# CDS 101/110: Lecture 3.1
# Linear Systems

**Goals for Today:**

- Revist and motivate linear time-invariant system models:
- Summarize properties, examples, and tools
  - Convolution equation describing solution in response to an input
  - Step response, impulse response
  - Frequency response
- Characterize performance of linear systems

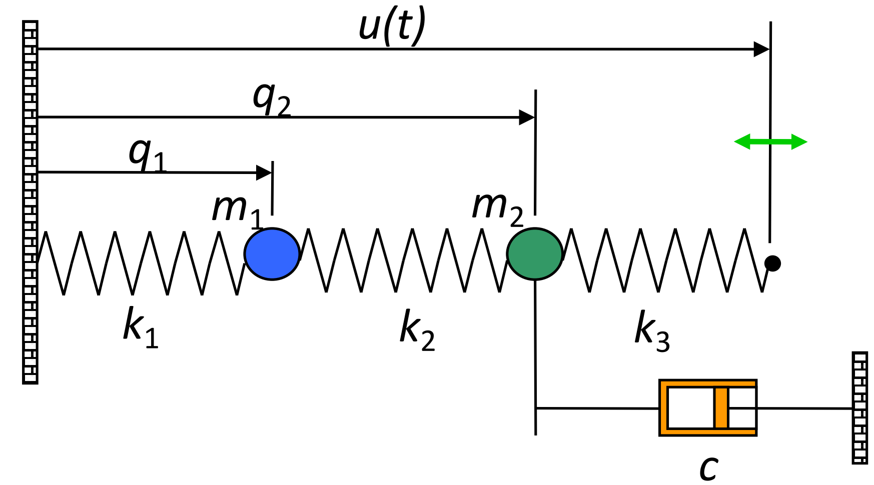
**Reading:**

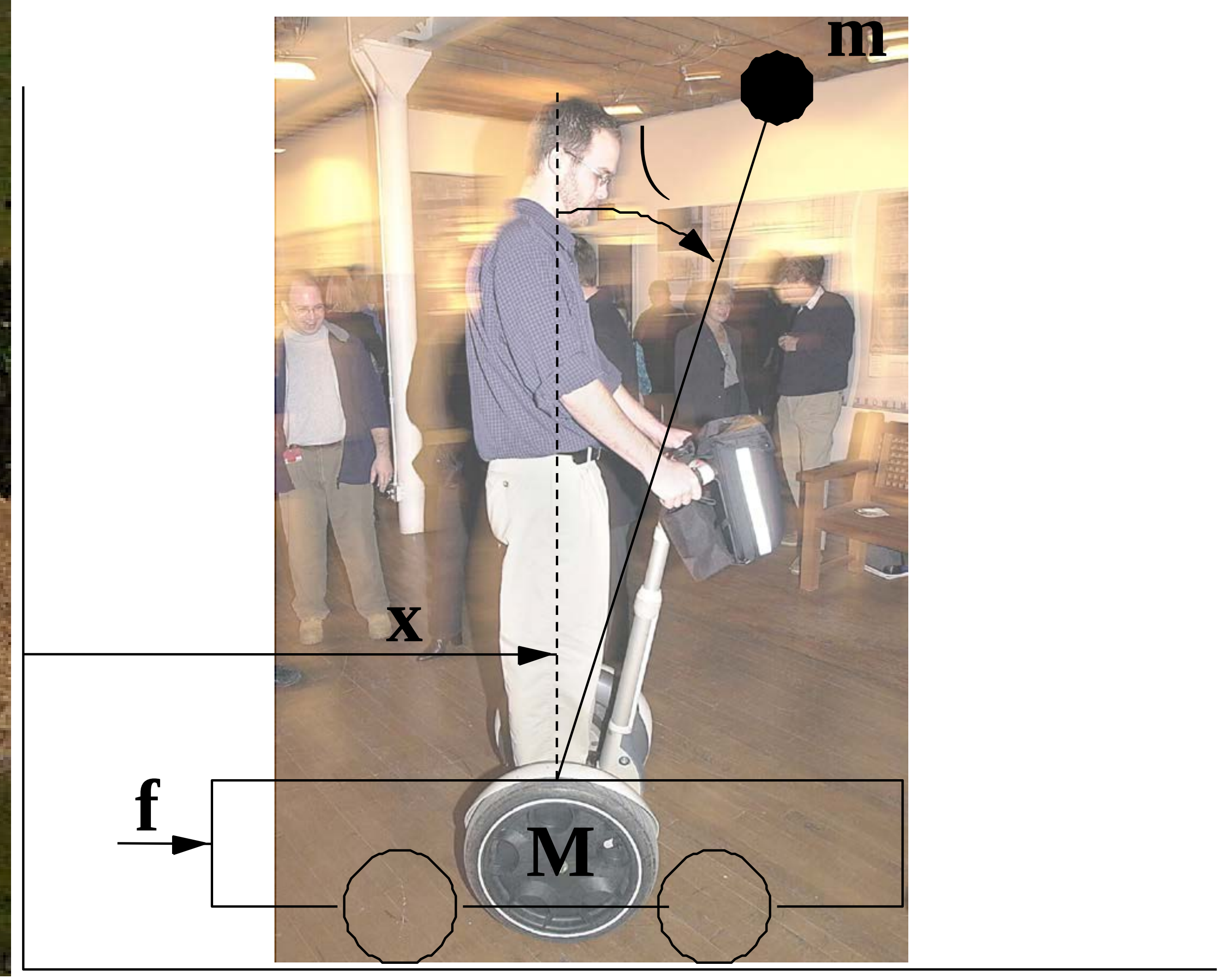
- Åström and Murray, FBS-2e, Ch 6.1-6.3

# Why are Linear Systems Important?

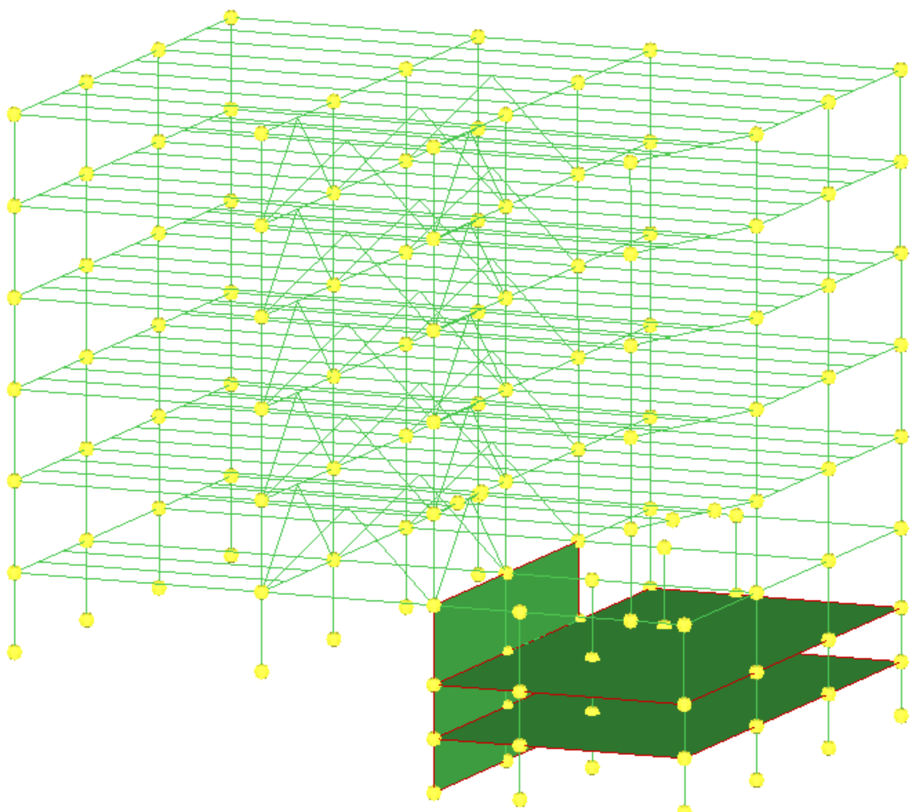
**Many important *examples***

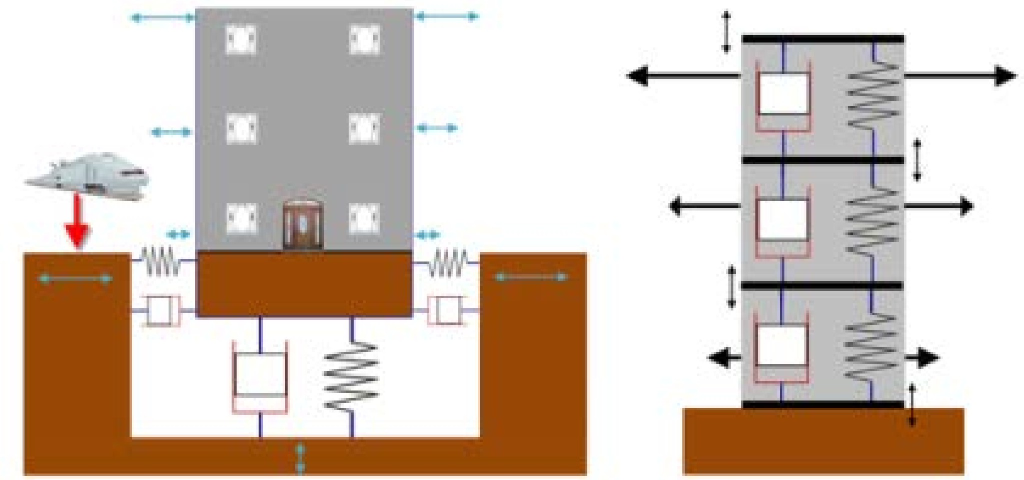
- Mechanical Systems

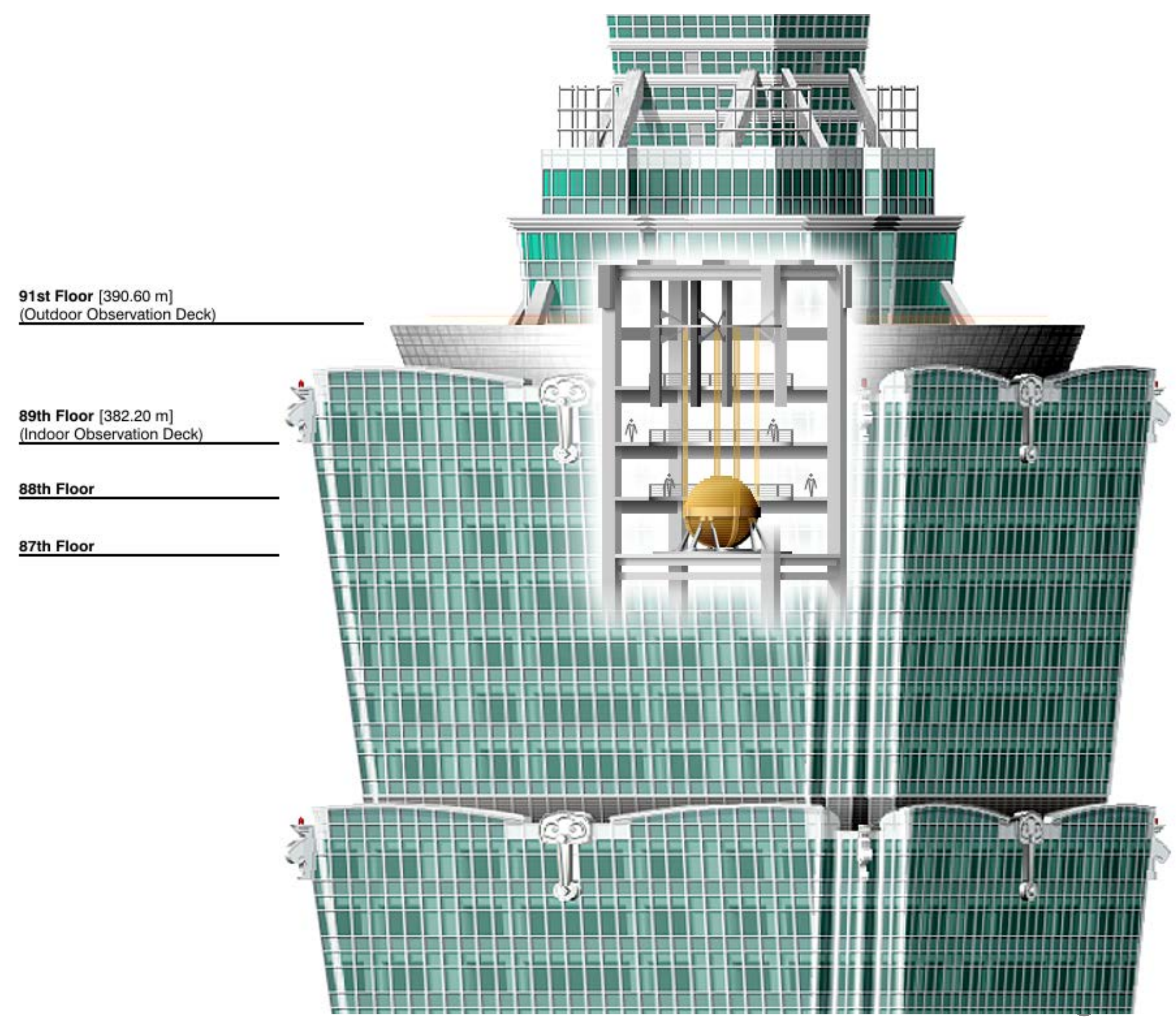


# Why are Linear Systems Important?



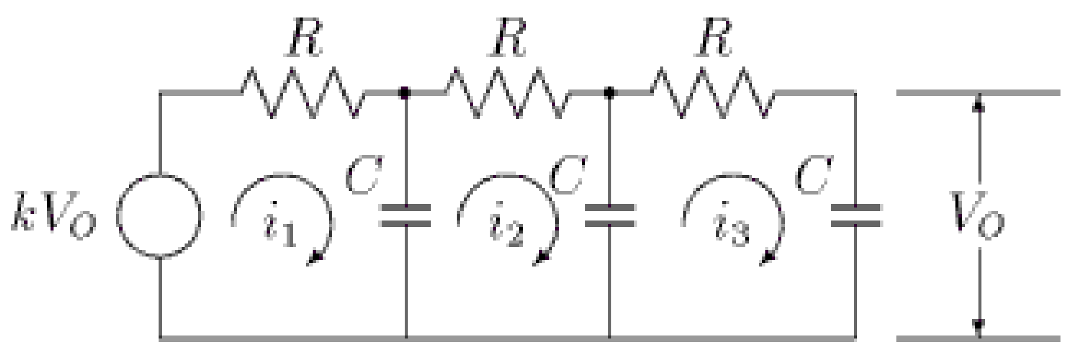


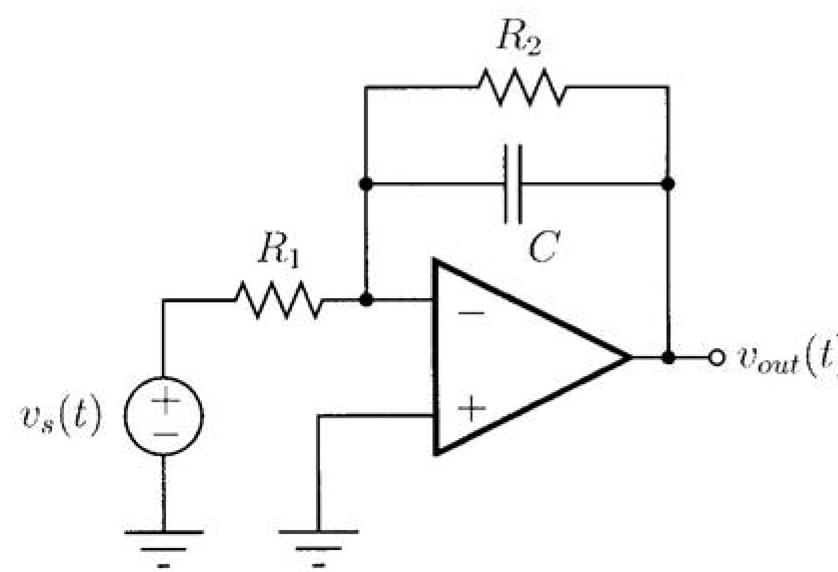


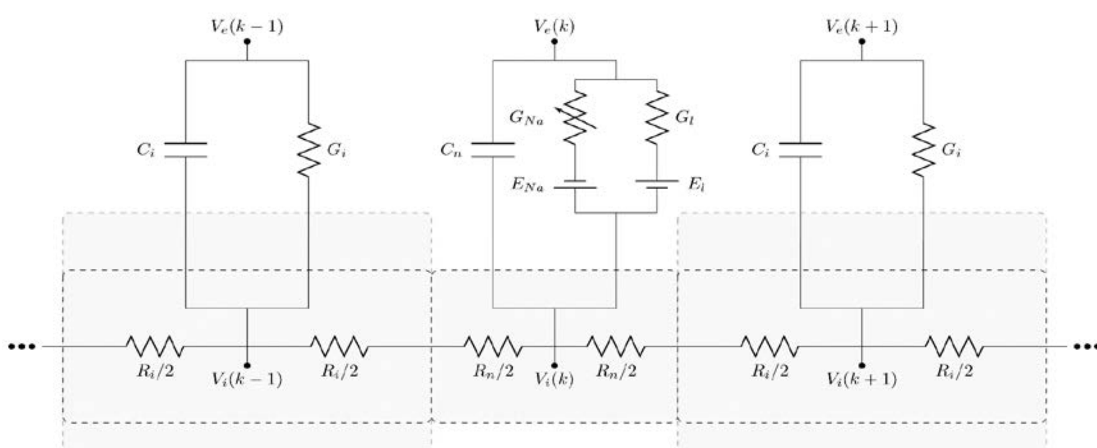
# Why are Linear Systems Important?

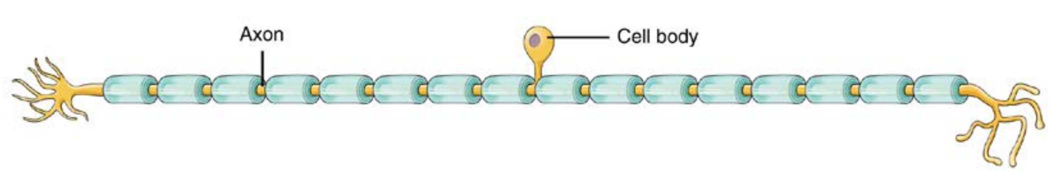
## Many important *examples*

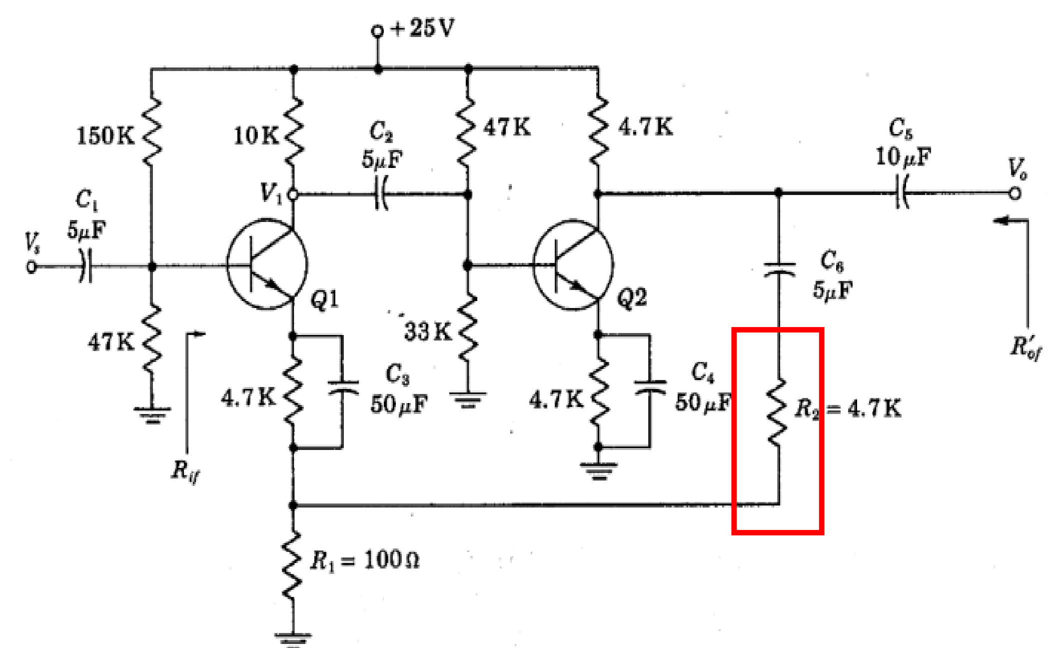
- Electronic circuits



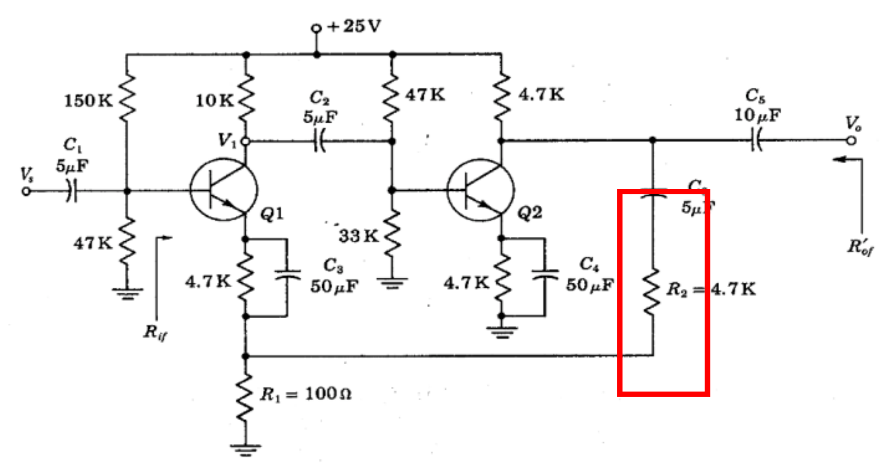



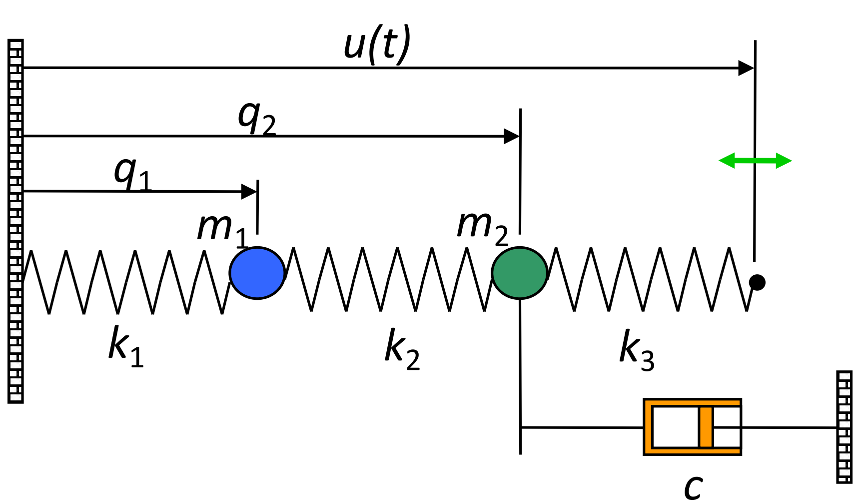




- Especially true after feedback
- Frequency response is key performance specification

# Why are Linear Systems Important?

## Many important *examples*




## Many important *tools*

- Frequency and step response,
  - Traditional tools of control theory
  - Developed in 1930's at Bell Labs
- Classical control design toolbox
  - Nyquist plots, gain/phase margin
  - Loop shaping

CDS 110

- Optimal control and estimators
  - Linear quadratic regulators
  - Kalman estimators

CDS 112

- Robust control design
  - $H_1$ control design
  - $\mu$ analysis for structured uncertainty

CDS 212/213

# Solutions of Linear Time Invariant Systems: "Modes"

Linear Time Invariant (LTI) System:

- If *Linear System,* input u(t) leads to output y(t)
- If u(t+T) leads to output y(t+T), the system is *time invariant*

- Matrix LTI system, with no input

$$\dot{x} = Ax \qquad y = Cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow y(t) = Ce^{At}x_0$$

- Let $\lambda_i$ and $v_i$ be eigenvalue/eigenvector of $A$. Then:

$$e^{At}v_i = \left(I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots\right)v_i = v_i + \frac{t}{1!}\lambda_i v_i + \frac{t\hat{}2}{2!}\lambda_i^2 v_i + \cdots$$
$$= e^{\lambda_i t}\, v_i$$

- If n distinct eigenvalues, then $x(0) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, and

$$e^{At}x(0) = \alpha_1 e^{\lambda_1 t}v_1 + \alpha_2 e^{\lambda_2 t}v_2 + \cdots + \alpha_n e^{\lambda_n t}v_n$$

} Sum of "modes"

# The Convolution Integral: Step 1

Let $H(t)$ denote the response of a LTI system to a ***unit step*** input at t=0.

- Assuming the system starts at Equilibrium

The response to the steps are:

- First step input at time t=0: $H(t-t_0)u(t_0)$
- Second step input at time $t_1$: $H(t-t_1)(u(t_1)-u(t_0))$
- Third step input at time $t_2$: $H(t-t_2)(u(t_2)-u(t_1))$

By linearity, we can add the response

$$y(t) = H(t-t_0)u(t_0) + H(t-t_1)\big(u(t_1)-u(t_0)\big) + \ \ldots$$

$$= \big(H(t-t_0) - H(t-t_1)\big)u(t_0) + \big(H(t-t_1) - H(t-t_2)\big)u(t_1) + \ldots$$

$$= \sum_{n=0}^{t_0<t} \frac{H(t-t_n) - H(t-t_{n+1})}{t_{n+1}-t_n} u(t_n)(t_{n+1}-t_n)$$

Taking the limit as $(t_{n+1}-t_n) \to 0$

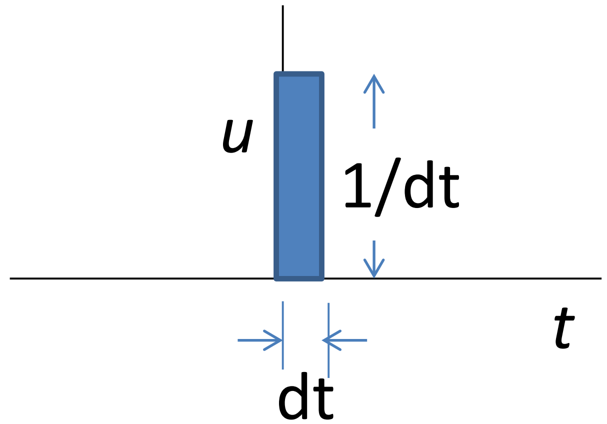
$$y(t) = \int_0^t H'(t-\tau)u(\tau)d\tau$$

# Impulse Response

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \quad \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- What is the "impulse response" due to u(t)=δ(t)?

*take limit as dt → 0 but keep unit area*




$$u(t) = p_\varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1/\varepsilon & 0 \le t < \varepsilon \\ 0 & t \ge \varepsilon \end{cases} \qquad \delta(t) = \lim_{\varepsilon \to 0} p_\varepsilon(t)$$

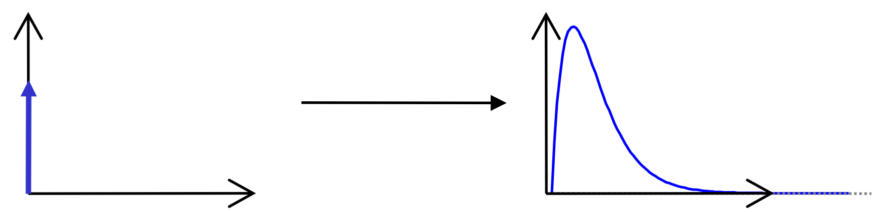
- Apply this unit impulse to the system (with x(0)=0):

$$x(0^+) = \int_{0^-}^{0^+} (Ax + Bu)\mathrm{d}t = B \quad \Rightarrow \quad \begin{aligned} x(t) &= e^{At}B \\ y(t) &= Ce^{At}B \end{aligned}$$

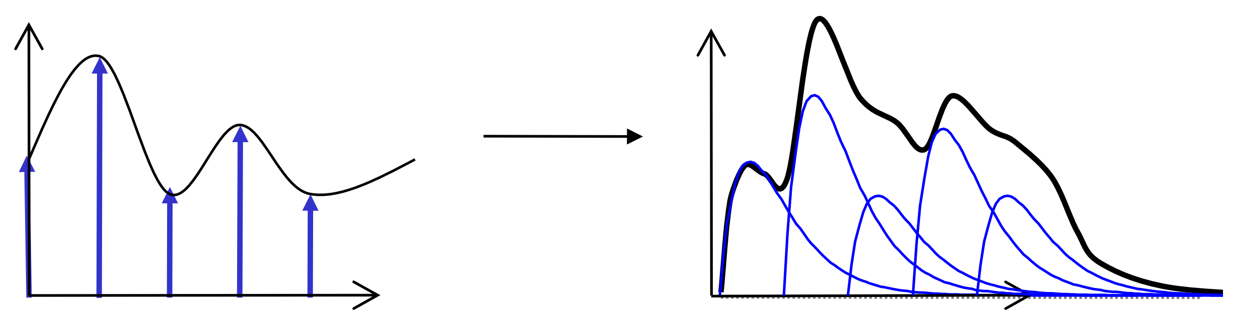
# Response to inputs: Convolution

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- *Impulse response, h*(*t*) = $Ce^{At}B$
  - Response to input "impulse"
  - Equivalent to "Green's function"



- Linearity $\Rightarrow$ compose response to arbitrary *u*(*t*) using *convolution*
  - Decompose input into "sum" of shifted impulse functions
  - Compute impulse response for each
  - "Sum" impulse response to find *y*(*t*)
  - Take limit as dt$\rightarrow$ 0



- Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input $\Rightarrow$ 2X output when $x(0) = 0$

Convolution Theorem

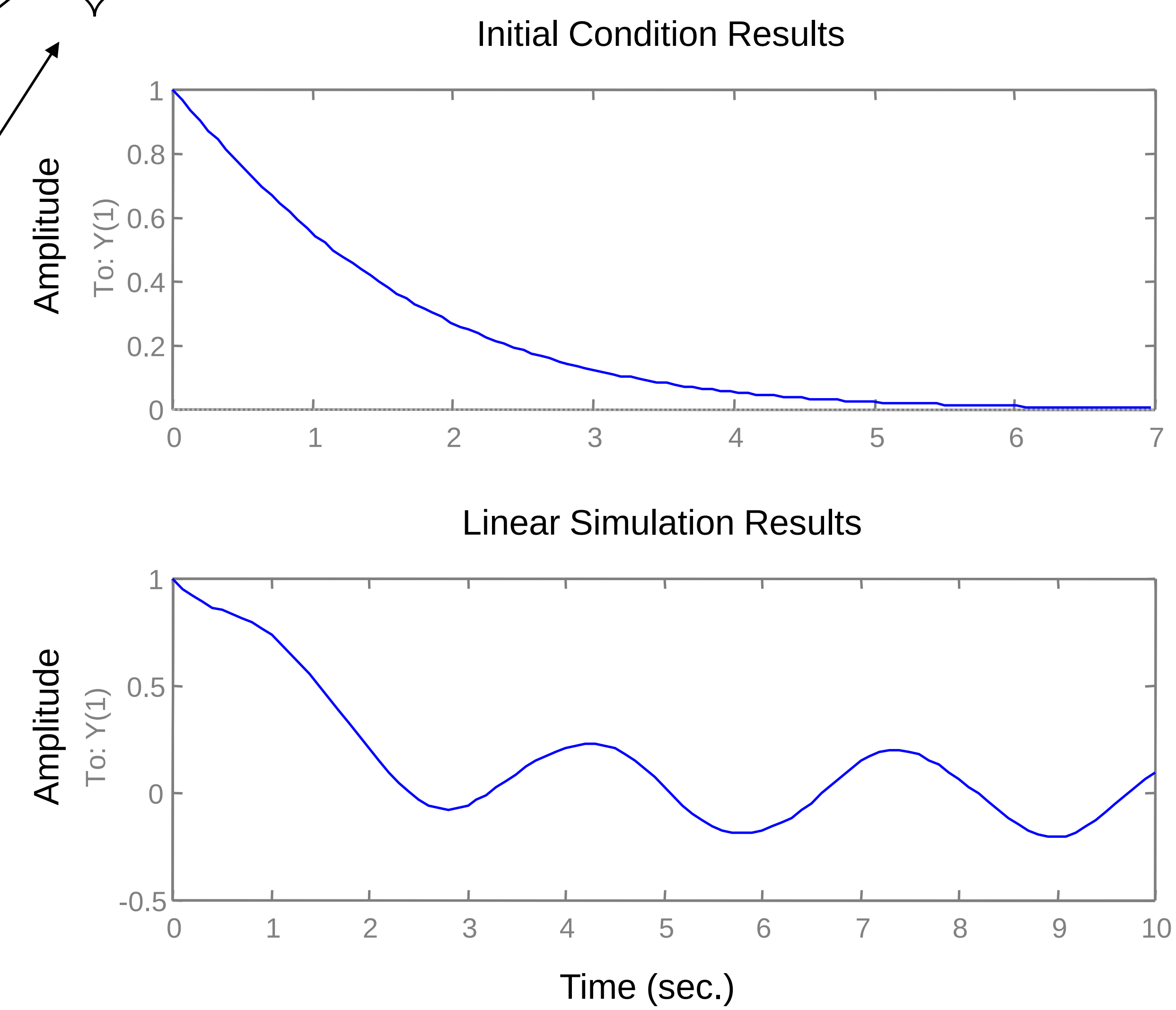
# MATLAB Tools for Linear Systems

$$y(t) = \underbrace{\underbrace{Ce^{At}x(0)} + \int_{\tau=0}^{t} \underbrace{Ce^{A(t-\tau)}Bu(\tau)}\mathrm{d}\tau + Du(t)}$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```




- Other MATLAB commands
  - gensig, square, sawtooth – produce signals of diff. types
  - step, impulse, initial, lsim – time domain analysis
  - bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

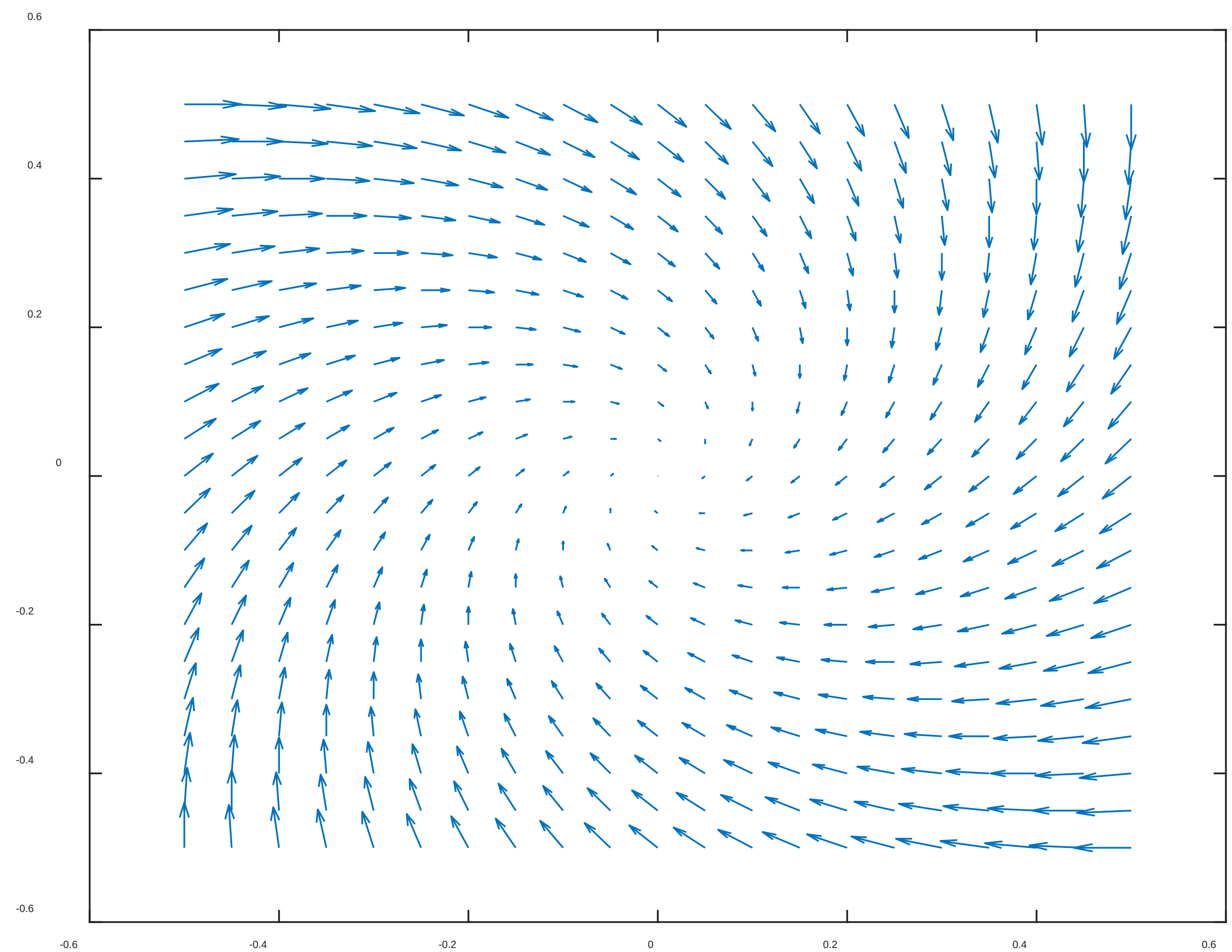
# MATLAB Tools for Phase Space

**System Equations**

$$\dot{x}_1 = -x_1 - 2x_2x_1^2x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

**MATLAB CODE**

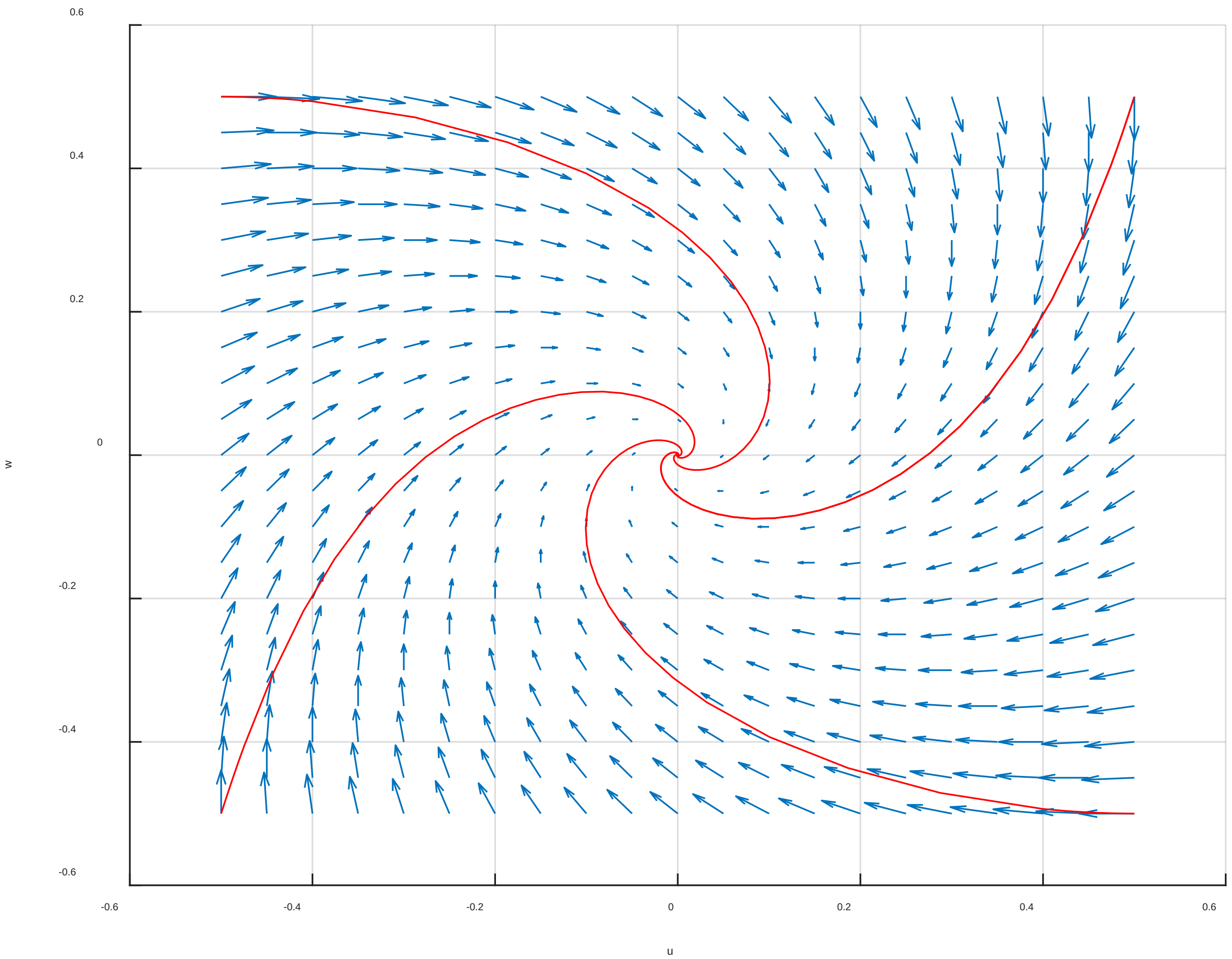
```
[x1, x2]=meshgrid(-0.5:0.05:0.5, -0.5:0.05:0.5);
x1dot=-x1 - 2*x2*x1^2 + x2;
x2dot=-x1-x2;
quiver(x1,x2,x1dot,x2dot);
x
```



# MATLAB Tools for Phase Space

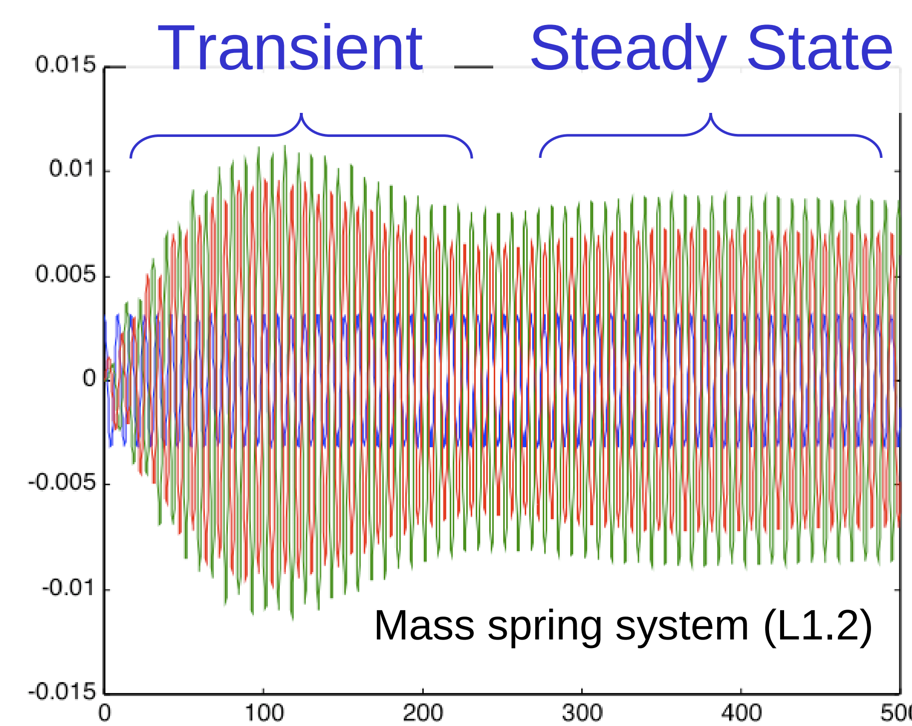
```
function my_phases(IC)
hold on
    [~,X]=ode45(@EOM,[0,50],IC);
    u=X(:,1);
    w=X(:,2);
    plot(u,w,'r');
xlabel('u')
ylabel('w')
grid
end

function dX=EOM(t,X)
dX=zeros(2,1);
x1=X(1);
x2=X(2);
x1dot=-x1 - 2*x2*x1^2 + x2;
x2dot=-x1-x2;
dX=[x1dot;x2dot];
end
```



# Input/Output Performance

- How does system respond to changes in input values?
  - Transient response:
  - Steady state response:
- Characterize response in terms of
  - Impulse response
  - Step response
  - Frequency response



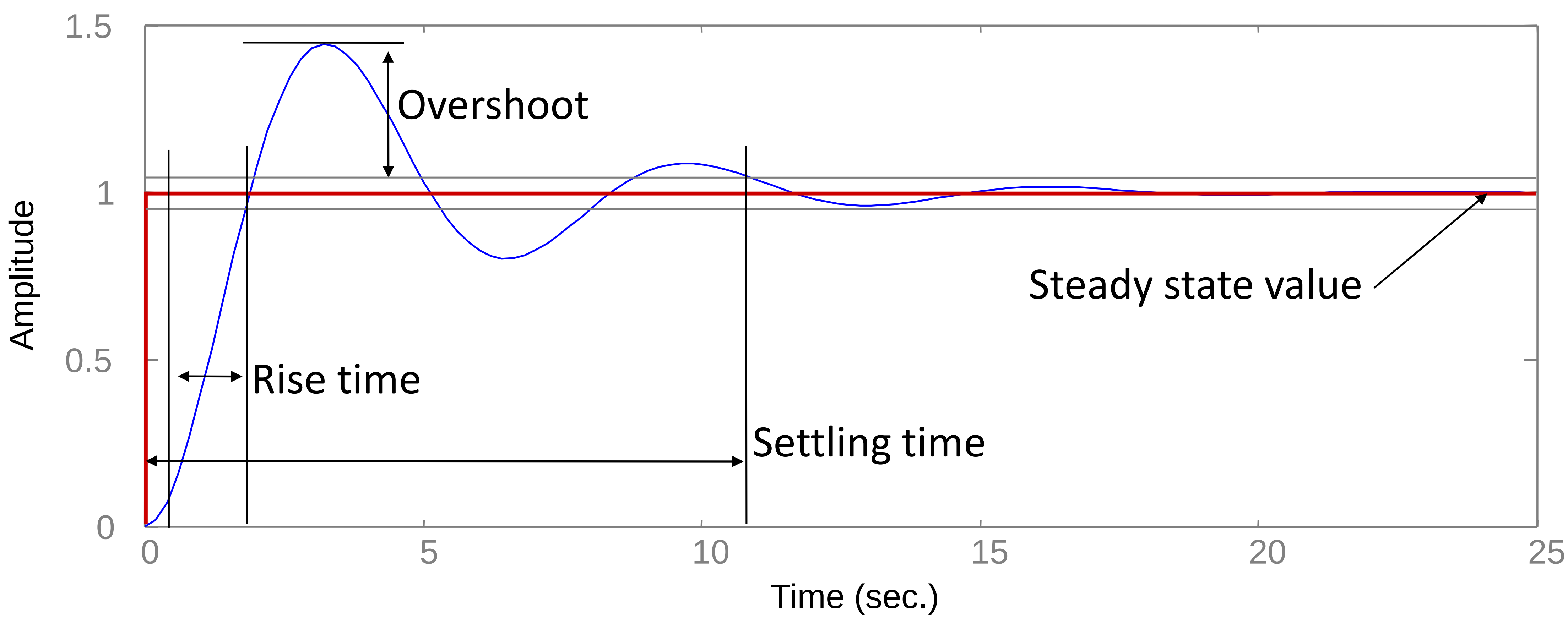

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Response to I.C.s, Transient response

Response to inputs, Steady State (if constant inputs)

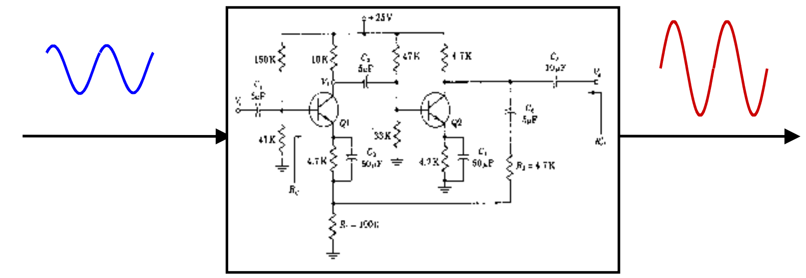
# Step Response

- Output characteristics in response to a "step" input
  - **Rise time:** time required to move from 5% to 95% of final value
  - **Overshoot:** ratio between amplitude of first peak and steady state value
  - **Settling time:** time required to remain w/in *p*% (usually 2%) of final value
  - **Steady state** value: final value at t = 1

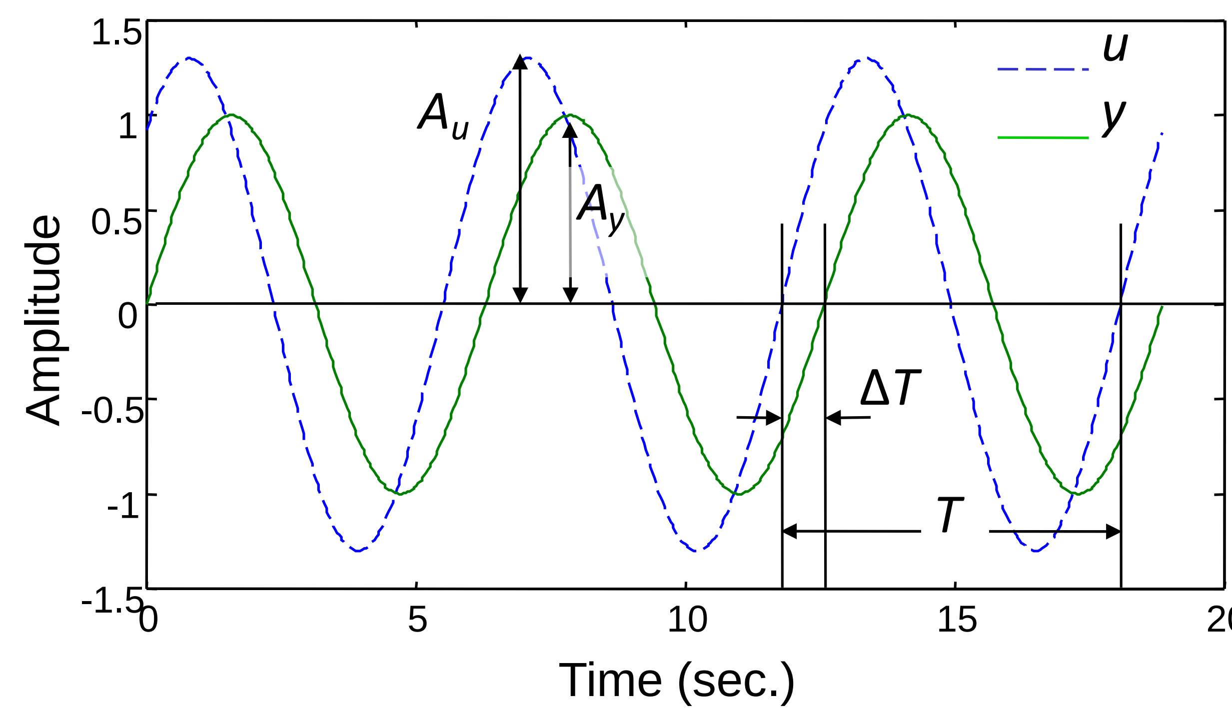


# Frequency Response

- Measure *steady state* response of system to sinusoidal input
  - Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
  - Individual sinusoids are good *test signals* for measuring performance in many systems



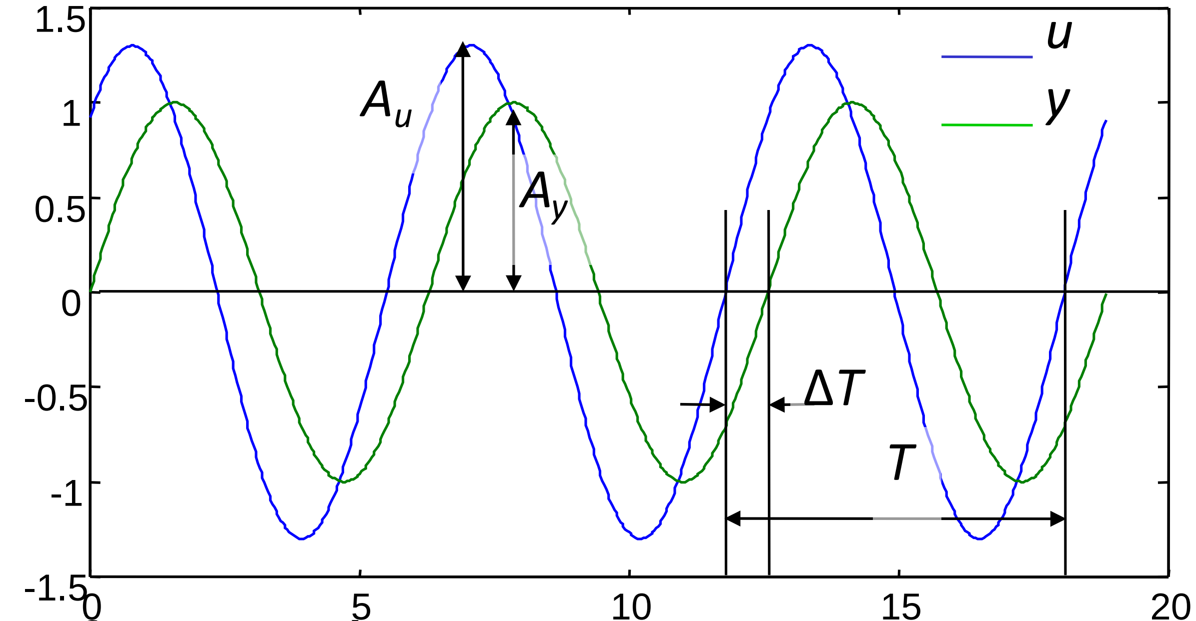
- Approach: plot input and output, measure *relative* amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = $A_y/A_u$
  - Phase = $2\pi \cdot \Delta T/T$




- May not work for *nonlinear* systems
  - System nonlinearities can cause *harmonics* to appear in the output
  - Amplitude and phase may not be well-defined
  - For *linear* systems, frequency response is always well defined

# Computing Frequency Responses

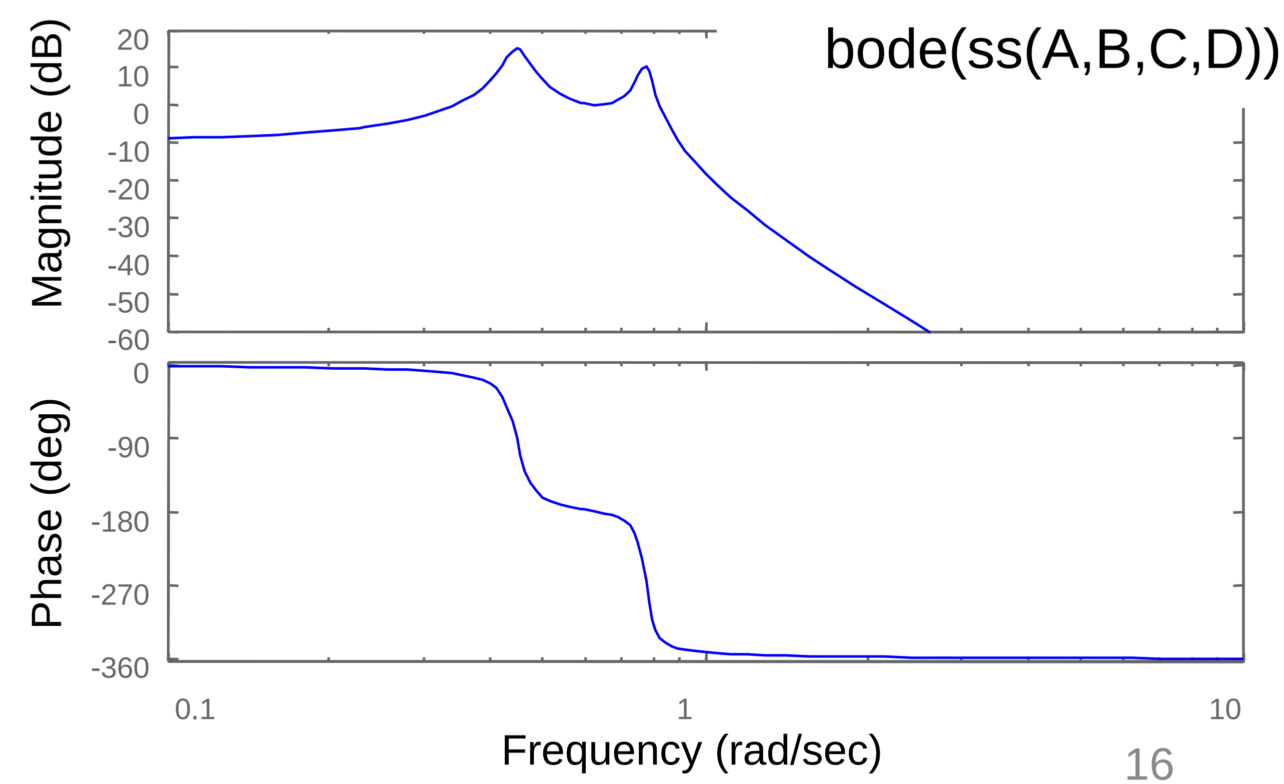
- Technique #1: plot input and output, measure relative amplitude and phase
  - Generate response of system to sinusoidal output
  - Gain = $A_y/A_u$
  - Phase = $2\pi \cdot \Delta T/T$
  - For *linear* system, gain and phase don't depend on the input amplitude




- Technique #2 (linear systems): use bode (or freqresp) command
  - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  - Gain plotted on log-log scale
    - dB = 20 $\log_{10}$ (gain)
  - Phase plotted on linear-log scale



# Calculating Frequency Response from convolution equation

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$u$(t)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B \boxed{e^{i\omega\tau}} \mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}\int_0^t e^{(i\omega I - A)\tau} B \mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1} e^{(i\omega I - A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I - A)t} - I\right) B \\
&= \underbrace{e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1}B}_{\text{Ratio of response/input}} e^{i\omega t}
\end{aligned}
$$

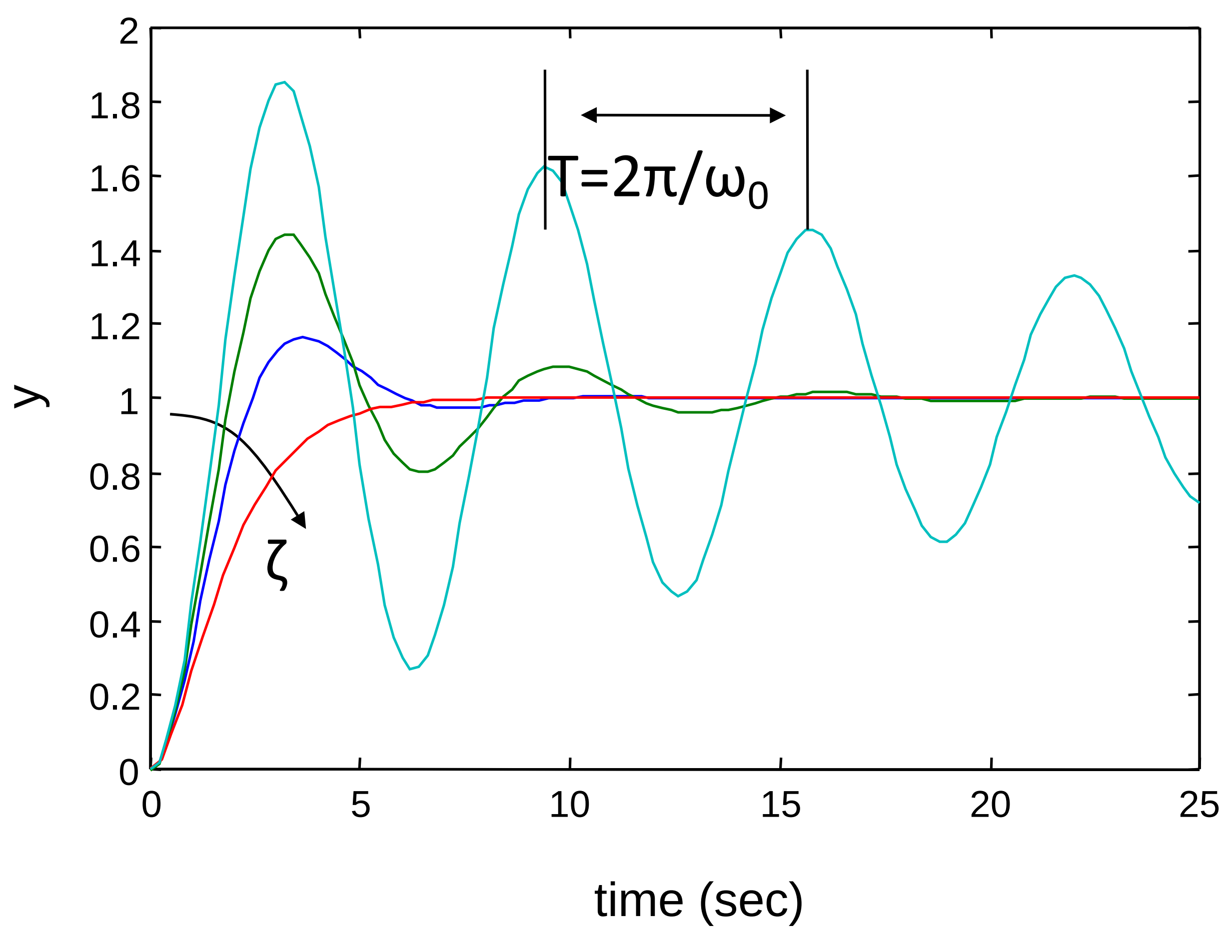
$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \boxed{\left(C(i\omega I - A)^{-1}B + D\right)} e^{i\omega t}
\end{aligned}
$$

"Frequency response"

# Second Order Systems

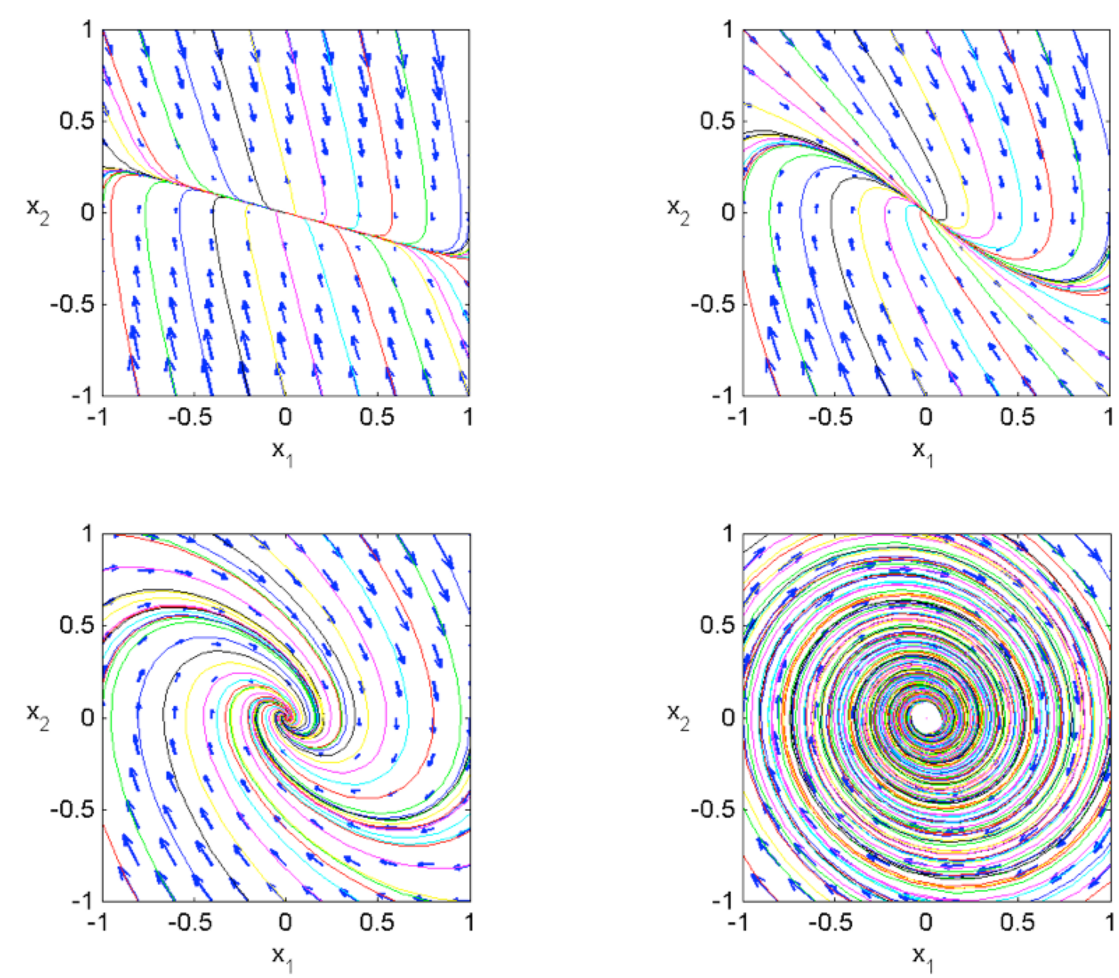
- Many important examples:
- Insight to response for higher orders (eigenvalues of A are either real or complex)
  - Exception is non-diagonalizable A (non-trivial Jordan form)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$




For $\zeta < 1$, eigenvalues at

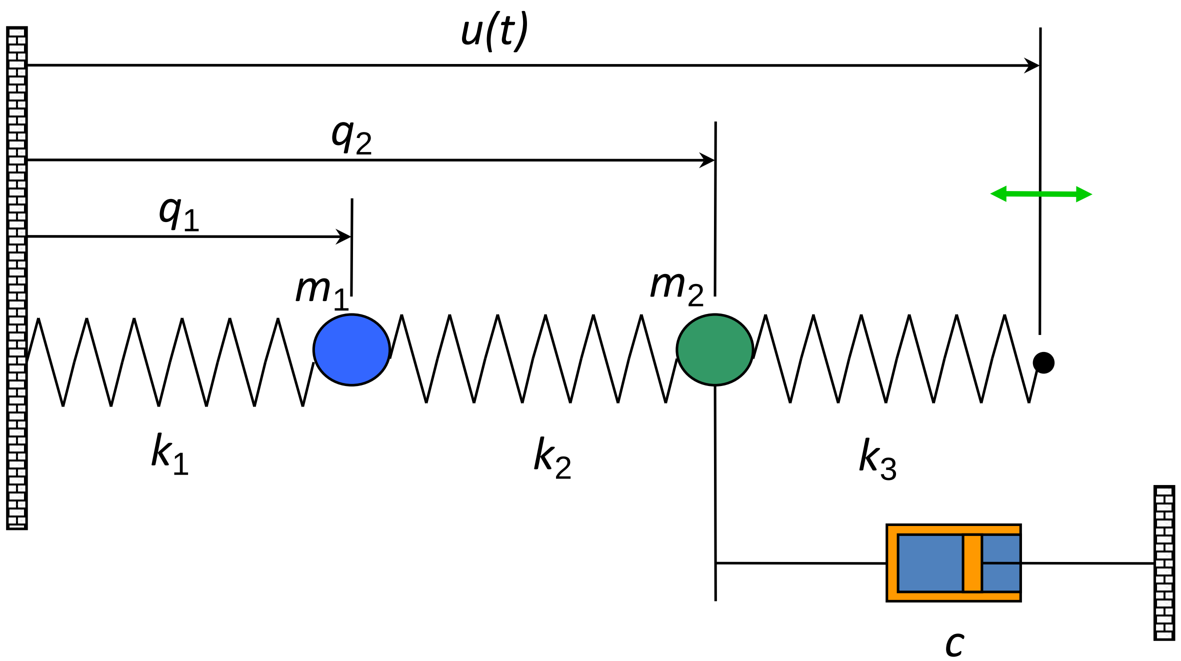
$$\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$




- Analytical formulas exist for overshoot, rise time, settling time, etc
- Will study more next week

# Spring Mass System

Frequency response: $C(j\omega I-A)^{-1}B+D$



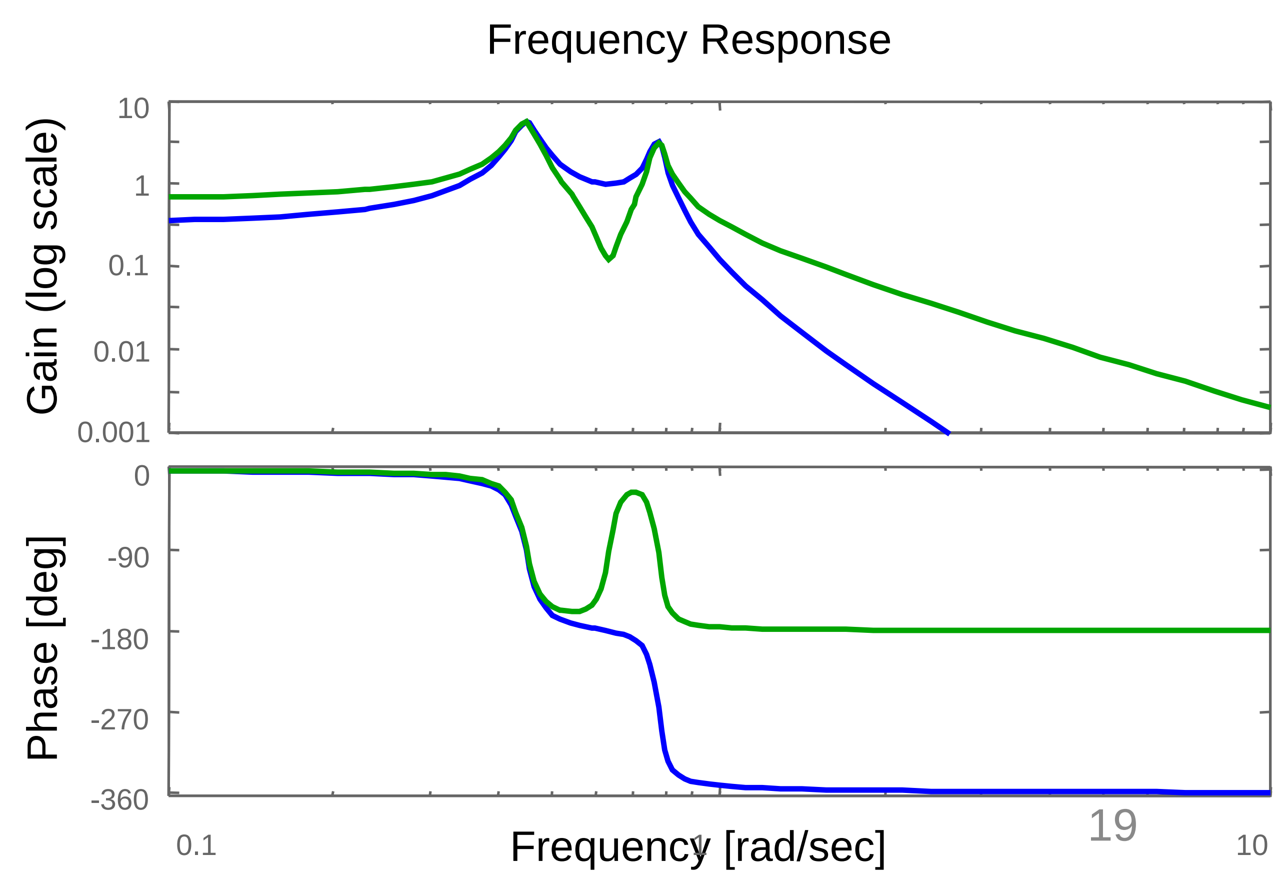

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot q_1 \\ \dot q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot q_1 \\ \dot q_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

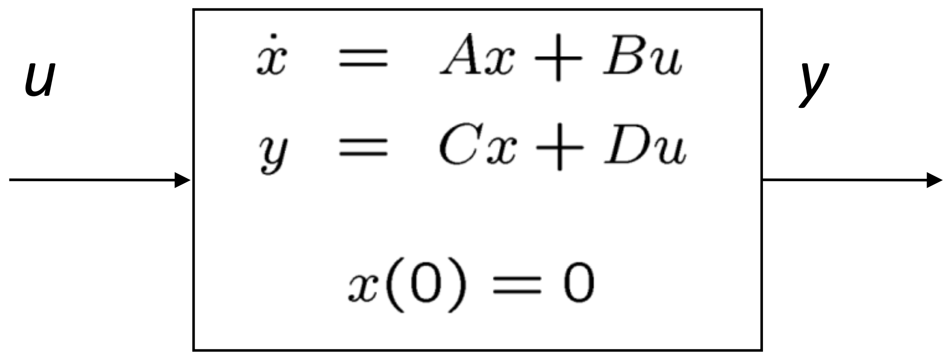
$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

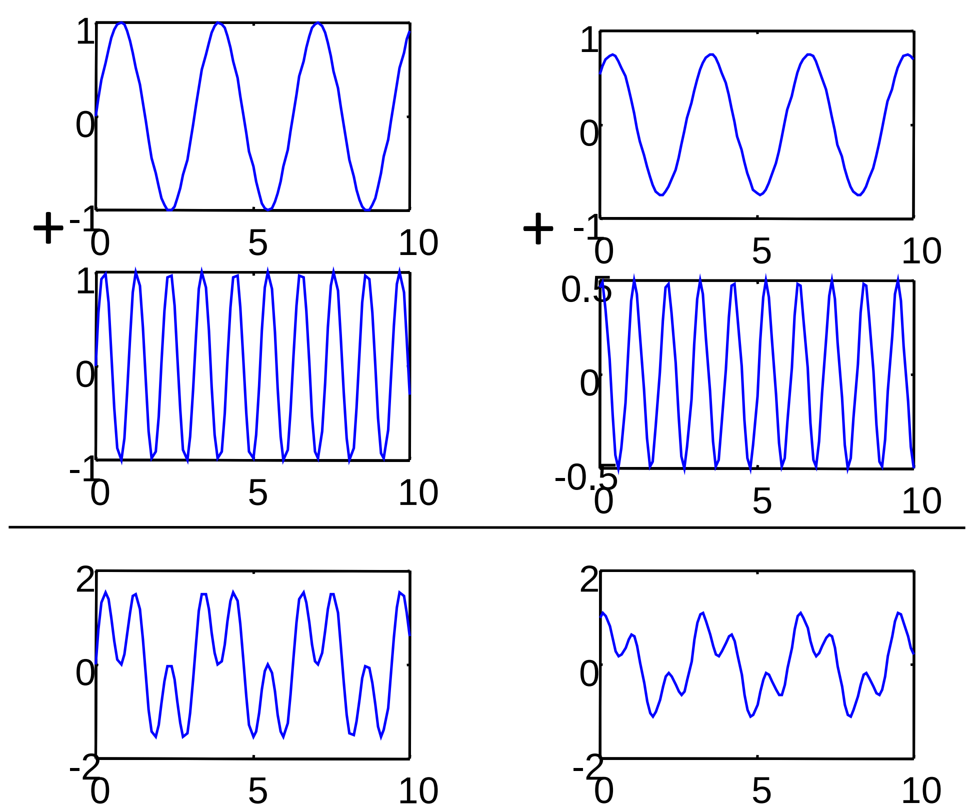
**Eigenvalues of A:**

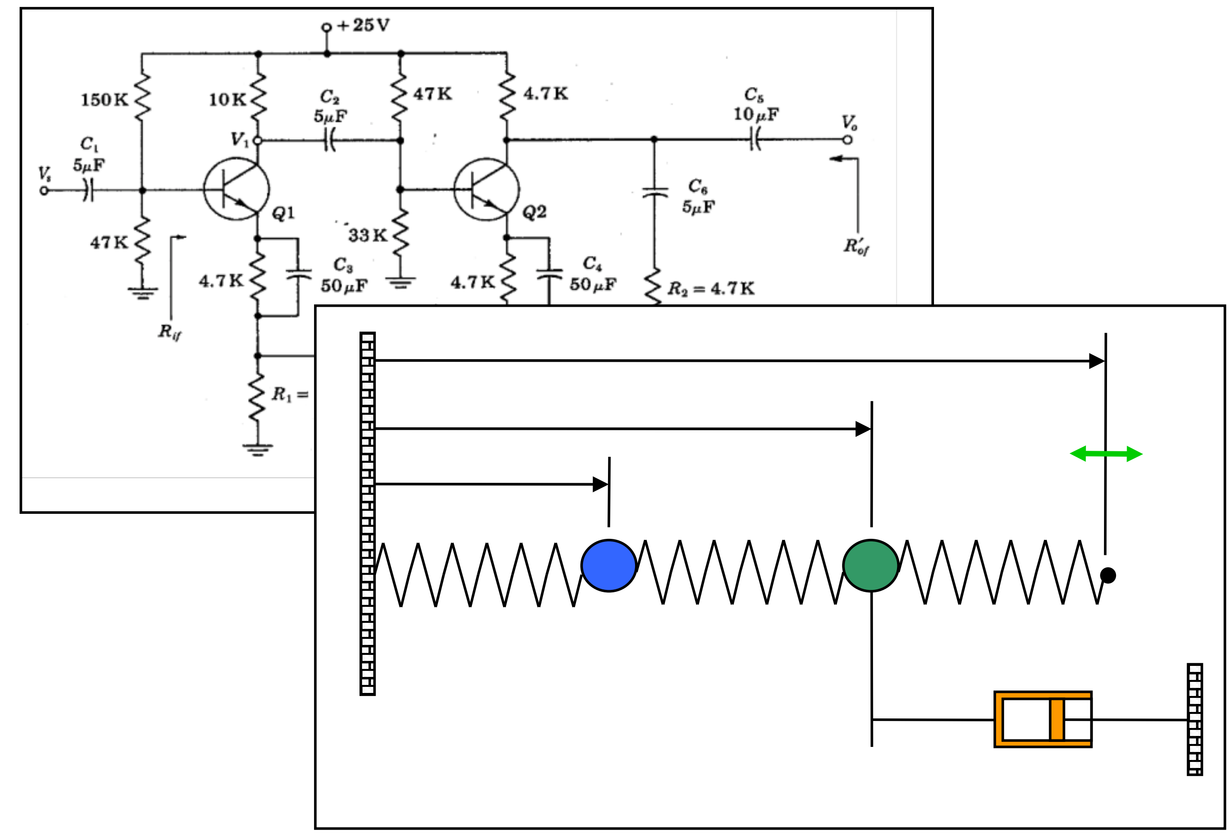
- For zero damping, $j\omega_1$ and $j\omega_2$
- $\omega_1$ and $\omega_2$ correspond frequency response peaks
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for lower freq.
  - Out-of phase motion for higher freq.



# Summary: Linear Systems



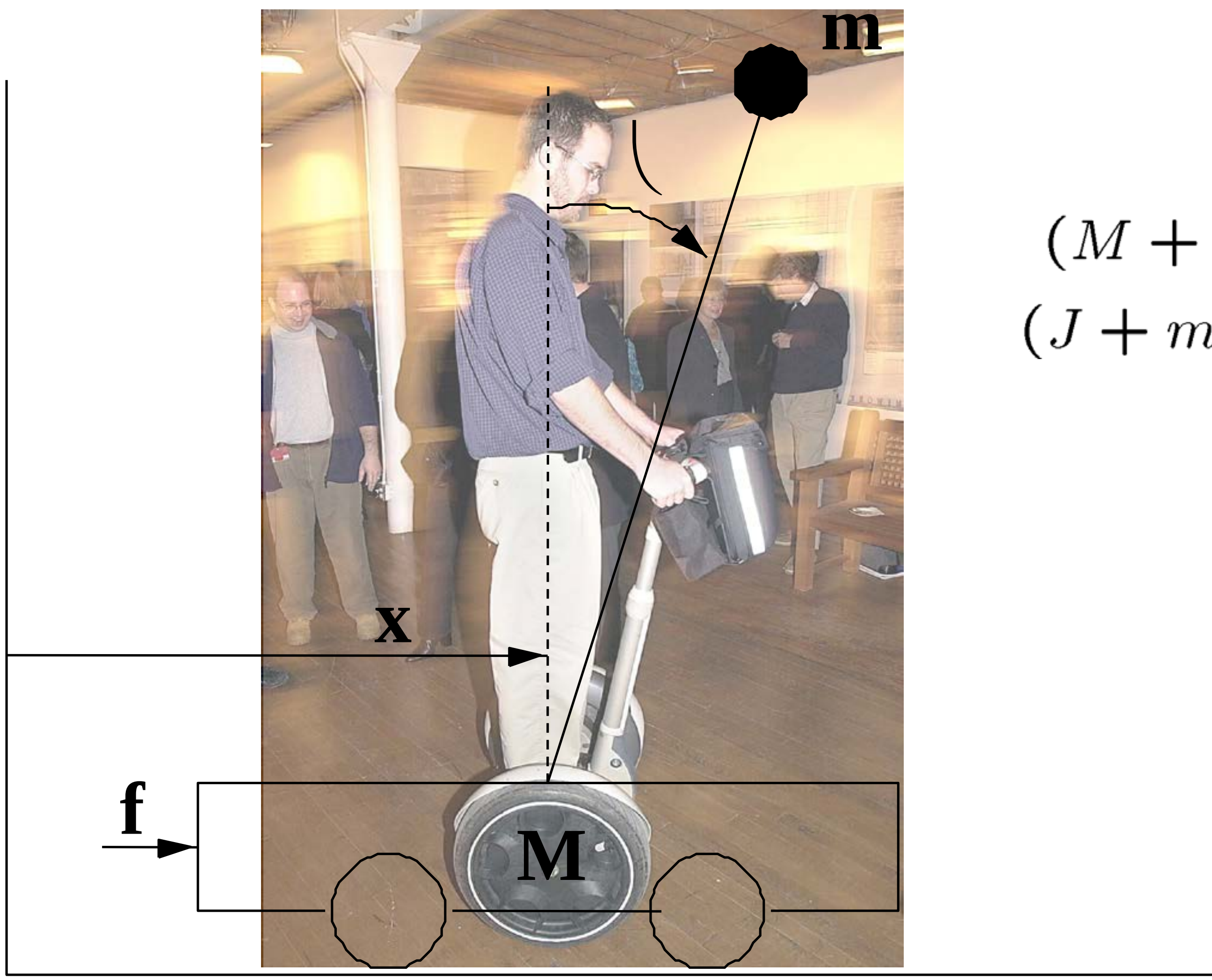





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- Properties of linear systems
  - Linearity with respect to initial condition and inputs
  - Stability characterized by eigenvalues
  - Many applications and tools available
  - Provide local description for nonlinear systems

# Example: Inverted Pendulum on a Cart




$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$

$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = 0$

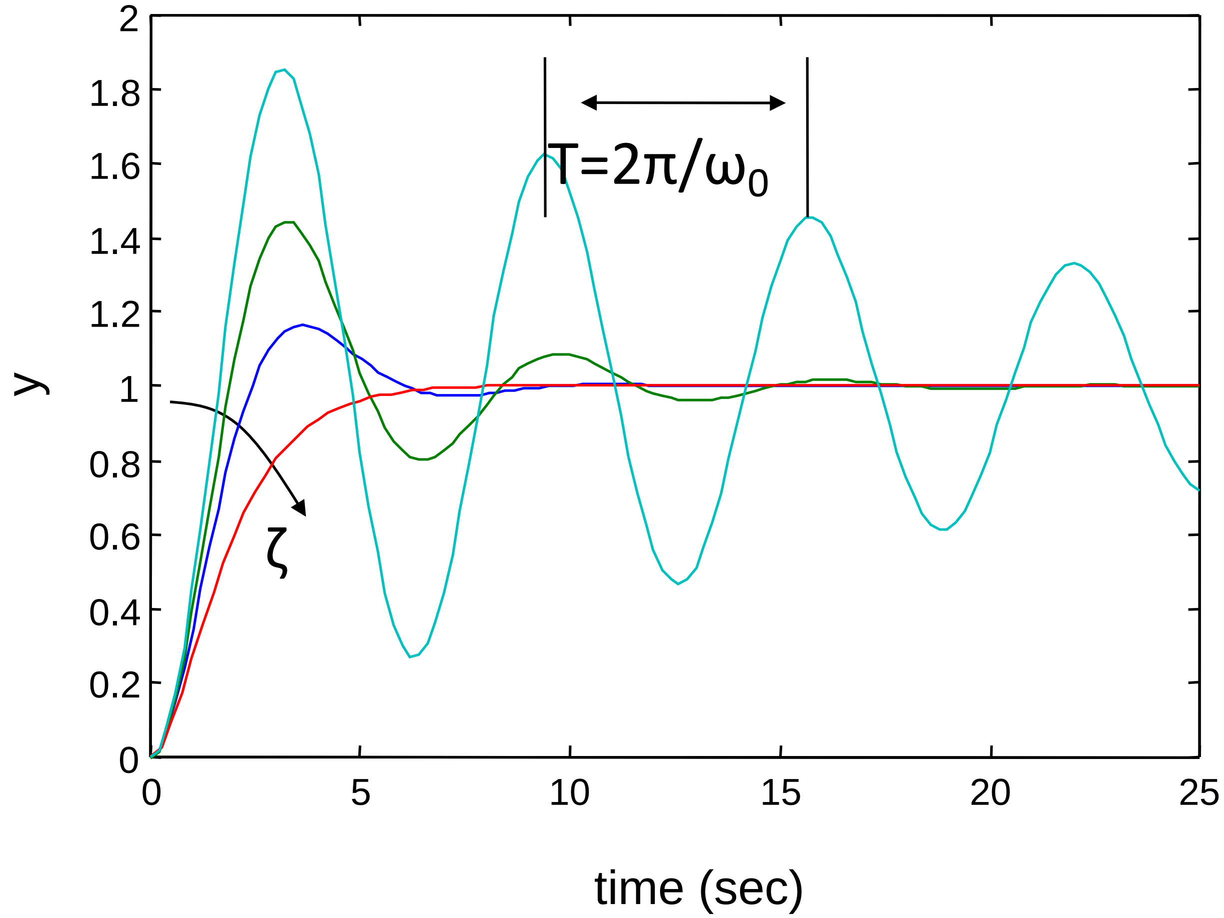
$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{J(M+m)+Mml^2} & \frac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m)+Mml^2} & \frac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{J(M+m)+Mml^2} \\ \frac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

# Second Order Systems

- Many important examples:
- Response of 1st and 2nd order systems -> insight to response for higher orders (eigenvalues of A are either real or complex)
  - Exception is non-diagonalizable A (non-trivial Jordan form)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \qquad \leftrightarrow \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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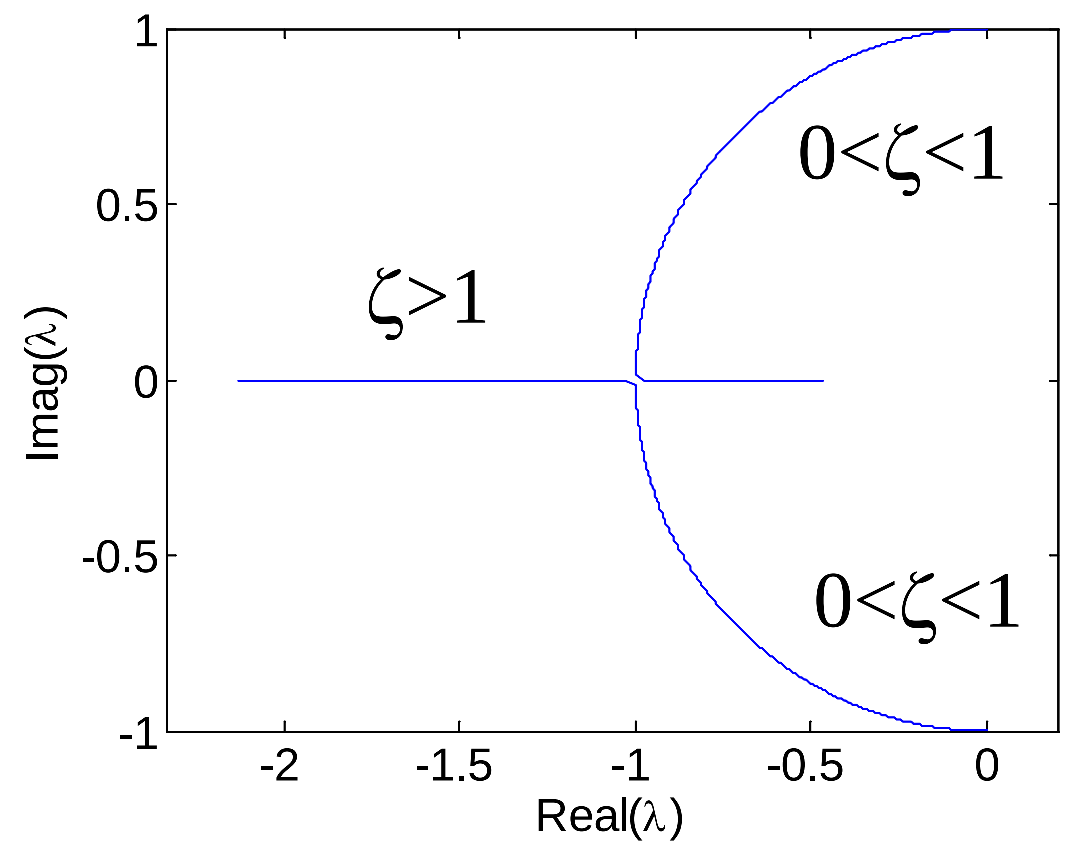


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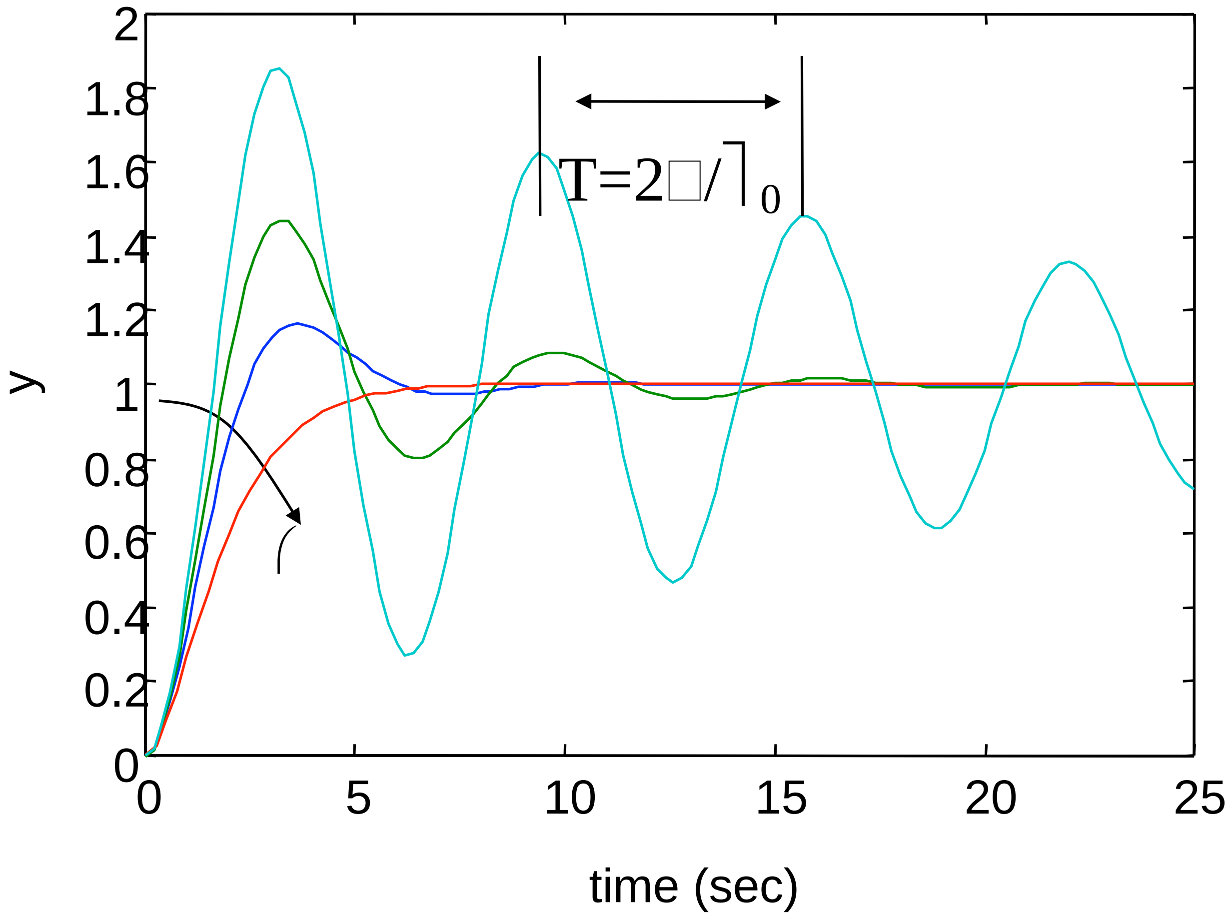
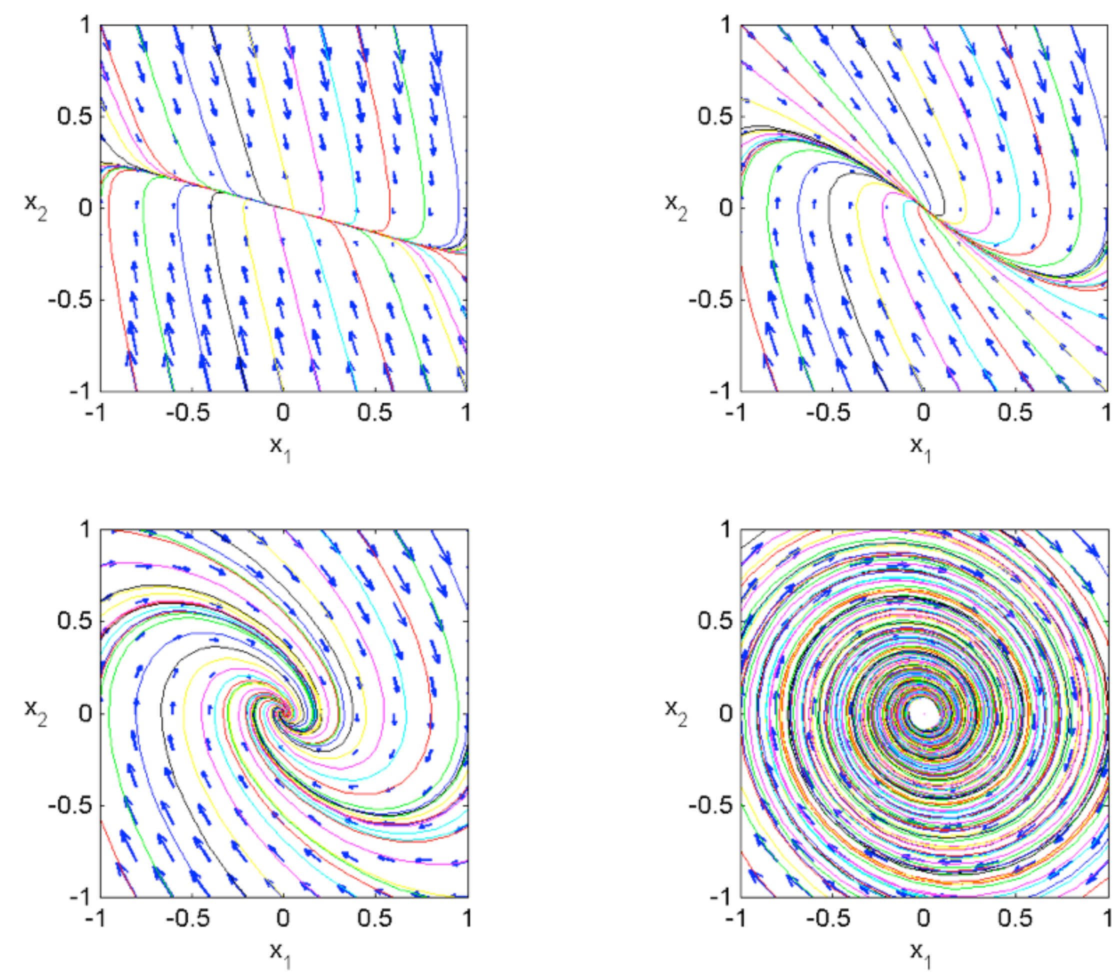
- Will study more next week

# Second Order Systems

**Important class of systems in many applications areas**

$$\ddot{q}+2\zeta\omega_0\dot{q}+\omega_0^2 q = u \quad \longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_0^2 & 2\zeta\omega_0\end{bmatrix} x + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix} u$$




- Analytical formulas exist for overshoot, rise time, settling time, etc
- Will study second order systems characteristics in more detail next week