Solution to Problem 1:

Let the $2 \times 1$ vectors $\vec{v}_1 = [v_x \ v_y]^T$ and $\vec{v}_2 = [v_x \ v_y]^T$ have associated complex representations $\tilde{v}_1 = v_x + i v_y$ and $\tilde{v}_2 = v_x + i v_y$ respectively (where $i^2 = -1$). Recall that the goal of this problem is to show that the complex number formula:

$$1\tilde{v} = d_{12} + e^{i\theta_{12}} 2\tilde{v}.$$  \hspace{1cm} (1)

is equivalent to the planar coordinate transformation:

$$1\vec{v} = \vec{d}_{12} + R(\theta_{12}) 2\vec{v}.$$  \hspace{1cm} (2)

Let’s evaluate the right hand side of expression (1) using the standard rules for multiplication of complex numbers$^1$:

$$\tilde{d}_{12} + e^{i\theta_{12}} 2\tilde{v} = (x + iy) + (\cos \theta_{12} + i \sin \theta_{12})(2v_x + i 2v_y)$$
$$= (x + 2v_x \cos \theta_{12} - 2v_y \sin \theta_{12}) + i(y + 2v_x \sin \theta_{12} + 2v_y \cos \theta_{12})$$  \hspace{1cm} (3)

where we have used Euler’s formula ($e^{i\theta} = \cos \theta + i \sin \theta$). Matching the real and complex portions of Equation (3) with the real and complex parts of $1\tilde{v}$ in the left hand side of Equation (1), we see that

$$1v_x = x + 2v_x \cos \theta - 2v_y \sin \theta$$  \hspace{1cm} (4)
$$1v_y = y + 2v_x \sin \theta + 2v_y \cos \theta.$$  \hspace{1cm} (5)

These equations are equivalent to

$$1\vec{v} = \tilde{d}_{12} + \begin{bmatrix} \cos \theta_{12} & -\sin \theta_{12} \\ \sin \theta_{12} & \cos \theta_{12} \end{bmatrix} 2\tilde{v}$$  \hspace{1cm} (6)

Solution to Problem 2: Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position $B\vec{p}$ as seen by an observer in the fixed $B$ frame. After displacement, the observer in the body fixed $C$ frame also sees the pole in his/her coordinates at point $\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coordinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$$B\vec{p} = \vec{d}_{12} + R_{12} B\vec{p}.$$  \hspace{1cm} (B1)

This equation can be solved to find the pole location:

$$B\vec{p} = (I - R_{12})^{-1} \vec{d}_{12}$$

\(1\)If $\tilde{a} = a_1 + i a_2$ and $\tilde{b} = b_1 + i b_2$, then $\tilde{a}\tilde{b} = (a_1 b_2 - a_2 b_2) + i(a_1 b_2 + a_2 b_1)$. 

Of course, the matrix \((I - R_{12})\) must be invertible, which will always be true except when \(R_{12} = I\). In this case, the motion is a pure translation, which is viewed as a rotation about the “pole at infinity.”

B) In Frame B, the pole is located at: \(B\overrightarrow{p} = (I - R_{12})^{-1}\overrightarrow{d}_{12}\)

C) In Frame C, the vector describing the pole has exactly the same value as seen by the observer in Frame B: \(C\overrightarrow{p} = (I - R_{12})^{-1}\overrightarrow{d}_{12}\)

A) In Frame A, the expression for the pole vector is obtained by a simple coordinate transformation of the expression in Frame B: \(A\overrightarrow{p} = \overrightarrow{d}_{01} + R_{01}B\overrightarrow{p} = \overrightarrow{d}_{01} + R_{01}(I - R_{12})^{-1}\overrightarrow{d}_{12}\)

Problem 3: To find the pole of the displacement: \(D_2 = (x, y, \theta) = (2, 0, 2, 0, 45.0^\circ)\), substitute into the above results:

\[
B\overrightarrow{p} = (I - R_{12})^{-1}\overrightarrow{d}_{12} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] - \left[ \begin{array}{cc} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{array} \right]^{-1} \left[ \begin{array}{c} 2.0 \\ 2.0 \end{array} \right] = \left[ \begin{array}{c} 1 - \frac{\sqrt{2}}{2} \\ - \frac{\sqrt{2}}{2} \end{array} \right]^{-1} \left[ \begin{array}{c} 2.0 \\ 2.0 \end{array} \right] = \left[ \begin{array}{c} -1.41421 \\ 3.4142 \end{array} \right] \quad (7)
\]

You could report this result in Frame B, or transform the results to frame A.

\[
A\overrightarrow{p} = \overrightarrow{d}_{01} + R_{01}B\overrightarrow{p} = \left[ \begin{array}{c} 1.0 \\ 2.0 \end{array} \right] + \left[ \begin{array}{cc} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{array} \right] \left[ \begin{array}{c} -1.414215 \\ 3.4142 \end{array} \right] = \left[ \begin{array}{c} -1.9319 \\ 4.2497 \end{array} \right] \quad (8)
\]

Problem 4: To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by \(D\), whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let \(\overrightarrow{p}\) denote the location of the pole, as seen by an observer in Frame B. The location of Frame B relative to Frame \(D\) is a pure translation of amount \(\overrightarrow{p}\), and therefore, \(D_{DB} = (-\overrightarrow{p}, I)\). The displacement of the body from the first position to the second position, as now observed in Frame \(D\), is obtained by a similarity transform \(D_{DB}D_{12}D_{DB}^{-1}\):

\[
D_{DB}D_{12}D_{DB}^{-1} = (-\overrightarrow{p}, I)(\overrightarrow{d}_{12}, R_{12})(-\overrightarrow{p}, I)^{-1} = (-\overrightarrow{p}, I)(\overrightarrow{d}_{12}, R_{12})(+\overrightarrow{p}, I) = (-\overrightarrow{p}, I)((\overrightarrow{d}_{12} + R_{12}\overrightarrow{p}), R_{12}) = ((\overrightarrow{d}_{12} + (R_{12} - I)\overrightarrow{p}), R_{12}) \quad (10)
\]

Hence, if \(\overrightarrow{p} = -(R_{12} - I)^{-1}\overrightarrow{d}_{12} = (I - R_{12})^{-1}\overrightarrow{d}_{12}\), then \(D_{DB}D_{12}D_{DB}^{-1} = (\overrightarrow{0}, R_{12})\). I.e., as viewed in reference Frame \(D\), the displacement is a pure rotation by amount \(R_{12}\).
Problem 5:

Part (a): There are many ways that one can prove that reflections preserve length. Here is one approach (see Figure 1).

![Figure 1: Geometry of Planar Rigid Body Reflection](image)

Select any two non-identical points, $A$ and $B$, in a rigid body. After reflection, those points become $A'$ and $B'$. Form the right triangle $ABD$, where the line $BD$ is chosen to be perpendicular to the line $AA'$. Similarly, in the reflected body, form the right triangle $A'B'D'$. Simple geometric arguments show that since the distance $|BD|$ and $|B'D'|$ are equal, and the distances $|AD|$ and $|A'D'|$ are equal, then $|AB| = |A'B'|$. Hence, the distance between $A$ and $B$ is preserved under reflection. Since $A$ and $B$ were chosen randomly, the result will hold for any non-identical pair of points in the body. Thus, distance is always preserved under reflection.

Part (b): Generally, physically meaningful planar displacements are not equivalent to a single reflection. To see this, define three points $(A, B, C)$ in the body of Figure 1. Because the body is rigid, one can think of points $(A, B, C)$ as forming a rigid triangle. Consider the triangle formed from the reflected points $(A', B', C')$. Note that it is impossible physically translate $(A, B, C)$ to $(A', B', C')$. Finally, note that any rigid body planar displacement can generally be realized as the result of two sequential reflections.

An alternative proof for problem 5:

Part (a):

Without loss of generality, we can select any coordinate system on the plane. We choose an $xy$-coordinate system such that the $y$-axis is coincident with the line of reflection. Under this coordinate system, the reflection of any point $(x, y)$ has coordinates $(-x, y)$. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two points on the rigid body, with corresponding reflections $A'$ and $B'$ respectively. Then, $A' = (-x_1, y_1)$ and $B' = (-x_2, y_2)$. To see that $|AB| = |A'B'|$, we can just plug their coordinates into the distance formula: $|AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = |A'B'| = \sqrt{(-x_2 - (-x_1))^2 + (y_2-y_1)^2}$. 


Part (b):  

As in part (a), without loss of generality, we can select an \( xy \)-coordinate system on the plane such that the \( y \)-axis is coincident with the line of reflection. Under this coordinate system, the reflection of any point \((x, y)\) has coordinates \((-x, y)\). Let \( D \) be the reflection operator, so that \( D : (x, y) \rightarrow (-x, y) \).

We prove by contradiction that operator \( D \) cannot be a planar displacement operator. Assume that \( D \) is represented by some planar displacement operator, such that \( D = (d, R) \) for some vector \( d \) and rotation matrix \( R \).

Let \( B \) be the set of points on the body. We assume that there exists a point \( P_0 \in B \) and a number \( \epsilon > 0 \) such that \( N_\epsilon(P_0) = \{ X \in \mathbb{R}^2 | ||X - P_0||_2 < \epsilon \} \subseteq B \). This just means that there exists some open set that is contained inside \( B \). Letting \( P_0 = (x_0, y_0) \), there must exist some points \( P_1 = (x_1, y_0), P_2 = (x_0, y_1), \) and \( P_3 = (x_1, y_1) \), where \( x_0 \neq x_1 \) and \( y_0 \neq y_1 \). For example, if you set \( x_1 = x_0 + \frac{\epsilon}{4} \) and \( y_1 = y_0 + \frac{\epsilon}{4} \), then clearly \( \{P_0, P_1, P_2, P_3\} \subseteq N_\epsilon(P_0) \subseteq B \).

The reflections of these points under our reflection operator are \( P'_0 = (-x_0, y_0), P'_1 = (-x_1, y_0), P'_2 = (-x_0, y_1), P'_3 = (-x_1, y_1) \).

Next, we note that \( D \) cannot represent a pure translation, i.e. \( R \neq I \). This is because under a pure translation, each point must have an equal distance of displacement under the operator; however, \( |P_0P'_0| = 2|x_0| \neq |P_1P'_1| = 2|x_1| \). This means that \( D \) must have a finite pole; in other words, the pole of the planar displacement is not located at infinity.

Consider the displacements \( P_0 \rightarrow P'_0 \) and \( P_1 \rightarrow P'_1 \). Knowing the movement of these points under \( D \) is enough to fully define the operator \( D \); as discussed in class, a planar displacement has 3 degrees of freedom. The movement of these 2 points is a pure rotation around the point \( P = (0, y_0) \). We can see that this must be the pole of the displacement, as (1) the pole is unique for any displacement that is not a pure translation, and (2) the reflections \( P'_0 \) and \( P'_1 \) are both achieved by rotating \( P_0 \) and \( P_1 \) about \( P = (0, y_0) \) by an angle of \( \pi \) radians.

Consider the displacements \( P_2 \rightarrow P'_2 \) and \( P_3 \rightarrow P'_3 \). By the same logic as that of the previous paragraph, the pole of the displacement must be at \((0, y_1)\) in order for the movement of these 2 points to be a pure rotation about the pole.

Thus, we have found that the pole of the displacement operator is at \( P = (0, y_0) = (0, y_1) \). Since \( y_0 \neq y_1 \), we have arrived at a contradiction! Thus, no planar displacement can equivalently perform a reflection.