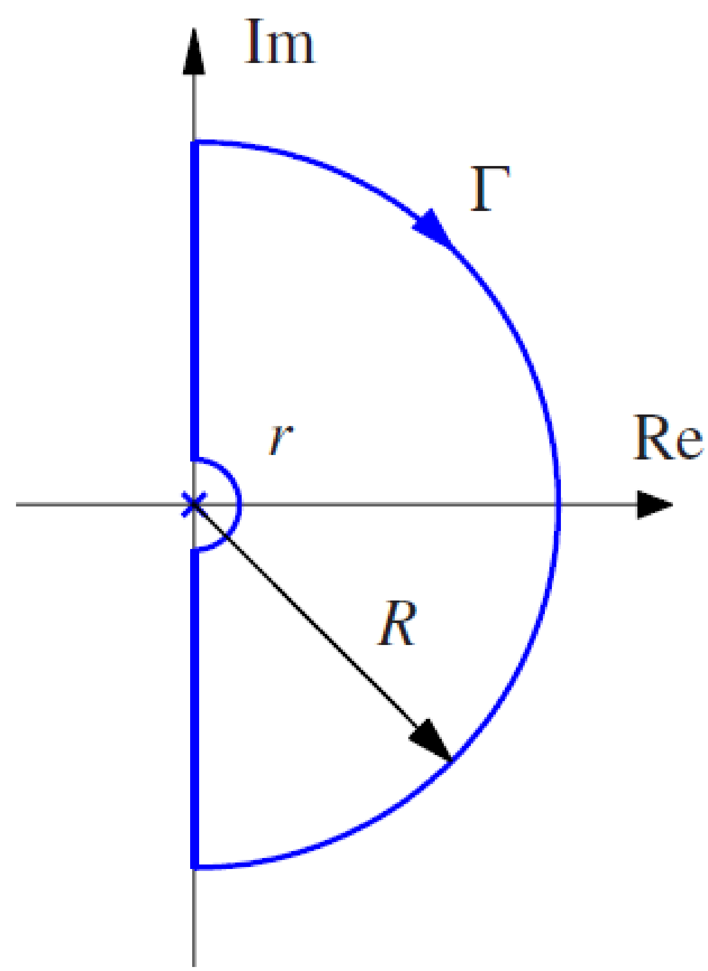
Naïve stability idea:

$$|L(\mathrm{i}\omega)| \leq 1$$

- Amplitude of signal at B is less than amplitude of injected signal at A.
- Reality is a bit more complicated

Nyquist Plot





- Nyquist "D" contour
- Take limit as $r \to 0$, $R \to \infty$
- Trace from −1 to +1 along imaginary axis

A different representation of frequency response of open loop transfer function, L(s) = P(s)C(s).

- Formed by tracing s around the Nyquist "D contour," Γ

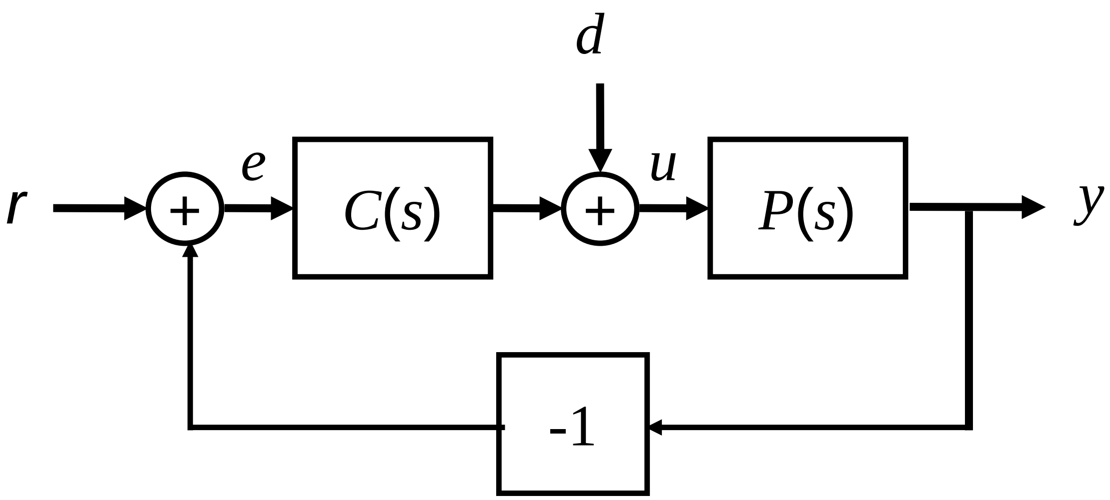
Nyquist Contour (Γ)**:**

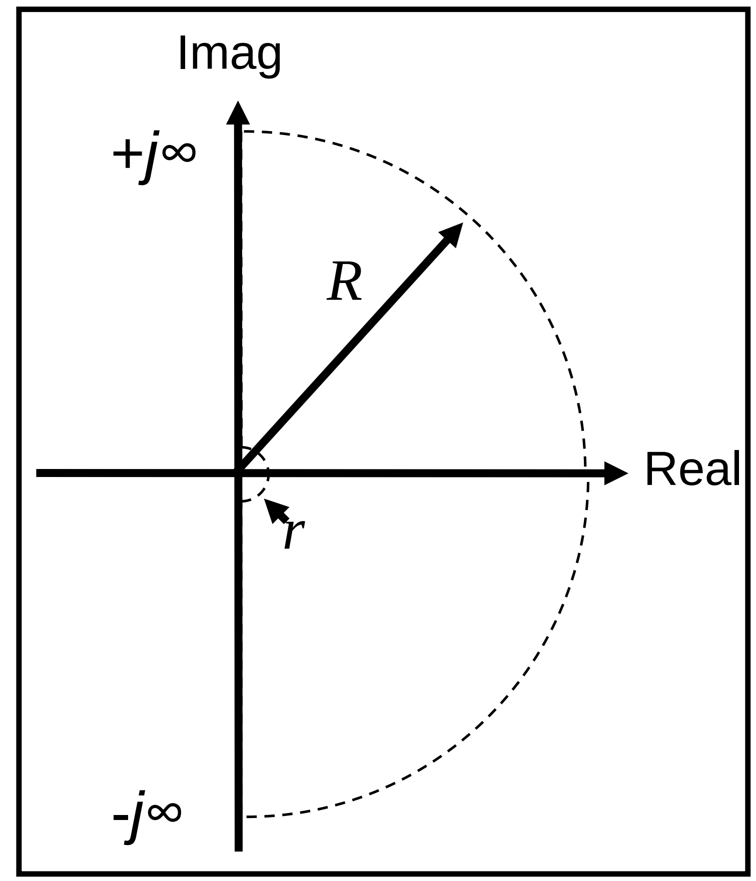
- Imaginary axis
- Semi-Circle, or arc, at infinity that connects endpoints of imaginary axis
- The image of $L(s)$ as s traverses Γ is the Nyquist plot
- Note, portion of plot corresponding to $\omega < 0$ is mirror image of $\omega > 0$

Nyquist Contour (Γ)**:**

- If pole of $L(s)$ on $j\omega$-axis, then create small semi-circular "detour" around the pole in RHP.
- Take limit as semi-circle radius $\to 0$
- Goal: from complex analysis, we're trying to find number of excess zeros in RHP, which leads to instability

Nyquist Criterion





- Nyquist “D” contour
- Take limit as $r \to 0$, $R \to \infty$
- Trace from −1 to +1 along imaginary axis

Determine stability from (open) loop transfer function, L(s) = P(s)C(s).

- Use “principle of the argument” from complex variable theory (see reading)

Thm (Nyquist). Consider the Nyquist plot for loop transfer function $L(s)$. Let

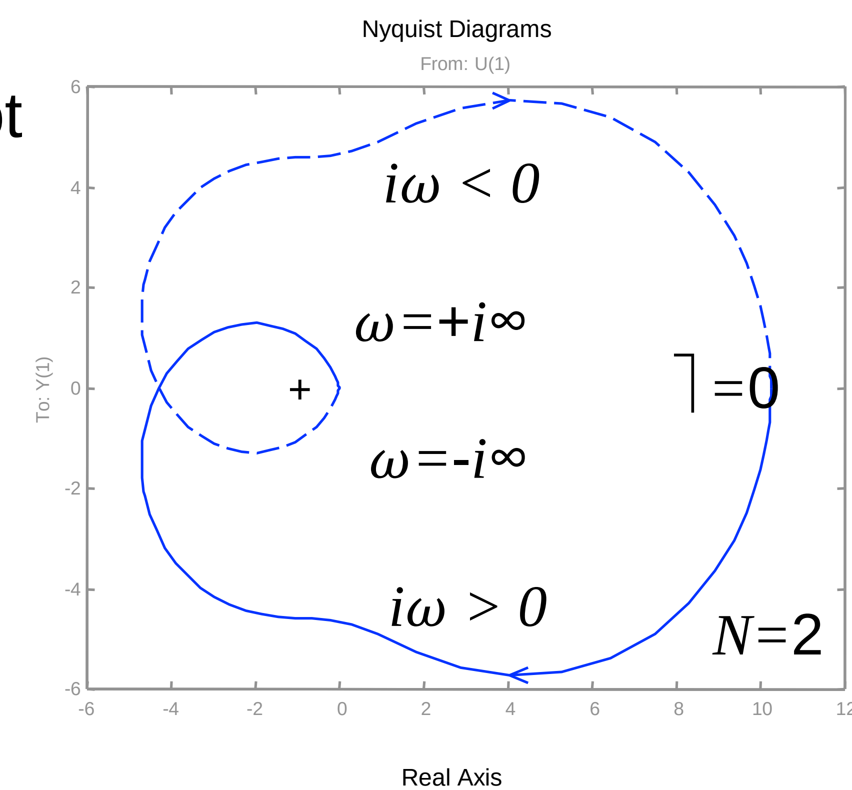
P # RHP poles of $L(s)$

N # clockwise encirclements of -1

Z # RHP zeros of $1 + L(s)$

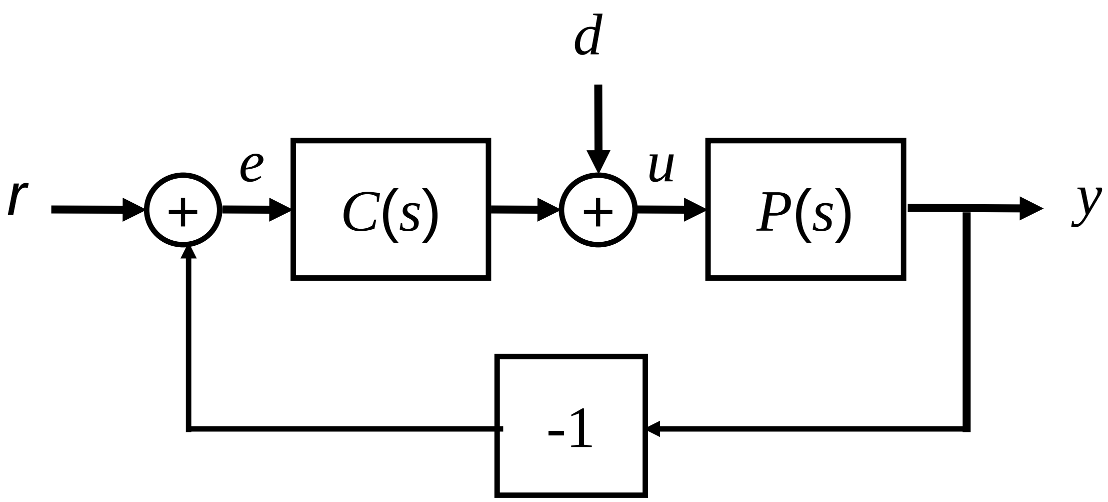
Then

$$Z = N + P$$



- Trace frequency response for $L(s)$ along the Nyquist “D” contour
- Count net # of clockwise encirclements of the -1 point

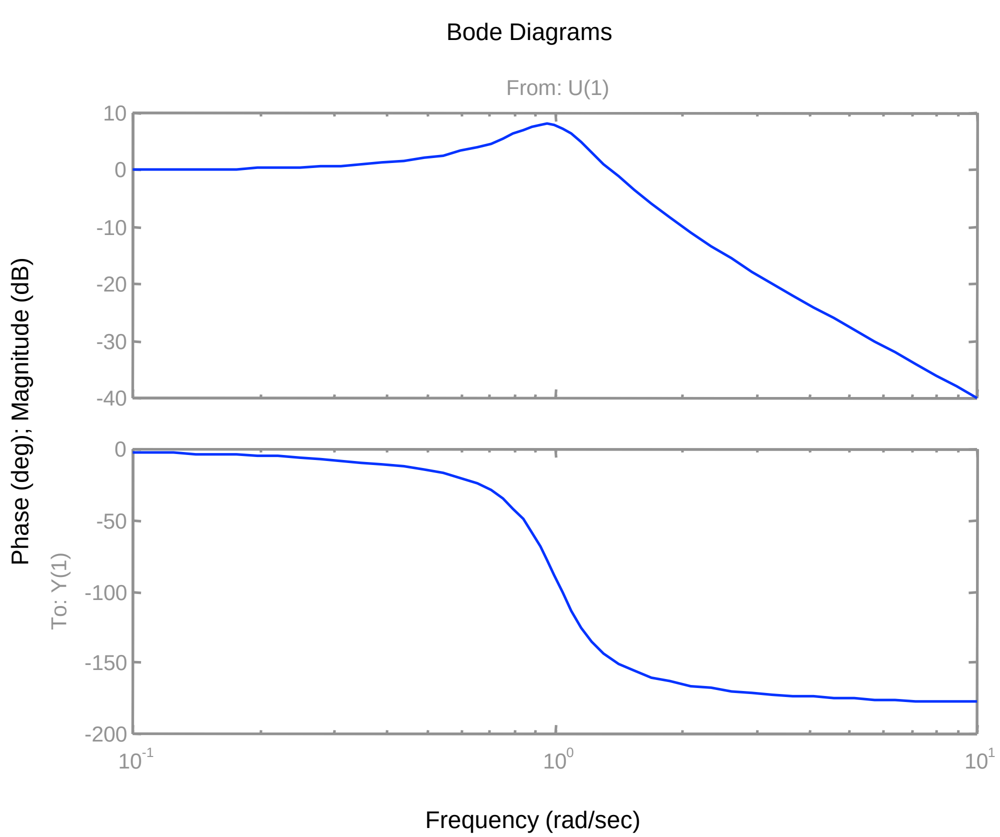
Simple Interpretation of Nyquist

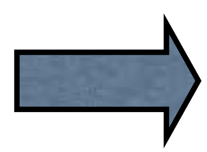


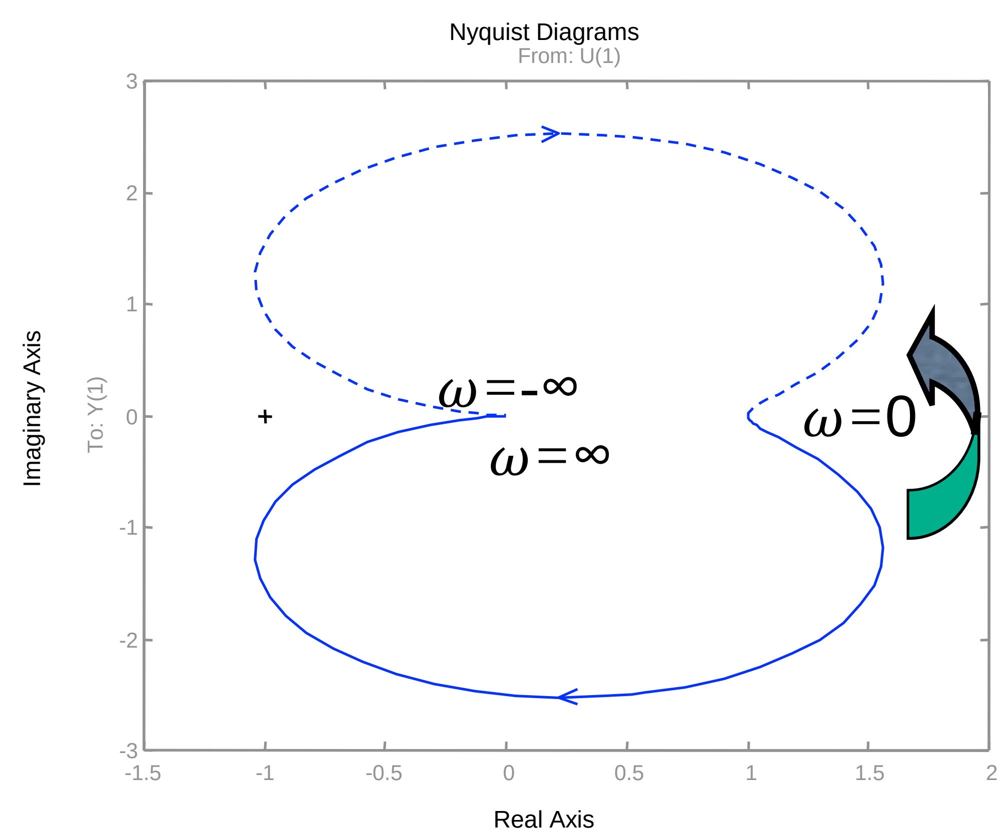
Basic idea: avoid positive feedback

- If L(s) has 180˚ phase (or greater) and gain greater than 1, then signals are amplified around loop
- Use when phase is monotonic
- General case requires Nyquist

Can generate Nyquist plot from Bode plot + reflection around real axis



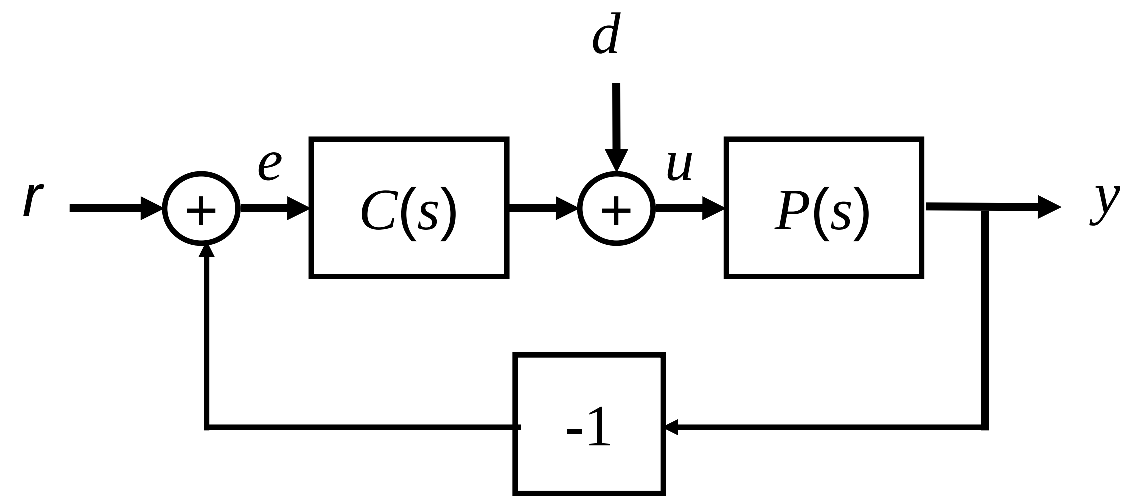


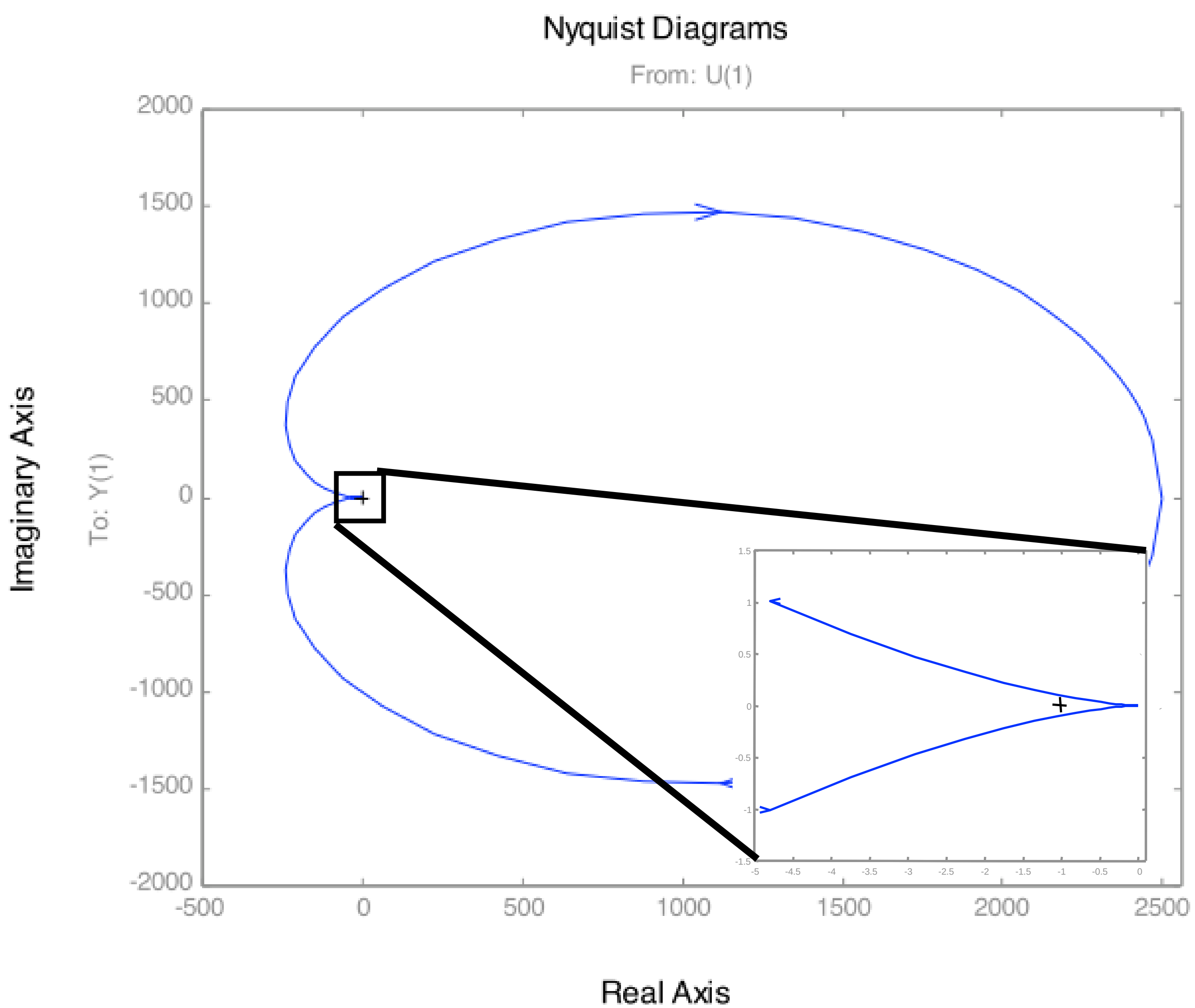


ambode(sys) [or bode(sys) in dB]

amnyquist(sys)

Example: Proportional + Integral* speed controller





$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

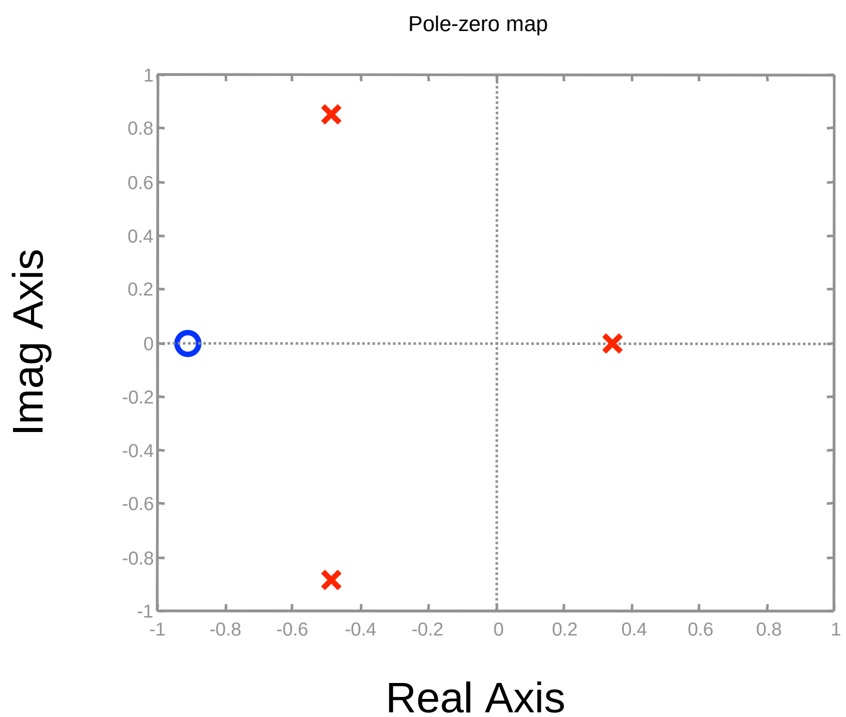
Remarks

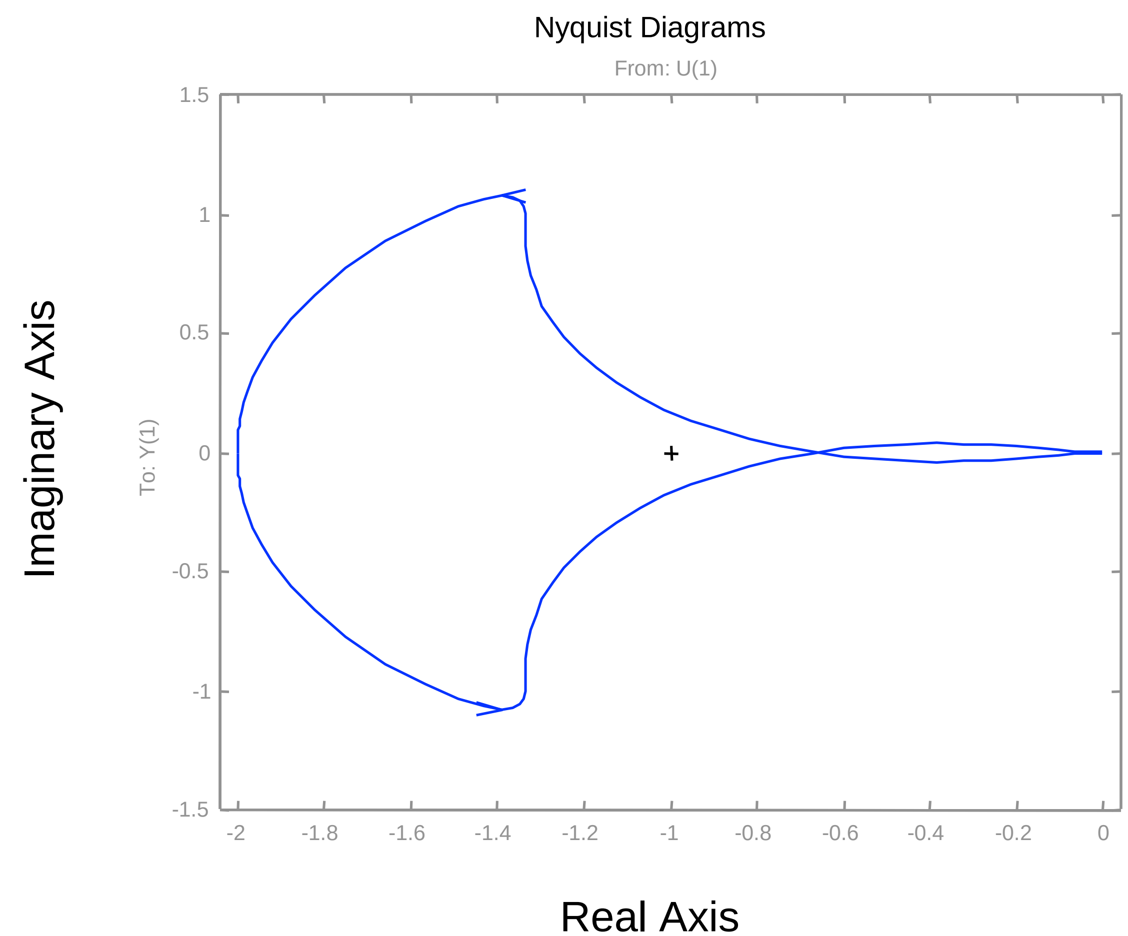
- N = 0, P = 0 ⇒ Z = 0 (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

More complicated systems

What happens when open loop plant has RHP poles?

- 1 + PC has singularities inside D contour ⇒ these must be taken into account





$$L(s) = \frac{s+1}{s-0.5} \times \frac{1}{s^2+s+1}$$

unstable pole

N = -1, P = 1 ⇒ $Z = N+P$ = 0 (stable)

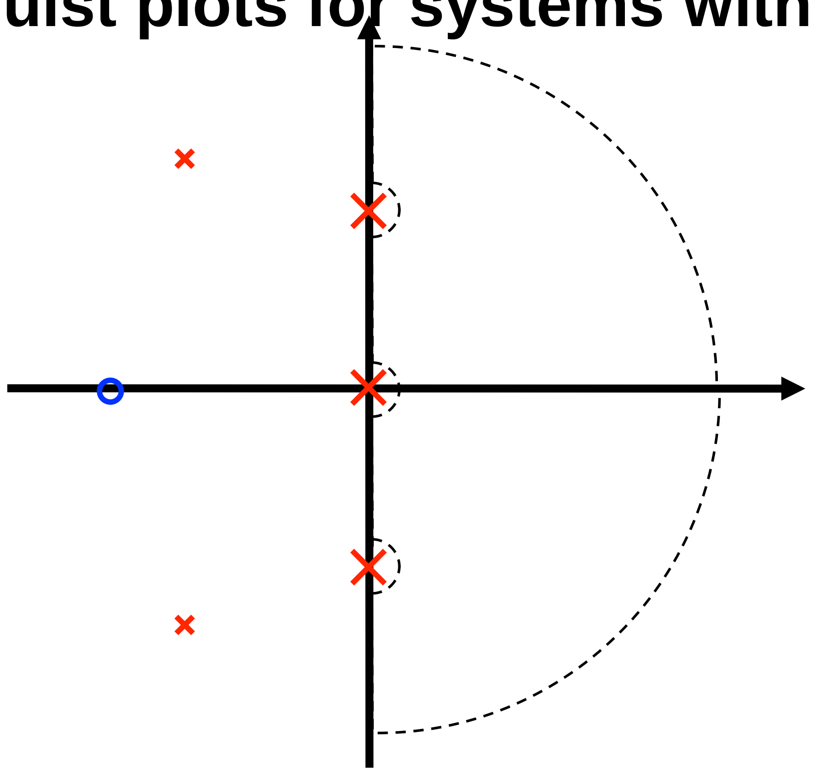
$$\frac{1}{1+L} = \frac{s+1}{(s+0.35)(s+0.07+1.2j)(s+0.07-1.2j)}$$ ✓

Comments and cautions

Why is the Nyquist plot useful?

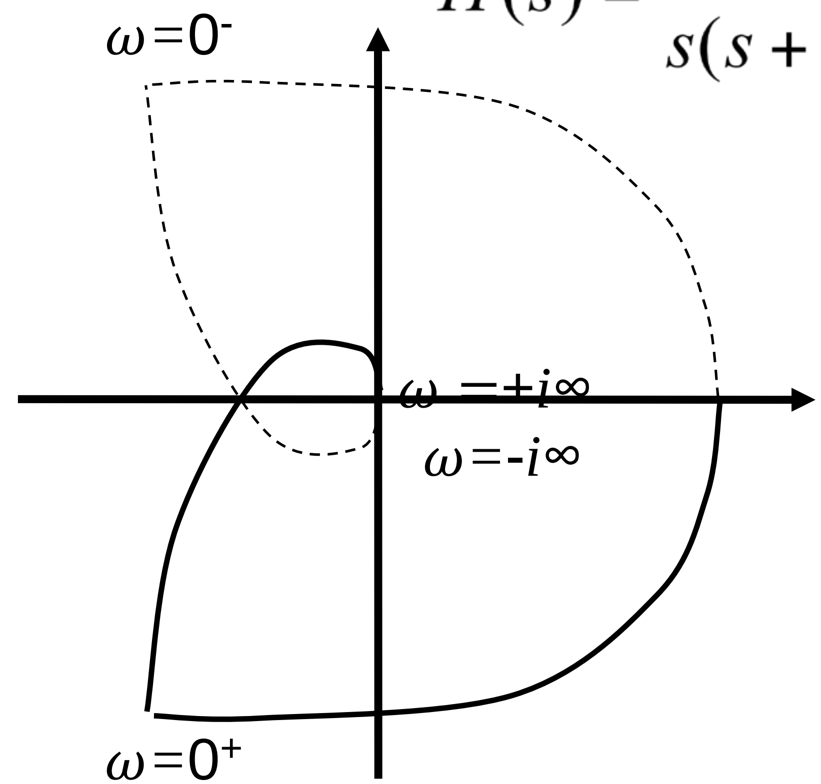
- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives insight into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the jω axis



- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\varepsilon+0i)$ to determine direction

$$H(s) = \frac{1}{s(s+1)^2}$$



Cautions with using MATLAB

- MATLAB doesn't generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)

Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not how stable

Gain margin

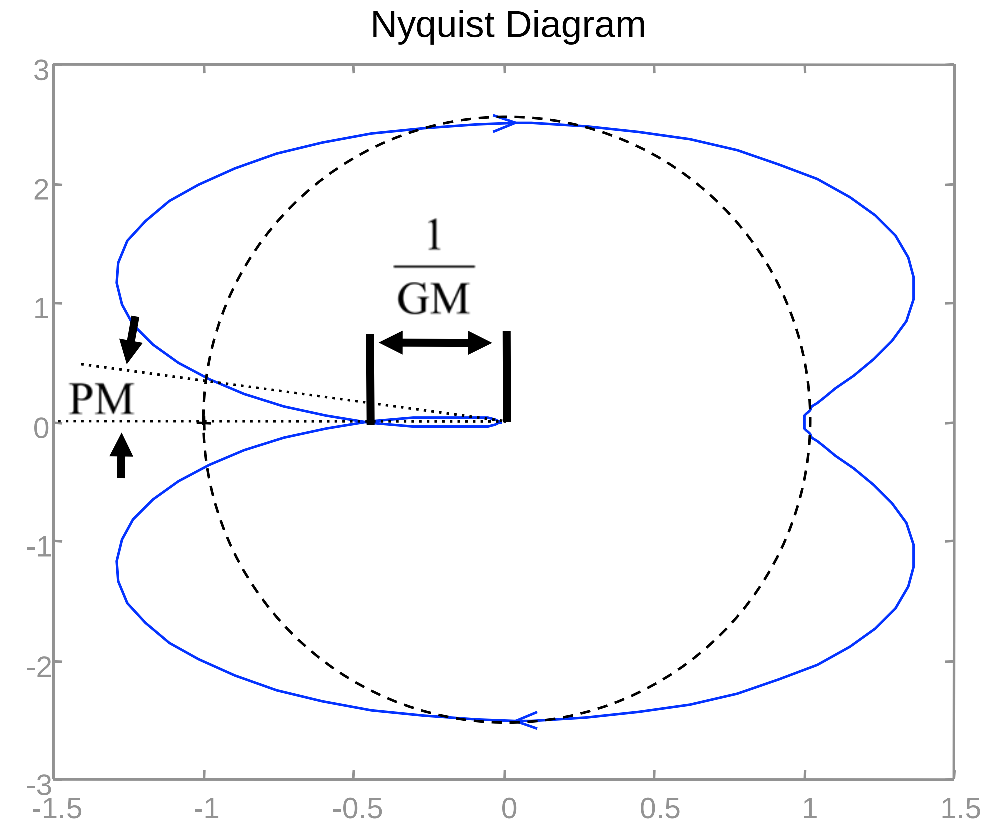
- How much we can modify the loop gain and still have the system be stable
- Determined by the location where the loop transfer function crosses 180˚ phase

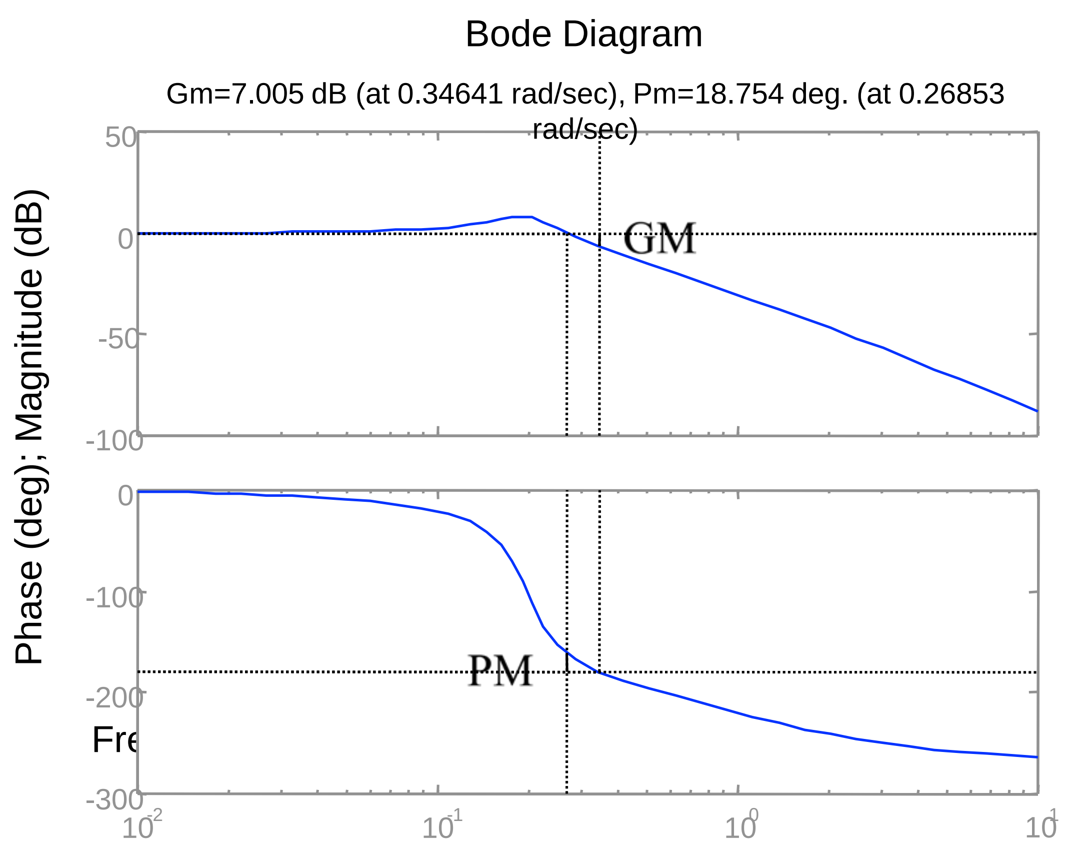
Phase margin

- How much "phase delay" can be addeded while system remains stable
- Determined by the phase at which the loop transfer function has unity gain

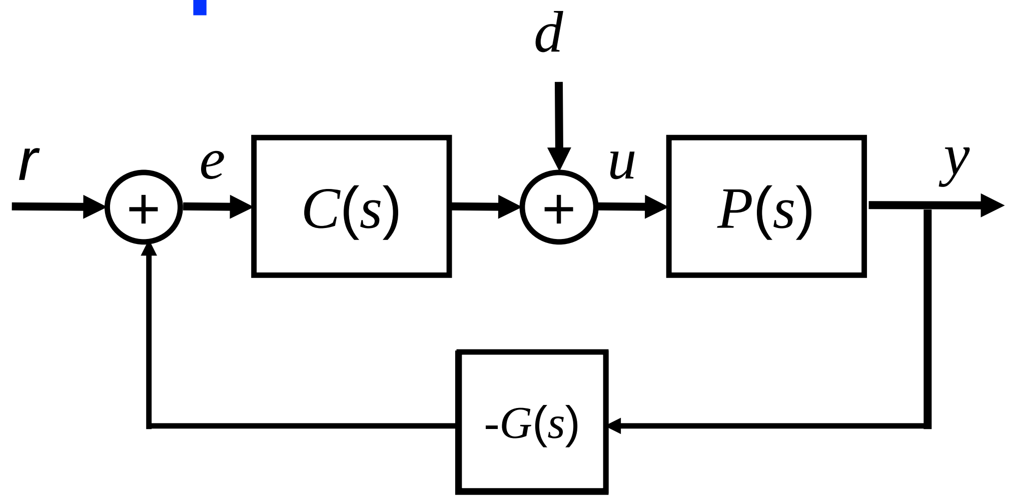
Bode plot interpretation

- Look for gain = 1, 180˚ phase crossings
- MATLAB: margin(sys)





Example: cruise control



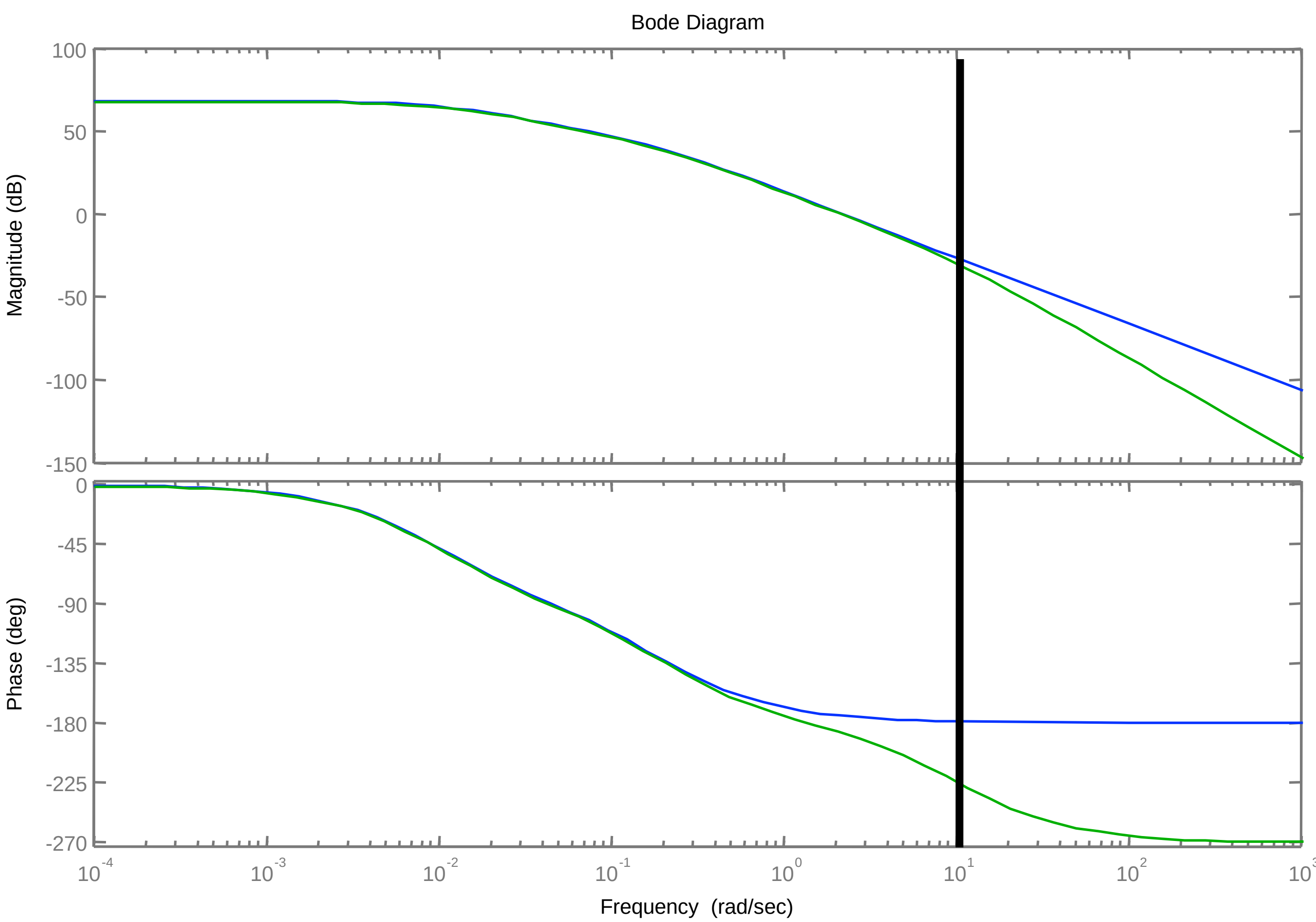
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

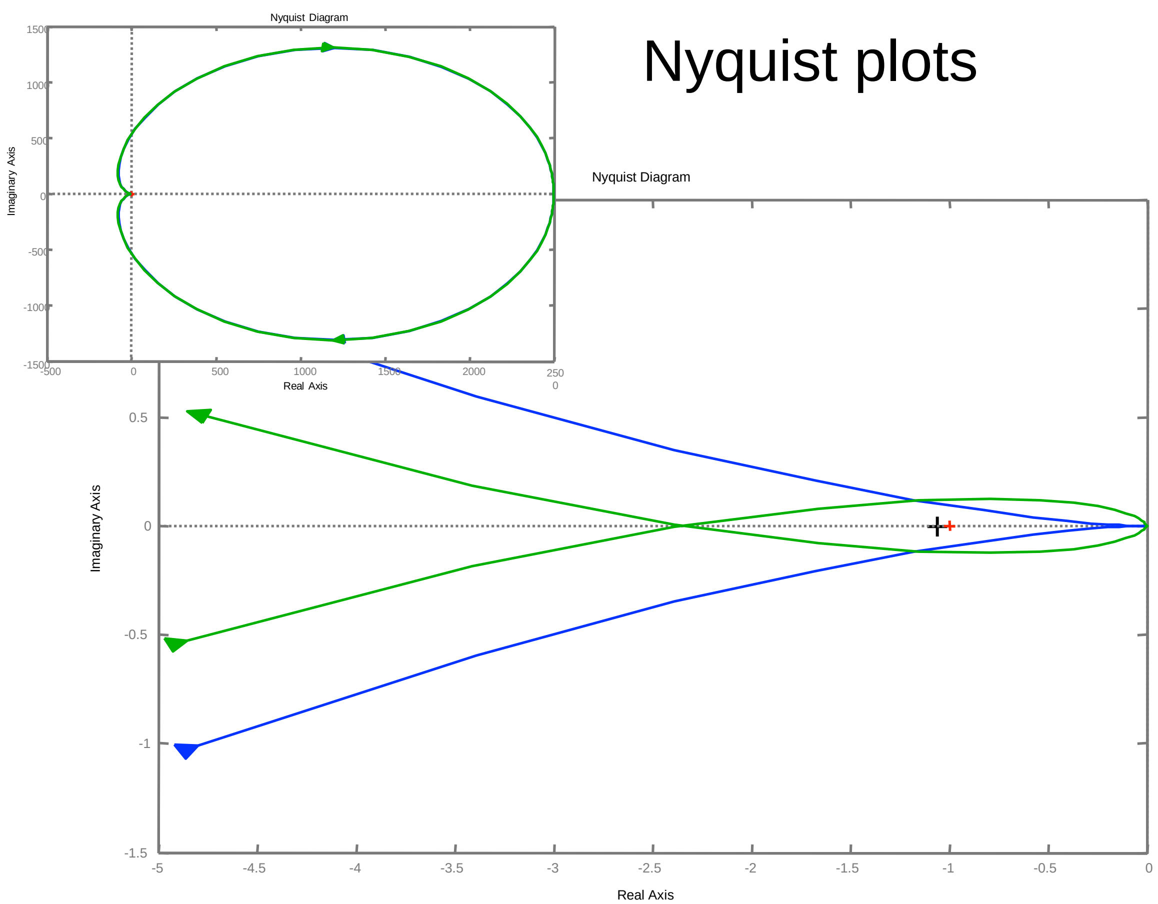
$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s+10}$$

Effect of additional sensor dynamics

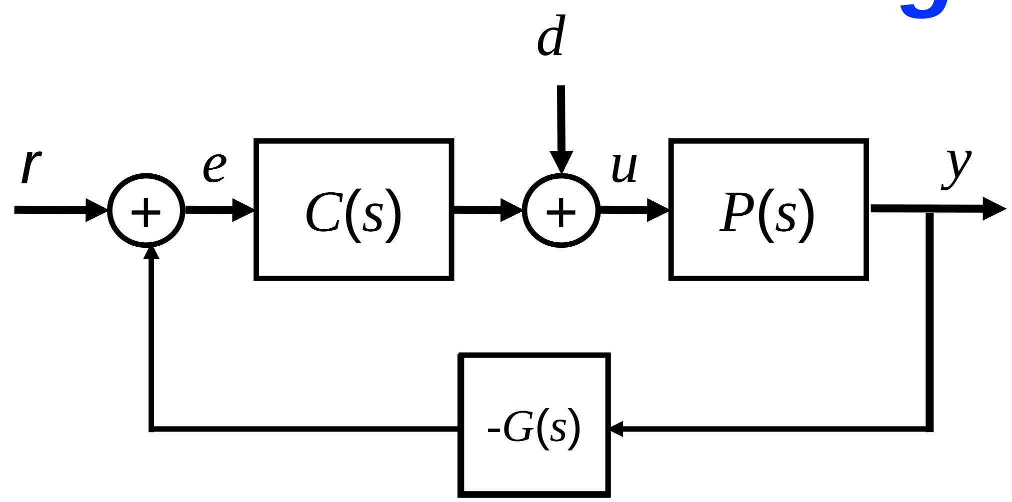
- New speedometer has pole at s = 10 (very fast); problems develop in the field
- What's the problem? A: insufficient phase margin in original design (not robust)





Nyquist plots

Preview: control *design*



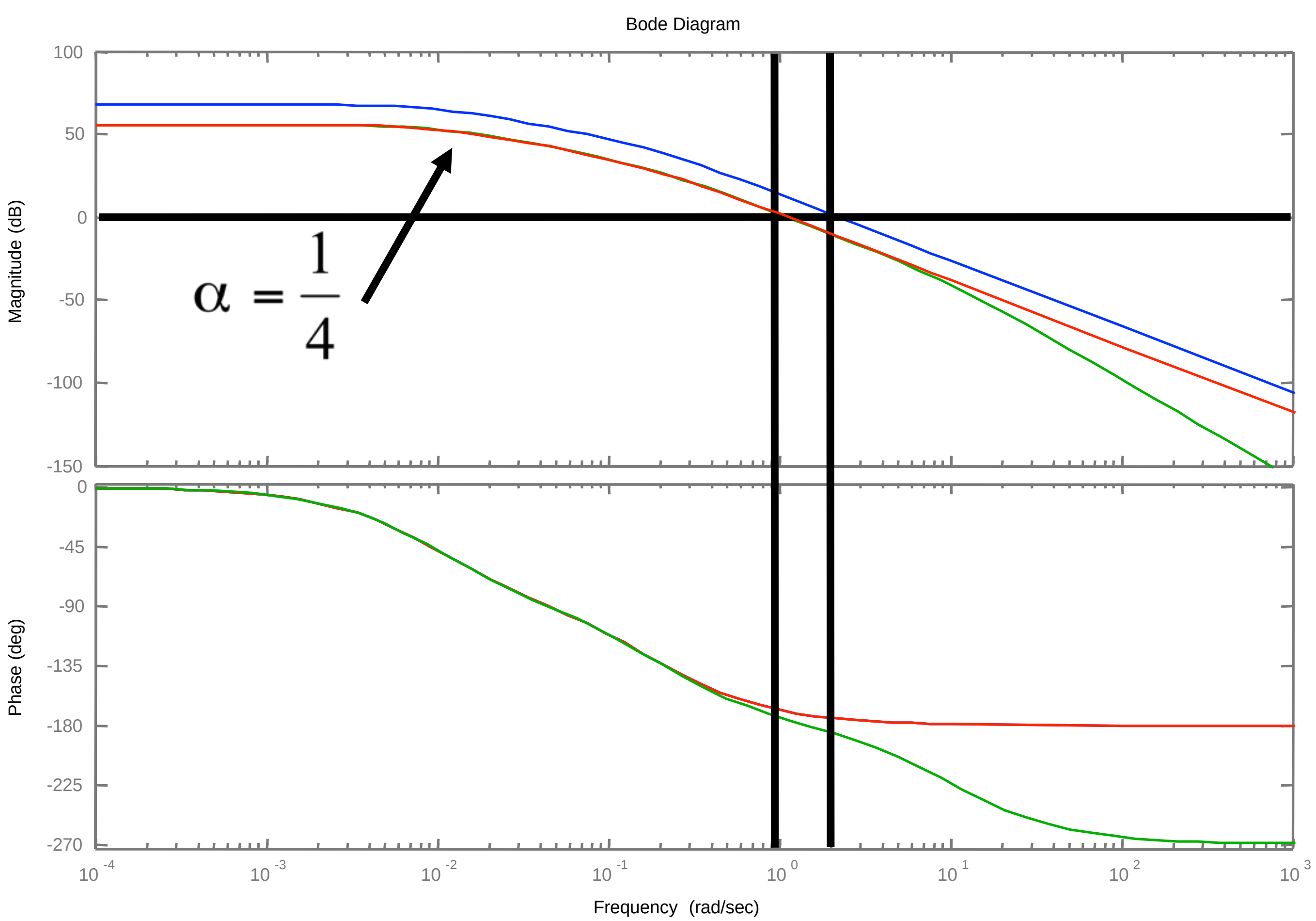
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

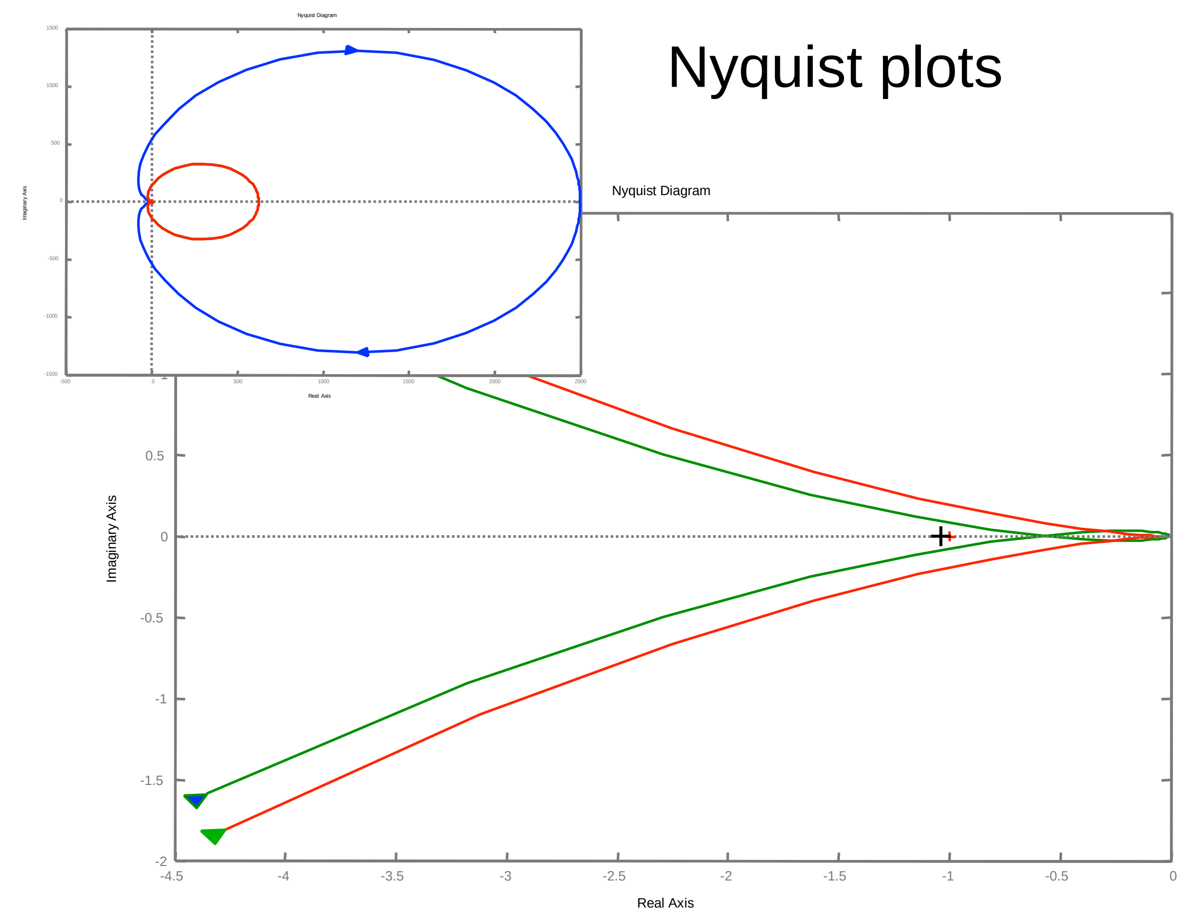
$$C(s) = \alpha \left(K_p + \frac{K_i}{s + 0.01} \right)$$

$$G(s) = \frac{10}{s+10}$$

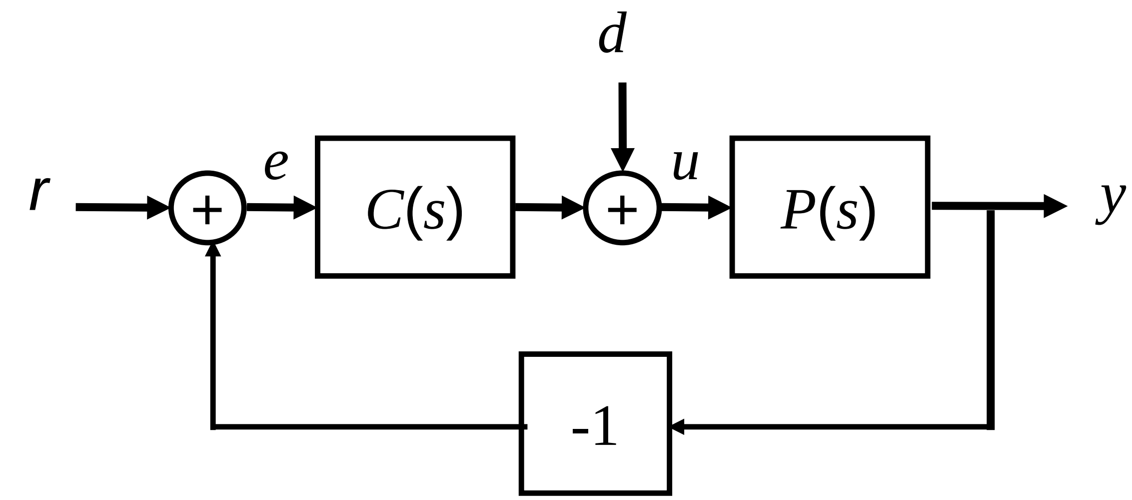
Approach: Increase phase margin

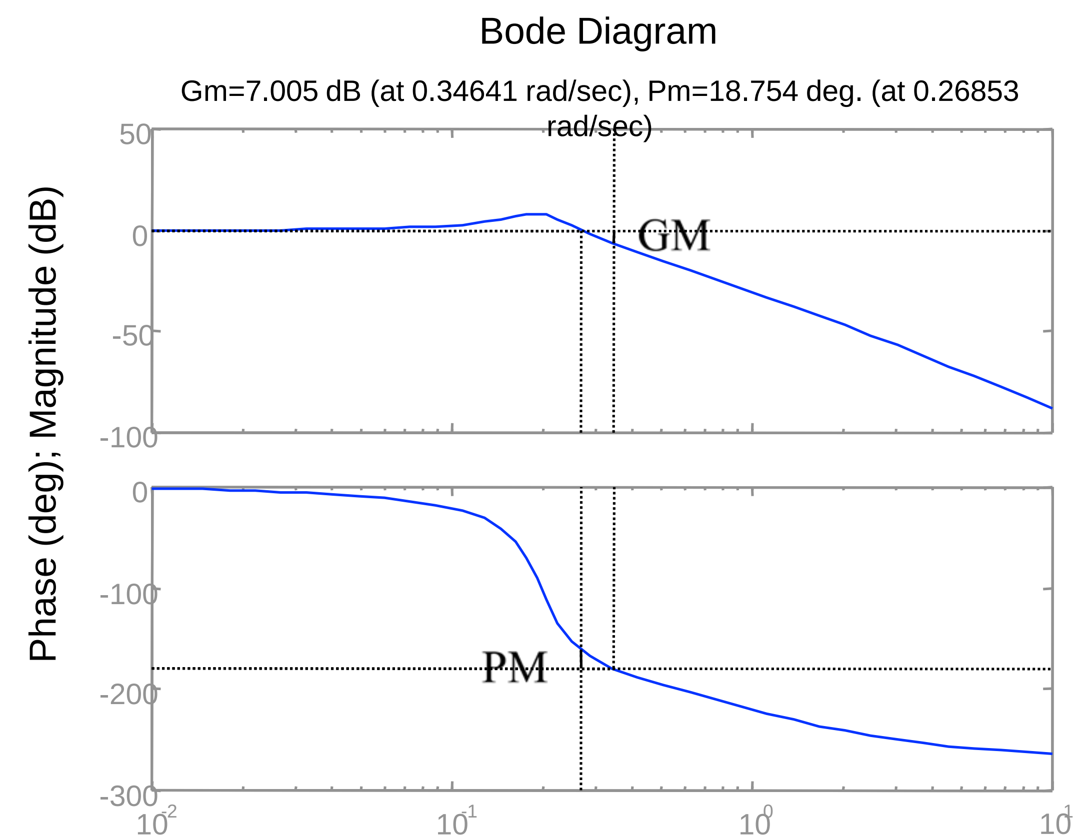
- Increase phase margin by reducing gain ⇒ can accommodate new sensor dynamics
- Tradeoff: lower gain at low frequencies ⇒ less bandwidth, larger steady state error



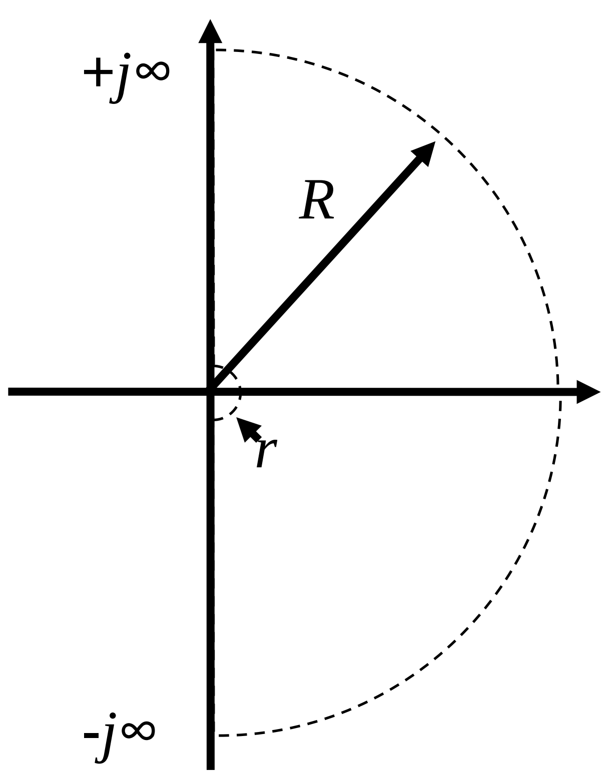


Summary: Loop Analysis of Feedback Systems





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



Thm (Nyquist).

P # RHP poles of $L(s)$

N # CW encirclements

Z # RHP zeros

$$Z = N + P$$