Problem 1: (10 points) Let $\mathcal{F}_1$ denote a fixed reference frame in the plane, with orthonormal basis vectors $\vec{x}_1$ and $\vec{y}_1$. Similarly, consider a second reference frame $\mathcal{F}_2$ with orthonormal basis vectors $\vec{x}_2$ and $\vec{y}_2$. Let $d_{12} = [x \ y]^T$ be the vector pointing from the origin of $\mathcal{F}_1$ to the origin of $\mathcal{F}_2$. Let $\theta_{12}$ denote the relative orientation of the two reference frames: $\theta_{12}$ is the angle between $\vec{x}_1$ and $\vec{x}_2$ (using the right hand rule, or RHR).

Let $2\vec{v} = [2v_x \ 2v_y]^T$ denote the coordinates of a point, $P$, as seen by an observer in $\mathcal{F}_2$. In class we developed a formula for the coordinate transformation of $P$ to its representation in $\mathcal{F}_1$:

$$1\vec{v} = d_{12} + R(\theta_{12}) \ 2\vec{v} \tag{1}$$

where $R(\theta_{12})$ is the $2 \times 2$ rotation matrix:

$$R(\theta_{12}) = \begin{bmatrix} \cos \theta_{12} & -\sin \theta_{12} \\ \sin \theta_{12} & \cos \theta_{12} \end{bmatrix}$$

For computational purposes, it is sometimes convenient to use different representations of coordinates, vectors, and rotations. This problem considers the use of complex numbers.

Let $\vec{w}$ be a $2 \times 1$ vector $\vec{w} = [w_1 \ w_2]^T$ in the plane. We can represent $\vec{w}$ as a complex number: $\vec{w} = w_1 + iw_2$ where $i$ is the complex number such that $i \cdot i = -1$. Show that if $2\vec{v}$ is the complex representation of $\vec{v}$, and $d_{12}$ is the complex representation of $d_{12}$, then the complex representation of the coordinate transform in Equation (1) is:

$$1\vec{v} = d_{12} + e^{i\theta_{12}} \ 2\vec{v}$$

Problem 2: (10 points) Every planar rigid body displacement is equivalent to a rotation about a unique point in the plane, known as the pole (see Figure 1).

Let $A$ be a fixed reference frame. A rigid body, $L$, which has local frame $B$ attached to it, is located relative to reference frame $A$ by $D_1 = (\vec{d}_{01}, R_{01})$. Body $L$ moves to position $C$, where the displacement to location $C$, as measured by an observer in frame $B$, is given by $D_2 = (\vec{d}_{12}, R_{12})$. Where is the pole of the body displacement from position $B$ to position $C$, as a function of $R_{01}$, $R_{12}$, $\vec{d}_{01}$, and $\vec{d}_{12}$?

a. As measured in Frame A
b. As measured in Frame B
c. As measured in Frame C
Problem 3: (5 points) In the above problem, suppose $D_1 = (x, y, \theta) = (2.0, 2.0, 20.0^\circ)$ and $D_2 = (x, y, \theta) = (3.0, 2.0, 45^\circ)$. Where is the pole of the displacement from B to C in this case?

Problem 4: (15 points) Using the set up of Problem 2, pick a coordinate system whose origin is located at the pole of the displacement, and show that in this coordinate system, the displacement of the body from B to C is a pure rotation. That is, in that coordinate frame, the displacement should take the form $D = (\text{displacement vector, rotation matrix}) = (\vec{0}, R)$, where $R$ is a $2 \times 2$ rotation matrix.

Problem 5: (15 points) A planar reflection is an operation wherein one “reflects” all of the particles in a body across a line (see Figure 2). Show (intuitively) that reflections “preserve length.” That is, reflections do not alter the distance relationship between particles in a rigid body. Can any planar displacement be equivalently performed by a reflection?
Problem 6: (15 points) Do parts (a,b,c) of Problem 10 in Chapter 2 of the Murray, Li, Sastry textbook. This problem is located on page 75-76.