ME/CS 133(a): Solution to Homework #2

Problem 1:(10 Points, Problem 4(a,b) in Chapter 2 of MLS).

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\ &= R(\vec{w} \times (R^T\vec{v})) \\ &= (R\vec{w}) \times (RR^T\vec{v}) \\ &= (R\vec{w}) \times \vec{v} \\ &= \widehat{(R\vec{w})}\vec{v}\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\ &= (R\hat{v}R^T)(R\vec{w}) \\ &= R\hat{v}R^TR\vec{w} \\ &= R(\hat{v}\vec{w}) \\ &= R(\vec{v} \times \vec{w})\end{aligned}$$

Problem 2: (15 points, Problem 5 of chapter 2 in the MLS text).

Part (a): This result was derived in class. Alternatively, you could show that $A = (I - \hat{a})^{-1}(I + \hat{a})$ is a matrix in $SO(3)$ for 3×3 skew symmetric matrix $\hat{a}$ by showing that A is orthogonal and that $det(A) = +1$. Let us first show that A is orthogonal.

$$\begin{aligned}AA^T &= (I-\hat{a})^{-1}(I+\hat{a})\big((I-\hat{a})^{-1}(I+\hat{a})\big)^T = (I-\hat{a})^{-1}(I+\hat{a})(I-\hat{a})(I+\hat{a})^{-1} \\ &= (I-\hat{a})(I+\hat{a})^{-1}(I-\hat{a})^{-1}(I+\hat{a})\ .\end{aligned}$$

Note that $(I+\hat{a})^{-1}(I-\hat{a})^{-1} = \big((I-\hat{a})(I+\hat{a})\big)^{-1} = (I-\hat{a}^2)^{-1} = \big((I+\hat{a})(I-\hat{a})\big)^{-1} = (I-\hat{a})^{-1}(I+\hat{a})^{-1}$ Therefore:

$$AA^T = (I-\hat{a})(I+\hat{a})^{-1}(I-\hat{a})^{-1}(I+\hat{a}) \ = (I-\hat{a})(I-\hat{a})^{-1}(I+\hat{a})^{-1}(I+\hat{a}) = I\ .$$

We just showed that $A \in \mathcal{O}(3)$. The orthogonal group has two subcomponents: $det(A) = +1$ and $det(A) = -1$. All of the matrices in each component are continuously deformable into another matrix in the respective component. In the limit that $\vec{a} \to 0$, $\hat{a} \to 0$. In that case, $A = I$, which has determine of $+1$. Hence, matrices with $\vec{a} \neq 0$ must in the same component as matrices with $\vec{a} = 0$, which is the component consisting of matrices in $SO(3)$.

Part (b): This is a calculation. The hard part is to derive an expression for $(I - \hat{a})^{-1}$:

$$(I-\hat{a})^{-1} = \frac{1}{1+||a||^2}\begin{bmatrix} (1+a_x^2) & (-a_z + a_x a_y) & (a_y + a_x a_z) \\ (a_z + a_x a_y) & (1+a_y^2) & (-a_x + a_y a_z) \\ (-a_y + a_x a_z) & (a_x + a_y a_z) & (1+a_z^2) \end{bmatrix}$$

where $\vec{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$.

Part (c): There are two ways to solve this. The simplest way is to use the result of part 5(b) quoted in the text:

$$R = \frac{1}{1+||a||^2}\begin{bmatrix} 1+a_1^2-a_2^2-a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3+a_2) \\ 2(a_1a_2+a_3) & 1-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_1) \\ 2(a_1a_3-a_2) & 2(a_2a_3+a_1) & 1-a_1^2-a_2^2+a_3^2 \end{bmatrix} \tag{1}$$

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$trace(R) = \frac{3-||a||^2}{1+||a||^2} \Rightarrow ||a||^2 = \frac{3-trace(R)}{1+trace(R)} = \frac{3-r_{11}-r_{22}-r_{33}}{1+r_{11}+r_{22}+r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of R yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1+||a||^2}{4}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

If you didn't use the results of 5(b) in the text, then you would have started with Cayley's formula $R = (I-\hat{a})^{-1}(I+\hat{a})$ and derived Equation (1).

Problem 3: (5 points, Problem 8(b) of chapter 2 in the MLS text).

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$

Problem 4: (15 points, Euler Angles)

Let Z-X-Y Euler angles be denoted by ψ, ϕ, and γ.

- **Part (a):** Develop an expression for the rotation matrix that describes the Z-X-Y rotation as a function of the angles ψ, ϕ, and γ.

  Rotation about the z-axis by angle ψ can be represented by a rotation matrix whose form can be determined from the Rodriguez Equation:

$$Rot(\vec{z},\psi) = I + \sin\psi\hat{z} + (1-\cos\psi)\hat{z}^2 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

  Using the Rodriguez equation, the rotations about the y-axis and x-axis can be similarly found as:

$$Rot(\vec{x},\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad Rot(\vec{y},\gamma) = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} .$$

  Multiplying the matrices yields the result:

$$\begin{aligned} R(\psi,\phi,\gamma) &= Rot(\vec{z},\psi)\ Rot(\vec{x},\phi)\ Rot(\vec{y},\gamma) \\ &= \begin{bmatrix} (c\psi\ c\gamma\ -\ s\psi\ s\phi\ s\gamma) & -s\psi\ c\phi & (c\psi\ s\gamma\ +\ s\psi\ s\phi\ c\gamma) \\ (s\psi\ c\gamma\ +\ c\psi\ s\phi\ s\gamma) & c\psi\ c\phi & (s\psi\ s\gamma\ -\ c\psi\ s\phi\ c\gamma) \\ -c\phi\ s\gamma & s\phi & c\phi\ c\gamma \end{bmatrix} \end{aligned} \tag{2}$$

  where $c\phi$ and $s\phi$ are respectively shorthand notation for $\cos\phi$ and $\sin\phi$, etc.

- **Part (b):** Given a rotation matrix of the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{3}$$

  compute the angles ψ, ϕ, and γ as a function of the r_{ij}.

  Direct observation of the matrices in Equations (2) and (3) show that:

$$\sin\phi = r_{32} \ .$$

  Because $\sin(\pi-\phi) = \sin\phi$, there are two solutions to this equation: $\phi_1 = \sin^{-1}(r_{32})$, and $\phi_2 = \pi - \phi_1$. Similar matchings of the matrix components yield:

$$\psi = Atan2[\frac{r_{22}}{\cos\phi}, \frac{-r_{12}}{\cos\phi}]$$

$$\gamma = Atan2[\frac{r_{33}}{\cos\phi}, \frac{-r_{31}}{\cos\phi}]$$

  where the value ϕ_1 or ϕ_2 is used consistently

Problem 5: (Problem 11(a,b) in Chapter 2 of the MLS text).

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi^{'} = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi^{'} = (h\hat{\xi}^{'} h^{-1})^{\vee} \tag{4}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}^{'} = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (4) gives:

$$\xi = (h\hat{\xi}^{'} h^{-1})^{\vee} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.