ME/CS 133(a,b)

Kinematics and Odometry for a Differential Drive Vehicle

1 Introduction

These notes develop the relevant equations of motion for a two-wheeled planar *differential drive* robot that are needed to implement *wheel odmetry*. *Odometry* is a process in which a robot uses data from on-board sensors (typically proprioceptors) to estimate its change in position over time, relative to a given starting configuration. In wheel odometry, the rotation of the robot's wheels (which we assume can be measured by sensors) provide the data needed to estimate the robot's displacement as a function of the wheel rotations. The equations developed below relate wheel motions to robot body motions. The detailed derivation is followed by a brief discussion of how to implement wheel odometry using these equations.

Figure 1 shows both a top view and a side view of an idealized differential drive vehicle.

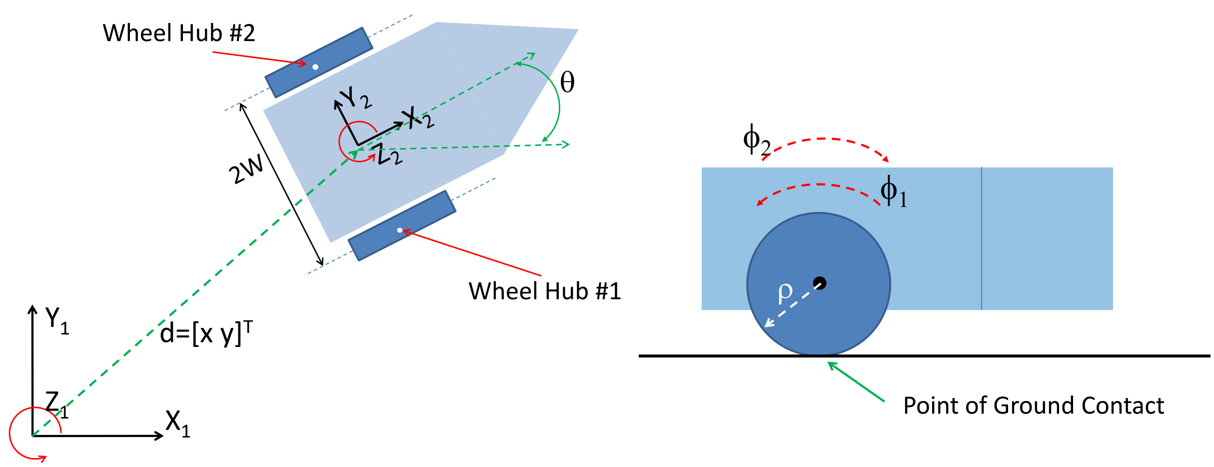

Figure 1: Schematic diagram of an idealized differential drive vehicle. **Left:** top view of the vehicle. **Right:** Side view of the vehicle.

1.1 Basic Kinematic Principles

The key equations will be developed from basic principles of rigid body kinematics. Let $\mathcal{F}_1$ be a fixed planar "observing" reference frame (whose unit basis vectors are $\{x_1, y_1, z_1\}$) that defines coordinates in the plane. Consider a rigid body moving in this plane (see Figure 2). Rigidly attach a reference frame, $\mathcal{F}_2$ (with unit basis vectors $\{x_2, y_2, z_2\}$), to the robot's moving body. Seee Fig. 1

Coordinate Transformations: Recall that the transformation between the coordinates of a point P in the moving rigid body (as described by an observer in $\mathcal{F}_2$) to its equivalent representation in the fixed observing reference frame $\mathcal{F}_1$ is:

$$\vec{\eta} = \vec{d} + R(\theta)\vec{r} \tag{1}$$

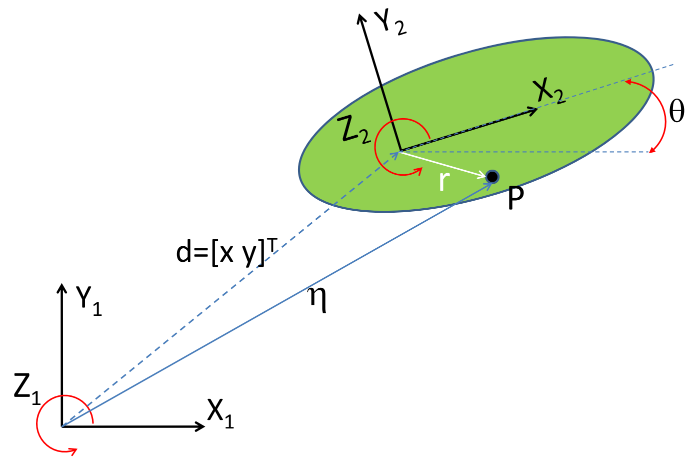

Figure 2: Geometry of a moving rigid body

where $\vec{d}$ is the vector from the origin of $\mathcal{F}_1$ to the origin of the $\mathcal{F}_2$, and where $\vec{r}$ is the vector to point P, as described in reference frame $\mathcal{F}_2$ (see Figure 2). The vector $\vec{\eta}$ points from the origin of $\mathcal{F}_1$ frame to that same point P, but its components are described in the fixed observing frame $\mathcal{F}_1$. The vectors $\vec{\eta}$ and $\vec{r}$ are two-dimensional for planar rigid bodies, and three-dimensional for general spatial rigid bodies. R is a rotation matrix, and θ parametrizes a rigid body rotation. In the case of planar rigid bodies, the rotation matrix takes the simple form:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Velocity Transformations: If the rigid body moves with respect to the fixed observing frame, then the velocity of the point P on the rigid body, as seen by the observer in the fixed frame, $\mathcal{F}_2$, is:

$$\vec{V} = \frac{d\vec{\eta}}{dt} = \dot{\vec{d}} + \vec{\omega} \times (R(\theta)\vec{r}) \tag{2}$$

where $||\vec{\omega}|| = \frac{d\theta(t)}{dt} = \dot{\theta}$ is the *spatial angular velocity* of the moving rigid body (the body's angular rate of rotation) and $||\vec{\omega}||^{-1}\vec{\omega}$ is the axis of rotation, as described in frame $\mathcal{F}_1$). For planar rigid body motion, the rotation axis is normal to the plane. The 2×1 vector $\dot{\vec{d}}$ describes the velocity (as seen by the fixed observer) of the point in the moving body which is coincident with the origin of frame $\mathcal{F}_2$. One can think of $\dot{\vec{d}}$ as the *translational velocity* of the vehicle.

2 Kinematic Analysis of the Differential Drive Vehicle

The goal of this section is to derive a relationship between the wheel speeds and the vehicle speed. In order to derive this relationship, we will make the following assumptions:

A1: each wheel contacts the ground at a single point,

A2: both wheels roll on the ground without slipping.

If the wheels roll without slipping, then the point on each wheel (whose Cartesian location is denoted by $\vec{C}_1$ for the 1^{st} (right) wheel, and by $\vec{C}_2$ for the 2^{nd} (left) wheel) which is instantaneously in contact with the ground must have zero velocity. Else, if that point has non-zero velocity, the wheel must be slipping with respect to the ground–in this case we need to implement a more complicated analysis that is beyond the scope of these notes..

2.1 Velocities of the wheel-ground contact points

To calculate the velocities of points $\vec{C}_1$ and $\vec{C}_2$ (denoted $\vec{V}_{C_1}$ and $\vec{V}_{C_2}$), we will resort to "diagram chasing." Diagram chasing involves a sequence of simple rigid body transformations of the types summarized in the last section.

Let $\vec{H}_1$ and $\vec{H}_2$ denote the location of the "hubs" on their respective wheels (see Fig. 1). The hub is the point that defines the center of the wheel's rotation. This point is rigidly affixed to the main vehicle body, and thus the velocity of the hub points (denoted $\vec{V}_{H_1}$ and $\vec{V}_{H_2}$) can be calculated from the vehicle's velocity via application of formula (2). Because $\vec{C}_1$ is a point on the rotating rigid body wheel, its velocity can also be calculated from Eq. (2) because we know the speed of the hub, and we assume that we know the rotational speed of the wheel. xo The velocity of hub 1 is:

$$\vec{V}_{H_1} = \vec{V}_R + \vec{\omega}_R \times R\vec{r}_{H_1} \tag{3}$$

where $\vec{V}_R$ and $\vec{\omega}_R$ are the linear and angular velocities of the robot:

$$\vec{V}_R = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \qquad \vec{\omega}_R = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \tag{4}$$

and $\vec{r}_{H_1}$ is the vector from origin of the body fixed frame to the hub point, and R denotes the relative orientation of the body fixed reference frame with respect to the fixed observing reference frame:

$$\vec{r}_{H_1} = \begin{bmatrix} 0 \\ -W \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where $2W$ is half of the width of the wheel base. Substituting Eq.s (4) and (5) into Eq. (3) yields:

$$\vec{V}_{H_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} . \tag{6}$$

Using a similar analysis, the velocity of the second hub is:

$$\vec{V}_{H_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta \\ 0 \end{bmatrix} . \tag{7}$$

Since the point $\vec{C}_1$ is rigidly affixed to moving wheel #1, and we know the velocity of the hub, $\vec{V}_{H_1}$, then:

$$\vec{V}_{C_1} = \vec{V}_{H_1} + \vec{\omega}_{W_1} \times \vec{r}_{H_1C_2} \tag{8}$$

where $\vec{\omega}_{w_1}$ is the angular velocity of wheel #1 and $\vec{r}_{H_1C_1}$ is the vector from the point of hub 1 to $\vec{C}_1$. Note that the angular velocity of the wheel has two components: the angular velocity due to the rotation of the robot's body, and the rotation due to the wheel relative to the body.

We can compute this angular velocity as follows.

$$\begin{aligned} \omega\vec{_{W_1}} &= [\frac{d}{dt}(R_{12}R_{2,W_1})(R_{12}R_{2,W_1})^T]^\vee = [(\dot{R}_{12}R_{12}^T) + R_{12}(\dot{R}_{2,W_1}R_{2,W_1}^T)R_{12}^T]^\vee \\ &= \vec{\omega}_{12} + R_{12}\vec{\omega}_{2,W_1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} + \dot{\phi}_1 \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \end{aligned} \tag{9}$$

Substituting Eq.s (9) and (6) into Eq. (8), and using the fact that $\vec{r}_{H_1C_2} = \begin{bmatrix} 0 & 0 & \rho \end{bmatrix}^T$, yields:

$$\vec{V}_{C_1} = \begin{bmatrix} \dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta \\ \dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\cos\theta \\ 0 \end{bmatrix} . \tag{10}$$

Using a similar derivation, the velocity of the contact point on the other wheel can be found as:

$$\vec{V}_{C_2} = \vec{V}_{H_2} + \vec{\omega}_{W_2} \times \vec{r}_{H_2C_2} = \begin{bmatrix} \dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta \\ \dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta \\ 0 \end{bmatrix} \tag{11}$$

2.2 Using the Nonholonomic Constraints for Odometry

From Equations (10) and (11), we see that the constraints, which require that the points on the wheels in contact with the ground must have zero velocity, result in these four equations:

$$\begin{aligned} \dot{x} + W\dot{\theta}\cos\theta + \rho\dot{\phi}_1\cos\theta &= 0 \, , & (12) \\ \dot{y} + W\dot{\theta}\sin\theta + \rho\dot{\phi}_1\sin\theta &= 0 \, , & (13) \\ \dot{x} - W\dot{\theta}\cos\theta - \rho\dot{\phi}_2\cos\theta &= 0 \, , & (14) \\ \dot{y} - W\dot{\theta}\sin\theta - \rho\dot{\phi}_2\sin\theta &= 0 \, . & (15) \end{aligned}$$

These constraints are said to be *nonholonomic* since they are not integrable (there is no function $f(x, y, \theta, \phi_1, \phi_2)$ that can be differentatiated with respect to time to yield these constraint equations). These equations give us four equations in the three unknowns $(\dot{x}, \dot{y}, \dot{\theta})$, assuming that $\dot{\phi}_1$ and $\dot{\phi}_2$ are known. In practice, we assume that there will be sensors on the wheels that measure the angular displacement of the wheel, and possibly its angular rate of rotation as well. If there is no sensor to directly measure the wheels' rates of rotation, we can numerically differentiate the angular measurements of the wheels' rotation angles over time to derive an approximation of their rotation rates.

One of these constraint equations is redundant, and thus we have 3 unique equations in 3 unknowns. Hence, we can solve for the vehicle's rigid body velocity in the plane as a function of the wheel velocities. To realize the desired relationship, Equations (12)-(15) can be manipulated in the following fashion. Add Equations (12) and (14)

$$2\dot{x} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta = 0 \;\Rightarrow\; \dot{x} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta \;. \tag{16}$$

Similarly, add Equations (13) and (15)

$$2\dot{y} - \rho(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta = 0 \;\Rightarrow\; \dot{y} = \frac{\rho}{2}(\dot{\phi}_2 - \dot{\phi}_1)\sin\theta \;. \tag{17}$$

Next, subtract Eq. (14) from Eq. (12) and subtract Eq. (15) from Eq. (13) to yield:

$$2W\dot{\theta}\cos\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta = 0 \tag{18}$$
$$2W\dot{\theta}\sin\theta + \rho(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta = 0 \;. \tag{19}$$

Multiply Eq. (18) by $\cos\theta$ and multiply Eq. (19) by $\sin\theta$. Add the two resulting equations to yield:

$$2W\dot{\theta} + \rho(\dot{\phi}_1 + \dot{\phi}_2) = 0 \quad\Rightarrow\quad \dot{\theta} = -\frac{\rho}{2W}(\dot{\phi}_1 + \dot{\phi}_2) \;. \tag{20}$$

In summary, the wheel motions $\dot{\phi}_1$ and $\dot{\phi}_2$ are related to the vehicle motion as follows:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \frac{\rho}{2}\begin{bmatrix}(\dot{\phi}_2 - \dot{\phi}_1)\cos\theta\\ (\dot{\phi}_2 - \dot{\phi}_1)\sin\theta\\ -(\dot{\phi}_2 + \dot{\phi}_1)/W\end{bmatrix} \tag{21}$$

which can be equivalently expressed as:

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \frac{\rho}{2}\begin{bmatrix}(\dot{\phi}_2 - \dot{\phi}_1)\\ (\dot{\phi}_2 - \dot{\phi}_1)\\ -(\dot{\phi}_2 + \dot{\phi}_1)/W\end{bmatrix} \;. \tag{22}$$

3 Wheel Odometry

Assume that at time $t = 0$ the vehicle is situated at configuration (x_0, y_0, θ_0). At any time $t > 0$ the robot's configuration can in theory be determined from Equation (21) as follows:

$$\begin{bmatrix}x(t)\\ y(t)\\ \theta(t)\end{bmatrix} = \begin{bmatrix}x(0)\\ y(0)\\ \theta(0)\end{bmatrix} + \int_0^t \begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{\theta}(t)\end{bmatrix} dt = \begin{bmatrix}x(0)\\ y(0)\\ \theta(0)\end{bmatrix} + \frac{\rho}{2}\int_0^t \begin{bmatrix}(\dot{\phi}_2(t) - \dot{\phi}_1(t))\cos\theta(t)\\ (\dot{\phi}_2(t) - \dot{\phi}_1(t))\sin\theta(t)\\ -(\dot{\phi}_2(t) + \dot{\phi}_1(t))/W\end{bmatrix} dt \tag{23}$$

In theory, if we knew the wheel velocities, $\dot{\phi}_1(t)$ and $\dot{\phi}_2(t)$, we could estimate the current robot position from this relationship. However, this integral equation is an *implicit* function of $\theta(t)$, and hence can only be solved numerically.

A straightforward numerical approach used in common practice is to approximate the derivatives (velocities) using a *backward difference* scheme:

$$\dot{x}(t_k) \simeq \frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}} = \frac{x(t_k) - x(t_{k-1})}{\Delta t} \tag{24}$$

$$\dot{y}(t_k) \simeq \frac{y(t_k) - y(t_{k-1})}{t_k - t_{k-1}} = \frac{y(t_k) - y(t_{k-1})}{\Delta t} \tag{25}$$

$$\dot{\theta}(t_k) \simeq \frac{\theta(t_k) - \theta(t_{k-1})}{t_k - t_{k-1}} = \frac{\theta(t_k) - \theta(t_{k-1})}{\Delta t} \tag{26}$$

where $\Delta t = t_k - t_{k-1}$ for all k. That is, we carry out the numerical approximation process over a *small* uniform time increment, Δt.

Substituting these approximations into Equation (21), using the concept of backward differentiation, and rearranging yields the following equation:

$$\begin{bmatrix} x(t_k) \\ y(t_k) \\ \theta(t_k) \end{bmatrix} = \begin{bmatrix} x(t_{k-1}) \\ y(t_{k-1}) \\ \theta(t_{k-1}) \end{bmatrix} + \frac{\rho}{2} \begin{bmatrix} (\Delta\phi_2 - \Delta\phi_1)\cos\theta(t_{k-1}) \\ (\Delta\phi_2 - \Delta\phi_1)\sin\theta(t_{k-1}) \\ -(\Delta\phi_2 + \Delta\phi_1)/W \end{bmatrix} \tag{27}$$

where $\Delta\phi_1 = \phi_1(t_k) - \phi_1(t_{k-1})$ and $\Delta\phi_2 = \phi_2(t_k) - \phi_2(t_{k-1})$. That is, if we know the robot's position, $(x(t_{k-1}), y(t_{k-1}), \theta(t_{k-1}))$, at time t_{k-1}, and if we know how much each wheel rotated between time t_{k-1} and time t_k, then we can estimate the robot's position at time t_k using formula (27). This is the basis of wheel odometry.

Clearly, the robot's estimate of its position will *drift* over time, as there are at least four sources of error which can corrupt the position estimation process:

1. The wheels slip on the ground, violating the *no-slip* assumption upon which the equations are based.

2. The equations of motion are only an approximation to reality, since real wheels do not have zero thickness, nor do they contact the ground at a single point.

3. The backwards-difference approximation introduces numerical error into the integration process.

4. The wheel angle measurements will always have some noise and uncertainty.