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Koopman Mode Decomposition for Periodic/Quasi-periodic Time Dependence ⋆

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Abstract: In this paper we propose an extension of Koopman operator framework for non-autonomous systems with periodic and quasi-periodic time dependence. Using a time parametrized family of Koopman operators and the associated time dependent eigenvalues and eigenfunctions, and concepts from Floquet theory, we extend the notion of the Koopman Mode Decomposition. We illustrate our framework on several examples.

Keywords: Nonlinear Dynamics, Operator Theory, Non-autonomous Systems

1. INTRODUCTION

In this paper we extend Koopman operator theoretic framework for analysis and decomposition of non-autonomous systems with periodic/quasi-periodic time dependence. Koopman operator is a linear but an infinite-dimensional operator that governs the time evolution of observables or outputs defined on the state space of a dynamical system, see Mezić (2012) and Budisic et al. (2012). Koopman operator being linear admits eigenvalues and eigenfunctions, and enables one to express time evolution of observables as a linear superposition of Koopman modes. This decomposition known as the Koopman Mode Decomposition (KMD), provides a powerful means for the analysis of nonlinear dynamical systems. KMD can be thought of as a generalized Fourier analysis, and offers several advantages over Discrete Fourier Transform (Chen et al. (2012)). Each Koopman mode represents only one frequency component, and thus is expected to decouple dynamics at different time scales more effectively than Proper Orthogonal Decomposition (Susuki et al. (2011)). Recent advances in computing KMD using techniques such as Dynamic Mode Decomposition (DMD) and it variants (see Tu et al. (2014), Williams et al. (2015a) and references there in) has enabled many high dimensional applications such as in: fluid mechanics (Rowley et al. (2009); Chen et al. (2012); Mezić (2012)), building diagnostics (Eisenhower et al. (2010)), power system stability analysis (Susuki and Mezić (2014)), data fusion (Williams et al. (2015b)), and computer vision (Grosek and Kutz (2013); Surana (2015)), to name a few.

The majority of work as discussed above assumes an autonomous setting for application of Koopman operator/DMD framework. A notable exception is recently proposed multi resolution DMD (mrDMD, see Kutz et al. (Unpublished)), which considers non-stationarity in the data. The mrDMD approach uses a standard DMD like procedure in a wavelet like fashion to separate a complex system into a hierarchy of multi-resolution time-scale components. While mrDMD provides an useful algorithm for decomposing data with multiple time scales, it assumes a linear time dependent system as an underlying generative process for the data at different scales, and it is unclear under what conditions it captures nonlinear behavior (as standard DMD does, under appropriate conditions, see Tu et al. (2014)).

In this paper we propose a rigorous Koopman operator theoretic framework for non-autonomous systems with periodic/quasi-periodic time dependence. We begin by introducing a time parametrized family of Koopman operators and associated time dependent eigenvalues/eigenfunctions, and propose a general form of time dependent KMD. We then develop the KMD form explicitly for a time periodic linear system using concepts from Floquet theory, and use the insights obtained to generalize KMD for nonlinear periodic/quasi-periodic time dependent setting. We also outline a numerical procedure for computing proposed time dependent KMD by adapting standard techniques such as DMD.

The paper is organized into seven sections. We start with a review of Koopman operator theoretic framework in Section 2 including notion of autonomous KMD. We outline a notion of KMD for general time dependent systems in Section 3, and further develop it for time-periodic systems in Section 4 and for quasi-periodic time dependence in Section 5. We describe a numerical procedure for computing time-dependent KMD in Section 6. Numerical examples are presented in Section 7, and paper is concluded in Section 8 with directions for future research.

2. KOOPMAN OPERATOR OVERVIEW

We start by reviewing, Koopman spectral decomposition as developed in Mezić (2005, 2012) in context of autonomous systems. Consider a flow $\Phi(t,\mathbf{x}) : \mathbb{R}\times\mathcal{M} \to \mathcal{M}$ on an arbitrary set $\mathcal{M} \subset \mathbb{R}^n$ which satisfies the group property $\Phi(0,\mathbf{x}) = \mathbf{x}$, and $\Phi(s,\Phi(t,\mathbf{x})) = \Phi(s+t,\mathbf{x})$.

Let $\mathcal{F}$ be space of scalar complex valued observables θ : $\mathcal{M} \to \mathbf{C}$ (where $\mathbf{C}$ is complex plane). We assume observables are atleast continuously differentiable i.e. $\mathcal{F} \subset C^1(\mathcal{M})$, see Mauroy and Mezić (2013) and Mohr and Mezić (Unpublished) for discussion on appropriate choice of $\mathcal{F}$.

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The Koopman (semi)group of operators $\mathcal{U}^t : \mathcal{F} \to \mathcal{F}$ associated with the flow Φ is defined by

$$(\mathcal{U}^t\theta)(\mathbf{x}) = \theta \circ \Phi(t, \mathbf{x}), \quad \theta \in \mathcal{F}. \tag{1}$$

Consider the flow induced by an autonomous dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2}$$

then, $\Phi(t, \mathbf{x}_0)$ is solution of above ODE with initial condition $\mathbf{x}_0$. If in addition θ and $\mathbf{f}$ are continuously differential, then $\tilde{\theta}(t, \mathbf{x}) = \mathcal{U}^t\theta$ is the solution to PDE

$$\frac{\partial\tilde{\theta}}{\partial t} = \mathbf{f} \cdot \nabla\tilde{\theta} \tag{3}$$

with appropriate boundary conditions.

An eigenfunction of the Koopman operator (or in short Koopman eigenfunction (KEF)) is an observable $\phi \in \mathcal{F}$ that satisfies:

$$\mathcal{U}^t\phi = e^{\lambda t}\phi, \tag{4}$$

where, $\lambda \in \mathbf{C}$ is corresponding Koopman eigenvalue (KE). It follows from (3) that the KEF satisfy the eigenvalue equation

$$\mathbf{f}.\nabla\phi = \lambda\phi. \tag{5}$$

Let ϕ_i be an eigenfunction for the Koopman operator corresponding to the eigenvalue λ_i. Give a vector valued observable $\mathbf{g} : \mathcal{M} \to \mathbb{R}^m$, the Koopman mode (KM), $\mathbf{v}_i$, corresponding to λ_i is the vector of the coefficients of the projection of $\mathbf{g}$ onto span$\{\phi_i\}$.

Note that KE/KEFs (λ, ϕ) depend only on the flow map Φ and the function space $\mathcal{F}$, and not on a particular observable, while the KMs $\mathbf{v}$ are specific to a given observable. We refer to KE, KEF and KM tuple as $(\lambda, \phi, \mathbf{v})$ as the Koopman tuple. Using the Koopman tuple, one can express time evolution of an observable as a linear superposition of Koopman modes leading to the notion of Koopman Mode Decomposition (KMD).

2.1 KMD for Autonomous Systems

To develop the intuition for KMD, consider a linear system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{6}$$

and let λ_k be eigenvalues of A with $\mathbf{v}_k$ and $\mathbf{w}_k^*$ as the corresponding right and left eigenvectors, s.t. $< \mathbf{v}_k, \mathbf{w}_k >= \delta_{ij}$. Then, λ_k are KE with $\mathbf{v}_k$ being the KM, and $\phi_k(\mathbf{x}) =< \mathbf{x}, \mathbf{w}_k >$ being the KEF. This follow from,

$$\dot{\phi} =< \dot{\mathbf{x}}, \mathbf{w} >=< A\mathbf{x}, \mathbf{w} >=< \mathbf{x}, A^*\mathbf{w} >= \lambda\phi \tag{7}$$

and thus

$$\tilde{\phi}(t, \mathbf{x}_0) = U^t\phi(\mathbf{x}_0) = e^{\lambda t}\phi(\mathbf{x}_0). \tag{8}$$

Assuming full set of eigenvectors with distinct eigenvalues, one obtains

$$\mathbf{x} = \sum_{k=1}^{n} < \mathbf{x}, \mathbf{w}_k > \mathbf{v}_k = \sum_{k=1}^{n} \phi_k(\mathbf{x})\mathbf{v}_k, \tag{9}$$

$$\mathcal{U}^t\mathbf{x}(\mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}_0) = \sum_{k=1}^{n} < \mathbf{x}, \mathbf{w}_k > \mathbf{v}_k \tag{10}$$

$$= \sum_{k=1}^{n} e^{\lambda_k t}\phi_i(\mathbf{x}_0)\mathbf{v}_k. \tag{11}$$

Next for a periodic orbit $\mathbf{p}(t) = \mathbf{p}(t + T)$ of (2), consider the Fourier expansion

$$\mathbf{p}(t) = \sum_{j=1}^{\infty} e^{i\omega_j t}\mathbf{v}_j. \tag{12}$$

Consider $\mathcal{U}_{\mathcal{S}}^t$ restricted to the invariant set $\mathcal{S} = \{\mathbf{p}(t) : t \in [0, T]\}$ and let $\Phi_{\mathbf{p}} \equiv \mathbf{p}(t) : [0, T] \to \mathcal{S}$. Then $\phi_j : \mathcal{S} \to \mathbf{C}$

$$\phi_j(\mathbf{s}) = e^{i\omega_j \Phi_{\mathbf{p}}^{-1}(\mathbf{s})} \tag{13}$$

is KEF of $\mathcal{U}_{\mathcal{S}}^t$ with KE $i\omega_j$, where note $\Phi_{\mathbf{p}}^{-1}(\mathbf{s}) = t$. Thus, $\mathbf{v}_j$ are the KMs, since

$$\mathbf{s} = \sum_{j=1}^{\infty} \phi_j(\Phi_{\mathbf{p}}^{-1}(\mathbf{s}))\mathbf{v}_j. \tag{14}$$

An analogous result in discrete time was given in Rowley et al. (2009).

The expansions (11) and (14) generalizes in some other cases. Assuming, that the flow map Φ is measure μ preserving and ergodic, then $\mathcal{U}^t$ is unitary operator and the spectral decomposition of any $\mathbf{g}(\mathbf{x})$ is given by Mezić (2005),

$$\mathcal{U}^t\mathbf{g}(\mathbf{x}) = \mathbf{v}_0 + \sum_{k=1}^{\infty} e^{\lambda_k t}\phi_k(\mathbf{x})\mathbf{v}_k + \int_0^1 e^{i2\pi\alpha t} dE(\alpha)(\mathbf{g}(\mathbf{x})) \tag{15}$$

where,

$$\mathbf{v}_0 = \int_{\mathcal{M}} \mathbf{g}(\mathbf{x})d\mu(\mathbf{x}), \tag{16}$$

and

$$\mathbf{v}_k = \int_{\mathcal{M}} \mathbf{g}(\mathbf{x})\overline{\phi}_k(\mathbf{x})d\mu(\mathbf{x}), \tag{17}$$

are the KMs. We will refer to the expansion (15) as Koopman Mode Decomposition (KMD) following Susuki and Mezić (2014), with $\mathbf{v}_i$ being the Koopman modes associated with eigenfunction ϕ_i and the observable $\mathbf{g}$. The modes capture correlations in the components of the observable, while the corresponding eigenvalues define growth/decay rates and oscillation frequencies for the mode.

Koopman operator in general could posses continuous (e.g as seen above) and residual parts of spectrum in addition to the point or singular spectrum. In whatever follows we will restrict our attention to the singular part of spectrum of $\mathcal{U}_s^t$. Note that one can write, the singular part as

$$\mathcal{U}_s^t \equiv \mathcal{P}_0^t + \sum_{k=1}^{\infty} e^{\lambda_i t}\mathcal{P}_{\lambda_k}^t, \tag{18}$$

where, $\mathcal{P}_0^t, \mathcal{P}_{\lambda_k}^t$ are projection operators

$$\mathcal{P}_0^t\mathbf{g}(\mathbf{x}) = \mathbf{v}_0 \tag{19}$$

$$\mathcal{P}_{\lambda_k}^t\mathbf{g}(\mathbf{x}) = \phi_k(\mathbf{x})\mathbf{v}_k. \tag{20}$$

3. TOWARDS KMD FOR NON-AUTONOMOUS SYSTEMS

Generalization of KMD for general non-autonomous systems remains an open problem. As a step forward, in this paper we develop KMD framework for special cases of time periodic/quasiperiodic non-autonomous systems. However, before we proceed to that, we outline a more general notion of non-autonomous KMD, (full development is beyond scope of this paper and will be treated elsewhere).

Consider a two parameter family of flow map Φ^{t,t_0}, such as induced by a non-autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \tag{21}$$

The flow map now satisfies the cocycle property $\Phi^{t+s,t_0} = \Phi^{t+s,s} \circ \Phi^{s,t_0}$. Let $\mathcal{U}^{t,t_0}$ be a two parameter family of Koopman operator parameterized by t, t_0 with action

$$\mathcal{U}^{t,t_0}\theta(\mathbf{x}) = \theta \circ \Phi^{t,t_0}(\mathbf{x}). \tag{22}$$

Analogous to (4), we define $\phi^{t,t_0}(\mathbf{x})$ to be non-autonomous Koopman eigenfunction, such that

$$\mathcal{U}^{t,t_0}\phi^{t,t_0}(\mathbf{x}) = e^{\lambda(t,t_0)}\phi^{t,t_0}(\mathbf{x}). \tag{23}$$

With this definition and considering only singular part of spectrum of $\mathcal{U}_s^{t,t_0}$, we propose a non-autonomous KMD of an observable $\mathbf{g}(\mathbf{x})$ as

$$\mathcal{U}_s^{t,t_0}\mathbf{g}(\mathbf{x}) \equiv \mathcal{P}_0^{t,t_0}\mathbf{g}(\mathbf{x}) + \sum_{k=1}^{\infty} e^{\lambda_k(t,t_0)}\mathcal{P}_{\lambda_k(t,t_0)}^{t,t_0}\mathbf{g}(\mathbf{x}), \tag{24}$$

where, $\mathcal{P}_0^{t,t_0}, \mathcal{P}_{\lambda_k(t,t_0)}^{t,t_0}$ are appropriate projection operators that project any observable onto the eigenspace of the eigenfunction $\phi_k^{t,t_0}(\mathbf{x})$ of the operator $\mathcal{U}^{t,t_0}$ associated with eigenvalue $\lambda_k(t, t_0)$. In next section we develop an exact form of (24) for time periodic systems.

4. KMD FOR TIME PERIODIC SYSTEMS

4.1 Linear Time Periodic Systems

To develop intuition for KMD in the time periodic case, we first start with a linear time periodic system

$$\dot{\mathbf{y}} = A(t)\mathbf{y}, \tag{25}$$

where, $A(t) = A(t+T)$. Let $M(t)$ be the fundamental matrix solution of (25). According to Floquet theory , there exists $P(t)$ and constant B s.t $M(t) = P(t)\exp(Bt)$, where $P(t)$ is invertible and T periodic. Under change of coordinate $\mathbf{y}(t) = P(t)\mathbf{z}(t)$,

$$\dot{\mathbf{z}} = B\mathbf{z}. \tag{26}$$

Assuming full set of eigenvalues μ_k of B with $\mathbf{v}_k$ and $\mathbf{w}_k^*$ as the corresponding right and left eigenvectors, and using (9), we get

$$\mathbf{z} = \sum_{k=1}^{n} \phi_k(\mathbf{z})\mathbf{v}_k \tag{27}$$

where, $\phi_k(\mathbf{z}) =< \mathbf{z}, \mathbf{w}_k >$. Substituting $\mathbf{z} = P(t)^{-1}\mathbf{y}$, we get

$$\mathbf{y} = \sum_{k=1}^{n} (P(t)\mathbf{v}_k)\phi_k(P^{-1}(t)\mathbf{y}). \tag{28}$$

In the above expansion, $< P^{-1}(t)\mathbf{y}, \mathbf{w}_k(t) >$ and $P(t)\mathbf{v}_k$ can be interpreted as non autonomous versions of KEF and KMs, respectively. Note the difference in the interpretation of the KEF for linear systems $\dot{\mathbf{y}} = A\mathbf{y}$ and that for $\dot{\mathbf{y}} = A(t)\mathbf{y}$. Namely, the KEFs/KMs in the latter case are functions of time!

To develop this interpretation further, consider the Fourier expansion,

$$P(t)\mathbf{v}_k(t) = \sum_{j=1}^{\infty} e^{i\omega_j t}\mathbf{v}_{k,j}, \tag{29}$$

and substitute it in (28) to obtain

$$\mathbf{y} = \sum_{k=1}^{n}\sum_{j=1}^{\infty} \phi_{k,j}(\mathbf{y}, t)\mathbf{v}_{k,j}, \tag{30}$$

where,

$$\phi_{k,j}(\mathbf{y}, t) = e^{i\omega_j t}\phi_k(P^{-1}(t)\mathbf{y}). \tag{31}$$

We next show that $\phi_{k,j}(\mathbf{y}, t)$ can be interpreted as the KEF of the suspended autonomous system

$$\begin{aligned} \dot{\mathbf{y}} &= A(s)\mathbf{y} \\ \dot{s} &= 1, \end{aligned} \tag{32}$$

with state space $(\mathbf{y}, s)$. Consider the time derivative

$$\begin{aligned} \frac{d}{dt}\phi_{k,j}(\mathbf{y}, s) &= i\omega_j e^{i\omega_j s}\phi_k(P^{-1}(s)\mathbf{y}) \\ &\quad + e^{i\omega_j s} < \frac{d}{ds}P^{-1}(s)\mathbf{y}, \mathbf{w}_k > \\ &= (i\omega_j + \mu_k)e^{i\omega_j s}\phi_k(P^{-1}(s)\mathbf{y}) \\ &= (i\omega_j + \mu_k)\phi_{k,j}(\mathbf{y}, s) \end{aligned} \tag{33}$$

where, we used $\frac{d}{ds}P^{-1}(s)\mathbf{y} = BP^{-1}(s)\mathbf{y}$, and

$$\begin{aligned} < BP^{-1}(s)\mathbf{y}, \mathbf{w}_k > &= < P^{-1}(s)\mathbf{y}, B^*\mathbf{w}_k > \\ &= \mu_k < P^{-1}(s)\mathbf{y}, \mathbf{w}_k > . \end{aligned}$$

Thus

$$\phi_{k,j}(t; \mathbf{y}_0, s_0) = \mathcal{U}^t\phi_{k,j}(\mathbf{y}_0, s_0) = e^{\lambda_{k,j}t}\phi_{k,j}(\mathbf{y}_0, s_0), \tag{34}$$

are KEFs for (32) with eigenvalues $\lambda_{k,j} = i\omega_j + \mu_k$, and

$$\begin{aligned} \mathcal{U}^t\mathbf{y}(\mathbf{y}_0, s_0) &= \mathbf{y}(t, \mathbf{y}_0, s_0) \\ &= \sum_{k=1}^{n}\sum_{j=1}^{\infty} \mathcal{U}^t\phi_{k,j}(\mathbf{y}_0, s_0)\mathbf{v}_{k,j}, \\ &= \sum_{k=1}^{n}\sum_{j=1}^{\infty} e^{\lambda_{k,j}t}\phi_{k,j}(\mathbf{y}_0, s_0)\mathbf{v}_{k,j}. \end{aligned} \tag{35}$$

Setting set $t_0 = s_0$, the above expansion can be rewritten as

$$\mathcal{U}^{t,t_0}\mathbf{y}(\mathbf{y}_0) = \sum_{k=1}^{n}\sum_{j=1}^{\infty} e^{\lambda_{k,j}t}\phi_{k,j}^{t_0}(\mathbf{y}_0)\mathbf{v}_{k,j}, \tag{36}$$

and, thus can be reinterpreted as KMD of full state observable for the time periodic linear system (25).

We next develop KMD like expansion (36) of a general observable $\mathbf{g}(\mathbf{y}, t)$ which is time periodic with same period T as the underlying dynamics. In doing so, we first introduce notion of principal Koopman eigenfunctions

$$\mathbf{z}(\mathbf{y}, s) = (z_1(\mathbf{y}, s), ..., z_n(\mathbf{y}, s))$$

which are defined by

$$\mathbf{z}(\mathbf{y}, s) = V^{-1}P^{-1}(s)\mathbf{y},$$

and where, V^{-1} is the diagonalizing matrix for B. Note that $z_k(\mathbf{y}, s) = \phi_k(P^{-1}(t)\mathbf{y}), k = 1, \cdots, n$ which are terms which appear in Eqn. (28). In the case when B is not diagonalizable, one can use the same construction to obtain generalized eigenfunctions of the Koopman operator. We have the following theorem:

Theorem 4.1. Let Floquet exponents $\boldsymbol{\mu} = (\mu_1, \cdot, \mu_n)$ of (eigenvalues of the Floquet matrix B) be distinct. Any vector-valued function $\mathbf{g}(\mathbf{y}, s)$ that is analytic in $\mathbf{y}$ and L^2 in s can be expanded into eigenfunctions of the Koopman operator associated with (32) as follows:

$$\mathbf{g}(\mathbf{y}, s) = \sum_{\mathbf{m}\in\mathbb{N}^n, k\in\mathbb{Z}} \mathbf{a}_{\mathbf{m}k}\mathbf{z}^{\mathbf{m}}(\mathbf{y}, s)e^{iks}, \tag{37}$$

where $\mathbf{a}_{\mathbf{m}k}$ are constant Koopman modes, $\mathbf{m} = (m_1, ..., m_n)$,

$$\mathbf{z}^{\mathbf{m}}(\mathbf{y}, s) = z_1^{m_1}(\mathbf{y}, s) \cdot ... \cdot z_n^{m_n}(\mathbf{y}, s)$$

and $\mathbf{z}^{\mathbf{m}}(\mathbf{y}, s)e^{iks}$ is an eigenfunction of the Koopman operator corresponding to the eigenvalue $e^{(\mathbf{m}\cdot\boldsymbol{\mu}+ik)t}$.

Proof : First, expand in Taylor series to obtain

$$\begin{aligned}\mathbf{g}(\mathbf{y}, s) &= \sum_{\mathbf{m}\in\mathbb{N}^n} b_{\mathbf{m}}(s)\mathbf{y}^{\mathbf{m}} \\ &= \sum_{\mathbf{m}\in\mathbb{N}^n} b_{\mathbf{m}}(s)(P(s)V\mathbf{z})^{\mathbf{m}} \\ &= \sum_{\mathbf{m}\in\mathbb{N}^n} \bar{a}_{\mathbf{m}}(s)\mathbf{z}^{\mathbf{m}},\end{aligned} \tag{38}$$

where $\bar{a}_{\mathbf{m}}(s)$ are T periodic functions. Fourier expansion of these L^2 coefficients and the relationship (34), completes the proof ■

4.2 Nonlinear Time Periodic Systems

Consider a nonlinear time periodic system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{v}(\mathbf{x}, t), \tag{39}$$

where, $\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t + T)$ is time periodic and $\mathbf{v}(\mathbf{x}, t) \sim O(\mathbf{x}^2)$ is a C^2-function, such that the linear system

$$\dot{\mathbf{y}} = A(t)\mathbf{y}, \tag{40}$$

is stable. Under these assumption, to develop KMD of observables defined on (39), we take the following steps:

1) Linearization of time-dependent flow: Transform (39) to (40) in the basin of attraction of $\mathbf{x} = 0$ using a C^1- diffeomorphism $\mathbf{y} = \mathbf{h}(\mathbf{x}, t)$.
2) Use KMD (37) for linear time periodic systems in the transformed space $\mathbf{y}$ and conjugacy in Step 1 to construct KMD expression for (39).

The system (40) with $\mathbf{y} = 0$ as a globally stable fixed point can be thought of as a system with a globally stable limit cycle given by $\mathbf{y} = 0, s \in [0, T)$. The first step then follows by reducing (39) close to its limit cycle to a form that is periodic in time with period T, and linear in variable transverse to the limit cycle. Since time can always be rescaled by $\tau = 2\pi t/T$, the dynamics close to the limit cycle can be considered periodic with period 2π.

Theorem 4.2. Consider a 2π-periodic system (39). If the associated linear system (40) is stable, then $\mathbf{x} = 0$ is stable for (39). Furthermore the system (39) is linearizable in basin of attraction $\mathbf{x} = 0$ to (40) by a C^1-diffeomorphism $\mathbf{h}(\mathbf{x}, t)$.

Proof

i) According to Floquet's theory (see Section 4.1), the solution of the linear system $\dot{\mathbf{y}} = A(t)\mathbf{y}$ can be written as
$$\mathbf{y}(t) = P(t)e^{Bt}\mathbf{y}(0),$$
where, as before $P(t) = P(t + 2\pi)$ is a C^1-matrix with $P(0) = I$. B is a constant matrix which determines the stability of the system. By assumption the origin is stable and thus all eigenvalues (Floquet exponents) of B have negative real parts.

ii) Define a linear Poincaré map by the 2π-time evolution of the linear system:
$$\mathbf{y}_{m+1} = e^{2\pi B}\mathbf{y}_m\,. \tag{41}$$
The corresponding Poincaré map for (39) at section $t = 0$ can be written as
$$\mathbf{x}_{m+1} = T(\mathbf{x}_m) = e^{2\pi B}\mathbf{x}_m + \mathbf{u}(\mathbf{x}_m)\,,$$
where $\mathbf{u}(\mathbf{x}) \sim O(\mathbf{x}^2)$ is a C^2-function. The origin $\mathbf{x}_m = 0$ is a stable fixed point and its basin of attraction $\mathcal{B}$ coincides with that of vector field (39). The map $T(\mathbf{x}_m)$ is a C^2-diffeomorphism defined on $\mathbb{R}^n$ since it is induced by the vector field (39). According to linearization results for maps in Lan and Mezić (2013), there exists a C^1-diffeomorphism $\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{h}}(\mathbf{x})$ defined on $\mathcal{B}$, such that $T(\mathbf{x}_m)$ is linearized into form (41). We denote the inverse diffeomorphism by $\mathbf{x}(\mathbf{y}) = \tilde{\mathbf{k}}(\mathbf{y})$. In the $\mathbf{y}$-space the corresponding basin of attraction of the stable fixed point $\mathbf{y} = 0$ is denoted as $\mathcal{B}' \subseteq \mathbb{R}^n$.

iii) Consider the extended phase space $\mathbb{R}^n \times S^1$, where S^1 represents a circle of length 2π parameterizing the time variable. We build a diffeomorphism between $\mathcal{B} \times S^1$ and $\mathcal{B}' \times S^1$ as follows. First, note that there is a diffeomorphism at section $\Sigma = 0$, $\tilde{\mathbf{h}}(\mathbf{x}) : \mathcal{B} \to \mathcal{B}'$ satisfying
$$\tilde{\mathbf{h}}(T(\mathbf{x})) = e^{2\pi B}\tilde{\mathbf{h}}(\mathbf{x}), \tag{42}$$
where $T = S_0^{2\pi}$. We define
$$\begin{aligned}(\mathbf{y}, s) \equiv \mathbf{h}(\mathbf{x}, s) &\equiv (P(s)e^{Bs}\tilde{\mathbf{h}}(S_s^0(\mathbf{x})), s) \\ &= (P(s)e^{Bs}\mathbf{y}(0), s),\end{aligned} \tag{43}$$
and thus $\mathbf{h}$ provides us with the desired conjugacy. Note that (42) is needed as $\mathbf{y}_0$ would otherwise not be uniquely defined when we increase s by 1.

■

To develop KMD of an analytic observable $\mathbf{g}(\mathbf{x})$ for system (39), we use conjugacy obtained by linearization theorem above, and the Theorem 4.2 from Section 4.1. Using the inverse $\tilde{\mathbf{k}}$ of diffeormorphism $\mathbf{h}$ as discussed above, define $\tilde{\mathbf{g}}(\mathbf{y}, s) = \mathbf{g}(\tilde{\mathbf{k}}(\mathbf{y}, s))$. The function $\tilde{\mathbf{g}}(\mathbf{y}, s)$ is time periodic function and can be expanded based on Theorem 4.2, providing the required formula for KMD of $\mathbf{g}(\mathbf{x})$.

5. KMD FOR SYSTEMS WITH QUASI-PERIODIC TIME-DEPENDENCE

In this section we outline a similar approach as described above for systems with quasi-periodic time dependence. The most general systems we consider are

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{44}$$

that have a quasi-periodic attractor - a torus on which the dynamics is conjugate to $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}, \boldsymbol{\theta} \in \mathbb{T}^m, \boldsymbol{\omega} \in \mathbb{R}^m$, and $\boldsymbol{\omega}$ is a constant and incommensurable vector of frequencies that satisfies $\mathbf{k} \cdot \boldsymbol{\omega} \geq c/|\mathbf{k}|^\gamma$ for some $c, \gamma > 0$. In addition, we ask that the quasi periodic linearization matrix $A(t) = D_{\mathbb{T}^m}$ at the torus has a full spectrum, where the spectrum $\sigma(A)$ of the quasi-periodic matrix is defined as a set of points $\lambda \in \mathbb{R}$ for which the shifted equation

$$\dot{\mathbf{y}} = (A(\boldsymbol{\theta} + \boldsymbol{\omega}t) - \lambda I)\mathbf{y},$$

does not have an exponential dichotomy. Provided $\sigma(A)$ is full - meaning it consists of m isolated points, and $A(\boldsymbol{\theta}+\boldsymbol{\omega}t)$ is sufficiently smooth in $\boldsymbol{\theta}$, in the paper Sell (1978) it is proven that there is a quasi-periodic transformation $P(t)$ and a constant matrix B - which we will call quasi-Floquet- such that the transformation $\mathbf{z}=P(t)\mathbf{y}$ reduces the system to $\dot{\mathbf{z}}=B\mathbf{z}$. Just like in the periodic system case we consider the skew-linear system [1]

$$\begin{aligned}\dot{\mathbf{y}} &= A(\boldsymbol{\theta})\mathbf{y},\\ \dot{\boldsymbol{\theta}} &= \boldsymbol{\omega},\end{aligned} \tag{45}$$

where $\mathbf{y}\in\mathbb{R}^n, \boldsymbol{\theta}\in\mathbb{T}^m$ and $A(\boldsymbol{\theta})$ is a 2π-periodic matrix.

Theorem 5.1. Let quasi-Floquet exponents $\boldsymbol{\mu}=(\mu_1,\cdot,\mu_n)$ of (45) (eigenvalues of the quasi-Floquet matrix B) be distinct. Any vector-valued function $\mathbf{g}(\mathbf{y},\boldsymbol{\theta})$ that is analytic in $\mathbf{y}$ and L^2 in $\boldsymbol{\theta}$ can be expanded into eigenfunctions of the Koopman operator associated with (45) as follows:

$$\mathbf{g}(\mathbf{y},\boldsymbol{\theta})=\sum_{\mathbf{m}\in\mathbb{N}^n,\mathbf{k}\in\mathbb{Z}^m}\mathbf{a}_{\mathbf{mk}}\mathbf{z}^{\mathbf{m}}(\mathbf{y},\boldsymbol{\theta})e^{i\mathbf{k}\cdot\boldsymbol{\theta}},$$

where $\mathbf{a}_{\mathbf{mk}}$ are constant Koopman modes, and

$$\mathbf{z}(\mathbf{y},\boldsymbol{\theta})=(z_1(\mathbf{y},\boldsymbol{\theta}),...,z_k(\mathbf{y},\boldsymbol{\theta}))$$

are as before the principal Koopman eigenfunctions defined by

$$\mathbf{z}(\mathbf{y},\boldsymbol{\theta})=V^{-1}P^{-1}(\boldsymbol{\theta})\mathbf{y},$$

where, V^{-1} is the diagonalizing matrix for B. For $\mathbf{m}=(m_1,...,m_n)$,

$$\mathbf{z}^{\mathbf{m}}(\mathbf{y},\boldsymbol{\theta})=z_1^{m_1}(\mathbf{y},\boldsymbol{\theta})\cdot...\cdot z_n^{m_n}(\mathbf{y},\boldsymbol{\theta})$$

, and $\mathbf{z}^{\mathbf{m}}e^{ik\boldsymbol{\theta}}$ is an eigenfunction of the Koopman operator corresponding to the eigenvalue $e^{(\mathbf{m}\cdot\boldsymbol{\mu}+i\mathbf{k}\cdot\boldsymbol{\omega})t}$.

In order to develop an expression for KMD of a general observable one can follow similar procedure as described in Sections 4.1 and 4.2.

6. TIME DEPENDENT KMD COMPUTATION

Computation of Koopman tuple is a challenging problem and is an active area of research. A variety of techniques have been proposed in literature, including harmonic averaging (Mezić (2005); Mezić and Banaszuk (2004)), generalized Laplacian analysis (Budisic et al. (2012)), and Dynamic Mode Decomposition (DMD) and its variants, (see Tu et al. (2014) and references there in), and extended DMD (Williams et al. (2015a)). These approaches enable computation of Koopman tuple in an autonomous setting of Section 2. As pointed in the introduction, a notable exception is multi resolution DMD (mrDMD, see Kutz et al. (Unpublished)), which considers non-stationarity in the data. The connection of mrDMD to Koopman operator spectral properties in a non-autonomous setting like introduced in this paper still need to be studied. Furthermore, mrDMD approach exploits time scale separation for decomposition, which may not be present in systems with periodic/quasi-periodic forcing. We next discuss how above approaches can be adapted for computing the time dependent KMD proposed in this paper. To develop this extension, we first discuss in more detail the extended DMD (EDMD) approach (Williams et al. (2015a)) and its kernel version (Williams et al. (Unpublished)).

[1] Note on terminology: these types of systems were classically called linear skew-product systems.

EDMD is a Galerkin weighted residual approach which uses a dictionary of basis functions to approximate KEFs and corresponding KEs. Let $\mathcal{F}_D\subset\mathcal{F}$ be a subset of observables spanned by a dictionary $\mathcal{D}\equiv\{\psi_1,\cdots,\psi_D\}$, where $\psi_i:\mathcal{M}\to\mathbf{C}$. Then $\theta,\hat{\theta}\in\mathcal{F}_D$ can be expressed as $\theta(\mathbf{x})=\Psi^*(\mathbf{x})\mathbf{a}$, $\hat{\theta}(\mathbf{x})=\Psi^*(\mathbf{x})\hat{\mathbf{a}}$, respectively, for some $\mathbf{a},\hat{\mathbf{a}}\in\mathbf{C}^D$, where $\Psi(\mathbf{x})=(\psi_1(\mathbf{x}),\cdots,\psi_D(\mathbf{x}))^*$. Under the action of Koopman operator

$$\begin{aligned}\mathcal{U}^t(\theta)(\mathbf{x}) &= (\Psi\circ\Phi(t,\mathbf{x}))^*\mathbf{a}=\Psi^*(\mathbf{x})\hat{\mathbf{a}}+r(\mathbf{x})\\ &= \Psi^*(\mathbf{x})U\mathbf{a}+r(\mathbf{x}),\end{aligned} \tag{46}$$

where, U is a finite dimensional approximation of $\mathcal{U}^t$, and $r(\mathbf{x})$ is the residual as $\mathcal{F}_D$ may not be invariant under action of $\mathcal{U}$. Note that $\hat{\theta}(\mathbf{x})$ is an approximation of $\tilde{\theta}(t,\mathbf{x})$ (see Eqn. (3)) for a given t.

Given a dataset of snapshot pairs $\{(\mathbf{x}_i,\overline{\mathbf{x}}_i)\}_{i=1}^N$, $\overline{\mathbf{x}}_i=\Phi(\mathbf{x}_i,t)$ generated from the ODE (2), one can formulate a least square problem of minimizing,

$$\sum_{i=1}^N|r(\mathbf{x}_i)|^2=\sum_{i=1}^N|\left(\Psi^*(\overline{\mathbf{x}}_i)-\Psi^*(\mathbf{x}_i)U\right)\mathbf{a}|,$$

to obtain

$$U=\boldsymbol{\Psi}_{\mathbf{x}}^{\dagger}\boldsymbol{\Psi}_{\overline{\mathbf{x}}}, \tag{47}$$

where, † is the pseudo inverse and

$$\boldsymbol{\Psi}_{\mathbf{x}}=\begin{bmatrix}\Psi^*(\mathbf{x}_1)\\ \Psi^*(\mathbf{x}_2)\\ \vdots\\ \Psi^*(\mathbf{x}_N)\end{bmatrix}_{N\times D},\quad \boldsymbol{\Psi}_{\overline{\mathbf{x}}}=\begin{bmatrix}\Psi^*(\overline{\mathbf{x}}_1)\\ \Psi^*(\overline{\mathbf{x}}_2)\\ \vdots\\ \Psi^*(\overline{\mathbf{x}}_N)\end{bmatrix}_{N\times D}.$$

Let $\lambda_i, i=1,\cdots,D$ be eigenvalues of U, with corresponding right/left eigenvectors ξ_i,γ_i, respectively. Then λ_i approximate KEs with corresponding KEFs given by $\phi_i(\mathbf{x})=\Psi^*(\mathbf{x})\xi_i$. Let the coordinate function $g_i(\mathbf{x})=\mathbf{x}_i$ be in span of $\mathcal{D}$ so that $\mathbf{x}_i=\Psi^*\mathbf{b}_i$ for some $\mathbf{b}_i\in\mathbb{R}^D$. Then, KMs can be obtained via $\mathbf{v}_i=B^*\gamma_i$, where $B=[\mathbf{b}_1,\cdots\mathbf{b}_n]$.

EDMD approach discussed above suffers from curse of dimensionality due to explosion in number of required dictionary elements D with the increase in state dimension n. To circumvent explicit construction of the dictionary, a kernel based EDMD approach has been proposed in Williams et al. (Unpublished). In this approach the computation of eigenvectors/eigenvalues of U is accomplished by forming an alternative matrix

$$\hat{U}=\hat{G}^{\dagger}\hat{A}, \tag{48}$$

where, $\hat{G}=\boldsymbol{\Psi}_{\mathbf{x}}\boldsymbol{\Psi}_{\mathbf{x}}^*$ and $\hat{A}=\boldsymbol{\Psi}_{\overline{\mathbf{x}}}\boldsymbol{\Psi}_{\mathbf{x}}^*$ are $N\times N$ matrices. Using the kernel trick, entries of matrices $\hat{G},\hat{A}$ can be computed directly (without forming $\Psi(\mathbf{x})$ for computing inner products of form $\Psi^*(\mathbf{x}_i)\Psi(\mathbf{x}_j)$, an $\mathcal{O}(D)$ operation) as

$$\hat{G}_{ij}=K(\mathbf{x}_i,\mathbf{x}_j),\quad \hat{A}_{ij}=K(\overline{\mathbf{x}}_i,\mathbf{x}_j), \tag{49}$$

where, $K(\mathbf{x},\mathbf{x}):\mathcal{M}\times\mathcal{M}\to\mathbb{R}$ is an appropriately chosen kernel function. K implicitly defines $\mathcal{F}_D$ (subspace of scalar observables spanned by elements of $\Psi(\mathbf{x})$), and evaluates the inner products implicity in $\mathcal{O}(n)$ rather than $\mathcal{O}(D)$ time. Optimal choice of $\mathcal{D}$ or equivalently K is an important, yet open question. These choices will most likely depend both on underlying dynamical system and strategy used to obtain the dataset. Examples of $\mathcal{D}$ include polynomials, spectral elements, radial basis functions etc. (Williams et al. (2015a)), while choices of

Algorithm 1 Kernel based EDMD

1: Input: Dataset with snapshot pairs $\{(\mathbf{x}_i, \overline{\mathbf{x}}_i)\}_{i=1}^N$, kernel function K
2: Output: Koopman tuple $(\{(\lambda_i, \phi_i, \mathbf{v}_i\}_{i=1}^N$
3: Compute $\hat{U}$ by forming $\hat{G}, \hat{A}$ using (49).
4: Compute eigenvalues $\lambda_i, i = 1, \cdots, N$, and corresponding right eigenvectors $\hat{\xi}_i$ of $\hat{U}$.
5: Let $\hat{\Xi} = [\hat{\xi}_1 \cdots \hat{\xi}_N]$, then i-th rows of

$$\Phi_{\mathbf{x}} = \hat{G}\hat{\Xi}, \qquad \Phi_{\overline{\mathbf{x}}} = \hat{A}\hat{\Xi}, \tag{50}$$

contains the numerically computed KEFs evaluated at $\mathbf{x}_i$ and $\overline{\mathbf{x}}_i$, respectively.
6: Compute KMs for full state observable as

$$[\mathbf{v}_1, \cdots, \mathbf{v}_N] = (\Phi_{\mathbf{x}})^\dagger X, \tag{51}$$

where, $X = [\mathbf{x}_1, \cdots, \mathbf{x}_N]^*$.

kernels include polynomial, Gaussian, Matern, etc. see Rasmussen and Williams (2006).

The steps for Koopman tuple computation based on kernel EDMD approach are summarized in Algo 1. Note that to compute KMs for any other observable $\mathbf{g}(\mathbf{x})$, one can replace X with $H_{\mathbf{x}} = [\mathbf{g}(\mathbf{x}_1), \cdots, \mathbf{g}(\mathbf{x}_N)]^*$ in (51). EDMD procedure being a weighted residual Galerkin method converges as $N \to \infty$. With randomly distributed samples, the convergence rate behaves as $\mathcal{O}(N^{-1/2})$ as in Monte Carlo integration techniques. Other sampling choices, e.g. uniform grid, effectively uses different quadrature rules and could lead to better convergence rates. The total computational cost of this approach is $\mathcal{O}(N^2 \max(n, N))$.

Using the interpretation of KEFs/KEs in the extended phase space (see Section 4.1), and standard techniques for computing Koopman tuple as discussed above, we propose following steps for computing time periodic KMD:

1) Simulate time periodic system (39)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, s)$$
$$\dot{s} = 1,$$

starting with different phases $s_1, s_2, \cdots s_M \in [0, T]$, where T is the period.

2) Apply standard DMD procedure (e.g. EDMD described in Algo 1) to the data generated at different phase $s_i, i = 1 \cdots, M$ and obtain the KE/KEF/KM for each phase s_i.

Note that one can apply Laplace analysis (Budisic et al. (2012)) or other variants of DMD in Step 2. For quasi-periodic case one can follow similar procedure, but rather than using different phases, one uses different points $\boldsymbol{\theta}$ on a torus (e.g. randomly chosen). It still remains to establish that above steps indeed lead to computation of time-periodic/quasi-periodic KMD as proposed in this paper, and will be considered in future work.

7. NUMERICAL EXAMPLES

In this section we demonstrate computation of time dependent KMD on some examples.

7.1 Example I

We first consider a linear time periodic system:

$$\dot{\mathbf{x}} = A(t)\mathbf{x},$$

with

$$A(t) = \begin{pmatrix} 1 + \dfrac{\cos(t)}{2+\sin(t)} & 0 \\ \dfrac{3\sin(t) - \cos(t) + 8}{4 + 2\sin(t)} & -1 \end{pmatrix}.$$

Following notation of the section 4.1, it can be shown that for above system

$$P(t) = \begin{pmatrix} 2 + \sin(t) & 0 \\ \dfrac{1}{2}(\sin(t) - \cos(t)) & 1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $P(t)\mathbf{v}_k(t) = \sum_{j=1}^3 e^{i\omega_j t}\mathbf{v}_{k,j}$, where $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$, and

$$\mathbf{v}_{1,1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{v}_{1,2} = \begin{pmatrix} -i/2 \\ -\dfrac{i+1}{4} \end{pmatrix}, \mathbf{v}_{1,3} = \begin{pmatrix} i/2 \\ \dfrac{i-1}{4} \end{pmatrix},$$

$$\mathbf{v}_{2,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{v}_{2,2} = \mathbf{v}_{2,3} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Also

$$\phi_{k,j}(\mathbf{x}, t) = e^{i\omega_j t}\phi_k(P^{-1}(t)\mathbf{x}),$$

for $k = 1, 2$ and $j = 1, 2, 3$ with corresponding $\lambda_{1,j} = 1 + i\omega_j$ and $\lambda_{2,j} = -1 + i\omega_j$, respectively, where

$$\phi_1(P^{-1}(t)\mathbf{x}) = \frac{1}{2 + \sin(t)}x_1,$$
$$\phi_2(P^{-1}(t)\mathbf{x}) = -\frac{\sin(t) - \cos(t) + 4}{4 + 2\sin(t)}x_1 + x_2.$$

With all the above quantities defined the KMD (36) for full state observable can be obtained as:

$$\mathcal{U}^{t,t_0}\mathbf{x}(\mathbf{x}_0) = \sum_{k=1}^{2}\sum_{j=1}^{3} e^{\lambda_{k,j}t}\phi_{k,j}^{t_0}(\mathbf{x}_0)\mathbf{v}_{k,j}.$$

7.2 Example II

As a second example, we consider periodically forced Van der Pol system,

$$\dot{x}_1 = x_1 \tag{52}$$
$$\dot{x}_2 = -x_1 + x_2 - x_1^2 x_2 + \Omega\cos(\omega t), \tag{53}$$

with $\omega = 2\pi$ and amplitude $\Omega = 0.5$. For these parameters there is a limit cycle with period ~ 8 sec for autonomous case, i.e. for $\Omega = 0$. To obtain time periodic KMD we generate sample data by simulating the suspended system using initial conditions uniformly spaced in $[-3, 3] \times [-3, 3]$ and with different initial phases $s \in \{0, T/3, 2T/3\}$, where $T = \frac{2\pi}{\omega}$. For data from each phase we compute Koopman tuple using kernel EDMD Algo. (1). After trail and error, we found Martern covariance kernel $(K(\mathbf{x}_1, \mathbf{x}_2) = (1 + \frac{\sqrt{5}r}{l} + \frac{5r^2}{3l^2})\exp(-\frac{\sqrt{5}r}{l})$, where, $r = ||\mathbf{x}_1 - \mathbf{x}_2||$, see Rasmussen and Williams (2006)) worked well for this problem.

Figure 1 shows a subset of Koopman eigenvalues recovered by kernel EDMD, and magnitude of Koopman eigenfunction corresponding to Koopman eigenvalue (denoted by red + in top row plots) whose period is close to the limit cycle period

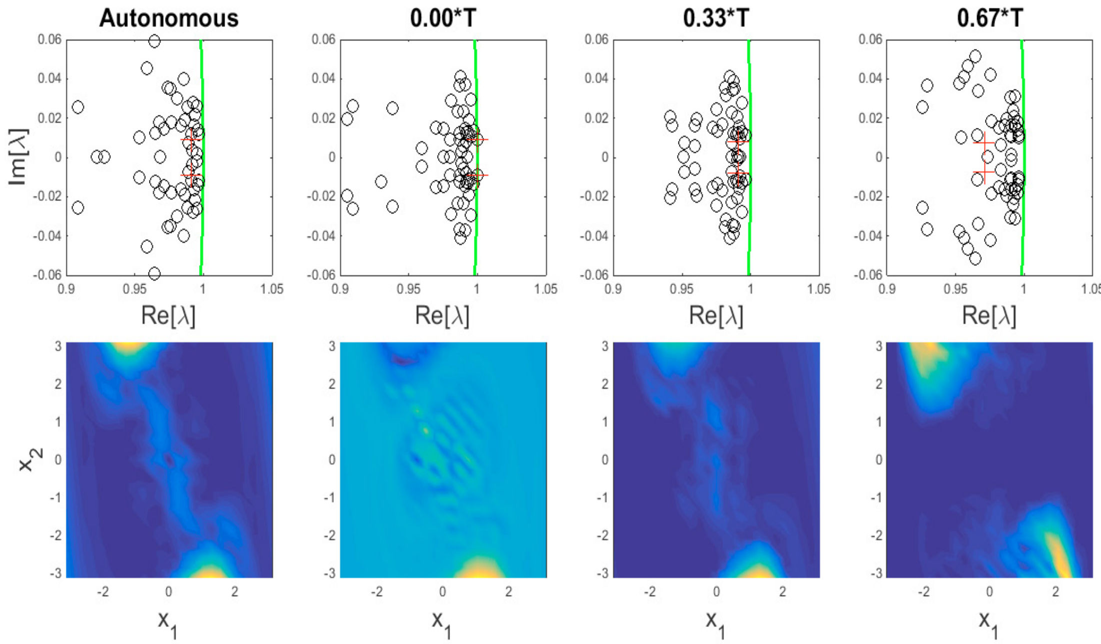

Fig. 1. Top row shows a subset of Koopman eigenvalues in complex plane, and bottom row shows magnitude of Koopman eigenfunction corresponding to Koopman eigenvalue (denoted by red + in top row) whose period is close to the limit cycle period. The green curve in top plots show a portion of a circle of unit radius. The first column shows results for autonomous case i.e. $\Omega = 0$. Other columns are for the case of time periodic forcing for different phases $s \in \{0, T/3, 2T/3\}$.

(which we found numerically). For reference, we also produce similar plots for autonomous case in the first column. As can be seen the Koopman eigenvalue distribution for periodically forced Van der Pol oscillator changes with phase, and is also different from the autonomous case. The bottom row shows that the eigenfunctions for different phases are close to the autonomous case but oscillate in magnitude.

8. CONCLUSION

In this paper we have developed a Koopman Mode Decomposition framework for non-autonomous systems with periodic/quasi-periodic time dependence, and proposed a numerical technique for computing it. In future it will be important to evaluate the framework on more complex and real datasets arising in practical problems such as fluid mechanics, power grid, etc. Also more work is needed to determine conditions under which the proposed numerical approach results in the theoretical time dependent KMD developed in the paper. In that regard it will also be worthwhile to explore connections to wavelet like approach used in multi resolution DMD (Kutz et al. (Unpublished)), and eventually generalization to non-autonomous systems with arbitrary time dependence, that we outlined in equation (24).

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