Problem 1: (10 Points, Problem 4(a,b) in Chapter 2 of MLS).

Part (a): Let’s assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a $3 \times 1$ vector and let $\vec{v}$ be any $3 \times 1$ vector. Then:

$$(R\hat{w}R^T)\vec{v} = R(\vec{w} \times (R^T\vec{v})) = (R\vec{w}) \times \vec{v} = (\hat{R}\vec{w})\vec{v}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (\hat{R}\vec{w})$.

Part (b): We can now assume that part (a) holds.

$$(R\vec{v}) \times (R\vec{w}) = (\hat{R}\vec{v})(\hat{R}\vec{w}) = (\hat{R}\vec{v}R^T)(\hat{R}\vec{w}) = R\hat{v}RT\hat{R}\vec{w} = R(\hat{v}\vec{w}) = R(\vec{v} \times \vec{w})$$

Problem 2: (15 points, Problem 5 of Chapter 2 in the MLS text).

Part (a): This result was derived in class. Alternatively, you could show that $A = (I - \hat{a})^{-1}(I + \hat{a})$ is a matrix in $SO(3)$ for $3 \times 3$ skew symmetric matrix $\hat{a}$ by showing that $A$ is orthogonal and that $\det(A) = +1$. Let us first show that $A$ is orthogonal.

$$AA^T = (I - \hat{a})(I + \hat{a}) = (I - \hat{a})(I + \hat{a})^{-1}(I + \hat{a})(I + \hat{a})^{-1} = (I - \hat{a})^{-1}(I + \hat{a})$$

Note that $(I + \hat{a})^{-1}(I - \hat{a})^{-1} = ((I + \hat{a})(I + \hat{a}))^{-1} = (I - \hat{a})^{-1} = ((I + \hat{a})(I + \hat{a}))^{-1} = (I + \hat{a})^{-1}$. Therefore:

$$AA^T = (I - \hat{a})(I + \hat{a})^{-1}(I + \hat{a})^{-1}(I + \hat{a}) = (I - \hat{a})(I - \hat{a})^{-1}(I + \hat{a})^{-1}(I + \hat{a}) = I.$$

We just showed that $A \in O(3)$. The orthogonal group has two subcomponents: $\det(A) = +1$ and $\det(A) = -1$. All of the matrices in each component are continuously deformable into another matrix in the respective component. In the limit that $\vec{a} \to 0$, $\hat{a} \to 0$. In that case, $A = I$, which has determinant of +1. Hence, matrices with $\vec{a} \neq 0$ must in the same component as matrices with $\vec{a} = 0$, which is the component consisting of matrices in $SO(3)$. 

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Part (b): This is a calculation. The hard part is to derive an expression for \((I - \hat{a})^{-1}\):

\[
(I - \hat{a})^{-1} = \frac{1}{1 + ||a||^2} \begin{bmatrix}
(a_x + a_y a_z) & (1 + a_y a_z) & (a_y + a_z a_x) \\
(a_x + a_y a_z) & (1 + a_y a_z) & (a_y + a_z a_x) \\
(a_x + a_y a_z) & (1 + a_y a_z) & (1 + a_z a_x)
\end{bmatrix}
\]

where \(\hat{a} = [a_x \ a_y \ a_z]^T\).

Part (c): There are two ways to solve this. The simplest way is to use the result of part 5(b) quoted in the text:

\[
R = \frac{1}{1 + ||a||^2} \begin{bmatrix}
1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1 a_2 - a_3) & 2(a_1 a_3 + a_2) \\
2(a_1 a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2 a_3 - a_1) \\
2(a_1 a_3 - a_2) & 2(a_2 a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2
\end{bmatrix}
\]

(1)

where \(||a||^2\) is shorthand notation for \(||a||^2 = a_1^2 + a_2^2 + a_3^2\). Noting that

\[
\text{trace}(R) = 3 - ||a||^2 \Rightarrow ||a||^2 = \frac{3 - \text{trace}(R)}{1 + \text{trace}(R)} = \frac{3 - r_{11} - r_{22} - r_{33}}{1 + r_{11} + r_{22} + r_{33}}
\]

so that an expression for \(||a||^2\) is known, simple algebraic manipulation of the off-diagonal term of \(R\) yield

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} = 1 + ||a||^2 \begin{bmatrix}
r_{32} - r_{23} \\
r_{13} - r_{31} \\
r_{21} - r_{12}
\end{bmatrix}
\]

If you didn’t use the results of 5(b) in the text, then you would have started with Cayley’s formula \(R = (I - \hat{a})^{-1}(I + \hat{a})\) and derived Equation (1).

Problem 3: (5 points, Problem 8(b) of chapter 2 in the MLS text).

\[
e^{g\Lambda g^{-1}} = I + \frac{1}{1!} g\Lambda g^{-1} + \frac{1}{2!} (g\Lambda g^{-1})^2 + \frac{1}{3!} (g\Lambda g^{-1})^3 + \cdots
\]

\[
= I + \frac{1}{1!} g\Lambda g^{-1} + \frac{1}{2!} (g\Lambda^2 g^{-1}) + \frac{1}{3!} (g\Lambda^3 g^{-1}) + \cdots
\]

\[
= g(I + \frac{1}{1!} \Lambda + \frac{1}{2!} \Lambda^2 + \frac{1}{3!} \Lambda^3 + \cdots) g^{-1}
\]

\[
= g e^{\Lambda} g^{-1}
\]

Problem 4: (15 points, Euler Angles)

Let Z-X-Y Euler angles be denoted by \(\psi, \phi, \text{ and } \gamma\).
**Part (a):** Develop an expression for the rotation matrix that describes the Z-X-Y rotation as a function of the angles $\psi$, $\phi$, and $\gamma$.

Rotation about the $z$-axis by angle $\psi$ can be represented by a rotation matrix whose form can be determined from the Rodriguez Equation:

$$\text{Rot}(\vec{z}, \psi) = I + \sin \psi \hat{z} + (1 - \cos \psi) \hat{z}^2 = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the Rodriguez equation, the rotations about the $y$-axis and $x$-axis can be similarly found as:

$$\text{Rot}(\vec{x}, \phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \quad \text{Rot}(\vec{y}, \gamma) = \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ -\sin \gamma & 0 & \cos \gamma \end{bmatrix}.$$

Multiplying the matrices yields the result:

$$R(\psi, \phi, \gamma) = \text{Rot}(\vec{z}, \psi) \text{Rot}(\vec{x}, \phi) \text{Rot}(\vec{y}, \gamma) = \begin{bmatrix} c\psi c\gamma - s\psi s\phi s\gamma \\ -s\psi c\phi \\ c\psi s\gamma + s\psi s\phi c\gamma \end{bmatrix} \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ -\sin \gamma & 0 & \cos \gamma \end{bmatrix}$$

where $c\phi$ and $s\phi$ are respectively shorthand notation for $\cos \phi$ and $\sin \phi$, etc.

**Part (b):** Given a rotation matrix of the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

compute the angles $\psi$, $\phi$, and $\gamma$ as a function of the $r_{ij}$.

Direct observation of the matrices in Equations (2) and (3) show that:

$$\sin \phi = r_{32}.$$

Because $\sin(\pi - \phi) = \sin \phi$, there are two solutions to this equation: $\phi_1 = \sin^{-1}(r_{32})$, and $\phi_2 = \pi - \phi_1$. Similar matchings of the matrix components yield:

$$\psi = \text{Atan}2\left[\frac{r_{22}}{\cos \phi}, -\frac{r_{12}}{\cos \phi}\right]$$

$$\gamma = \text{Atan}2\left[\frac{r_{33}}{\cos \phi}, -\frac{r_{31}}{\cos \phi}\right]$$

where the value $\phi_1$ or $\phi_2$ is used consistently.
Problem 5: (Problem 11(a,b) in Chapter 2 of the MLS text).

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi \hat{\xi}} = I + \frac{\phi}{1!} \hat{\xi} + \frac{\phi^2}{2!} \hat{\xi}^2 + \frac{\phi^3}{3!} \hat{\xi}^3 + \cdots$$

First, let’s consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi \hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \hat{\omega} \hat{\nu} \\ \hat{\omega} & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where $I$ is the $2 \times 2$ identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \hat{\nu} \\ \hat{\omega} & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega} \hat{\nu} \\ \hat{\omega} & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\hat{\xi}' = g^{-1} \hat{\xi} g = \begin{bmatrix} I & -\hat{\omega} \hat{\nu} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \hat{\nu} \\ \hat{\omega} & 0 \end{bmatrix} = \begin{bmatrix} I & \hat{\omega} \hat{\nu} \\ \hat{\omega} & 0 \end{bmatrix}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi \hat{\xi}'} = \begin{bmatrix} e^{\phi \hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi \hat{\xi}} = ge^{\phi \hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi \hat{\omega}} (I - e^{\phi \hat{\omega}}) \hat{\omega} \hat{\nu} \hat{\phi} \\ 0 & 1 \end{bmatrix}$$

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which is clearly an element of $SE(2)$.

**Part (b):** It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation $(v_x, v_y)$.

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the $\text{Ad}$-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, $g$, which is pure translation by $\vec{q}$:

$$\xi = \text{Ad}_h \xi' = (h \xi' h^{-1})^\vee$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{x}' = \begin{bmatrix} \hat{\omega} & 0 \\ \hat{\omega} & 0 \end{bmatrix}.$$

Expanding Eq. (4) gives:

$$\xi = (h \xi' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega} \vec{q} \\ \hat{\omega} & 0 \end{bmatrix}^\vee = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$. 

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