CDS 101/110: Lecture 6.1
Observability Wrap-Up
Intro to Transfer Functions

October 31, 2016

Goals:
• Present simple computational study of observability.
• Hand out and discuss Midterm exam.
• Define the input/output transfer function of a linear system.
• Describe Bode plots for frequency response investigation

Reading:
• Åström and Murray, Feedback Systems-2e, Sections 9.1-9.2
Double Integrator Example

Double Integrator Model: \( \ddot{x} = u, \quad y = x. \)

- 1st-order equivalent: \( \dot{x} = Ax + Bu \) with

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = [0]
\]

Check Controllability & Observability

- \( W_r = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)
- \( W_o = \begin{bmatrix} C \\ AC \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

State Feedback Controller: \( u = -Kx + k_r r \)

- Place poles at \( \lambda_{1,2} = -\zeta \omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1} \)
  \( = -7.0 \pm 7.0i \)
- \( K = [98 \quad 14] \)
- \( k_r = -[C(A - BK)^{-1}B]^{-1} = 98 \)
Double Integrator Example (continued)

MATLAB:

\[
A=[0 \ 1; \ 0 \ 0 ]; \\
B=[0; \ 1]; \quad C=[1 \ 0]; \quad D=[0]; \\
sys0=ss(A,B,C,D);
\]

State Feedback Controller: \( u = -Kx + kr \cdot r \)

\[
FdbkPoles=[(-7 + 7i) (-7 - 7i)]; \\
K=place(A,B,FdbkPoles); \\
BK= B*K; \\
AK=A-BK; \quad \% \text{closed loop system} \\
kr=-inv(C*inv(AK)*B); \\
Bref=kr*B; \\
sysref=ss(AK,Bref,C,D);
\]

Square Wave Reference:

\[
[usquare,tsquare]= \\
gensig('square',1.5,7.5,0.01); \\
[yout,tout,xout]= \\
lsim(sysref,usquare,tsquare); \\
plotout=[yout usquare(:,1)]; \\
plot(plotout);
\]
Design an Observer: \[ \dot{x} = A\hat{x} + Bu + L(y - C\hat{x}) \]

- Place observer poles at \[ \lambda_{1,2} = -\zeta \omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1} \]
  \[ = -10.0 \pm 10.0i \]

- To calculate \( L \), use MATLAB: \[ L^T = \text{place}(A^T, C^T, \lambda_{1,2}) \]
  
  ```matlab
  LT=place(A',C',[-10.0+10i;-10.0-10i]);
  L=LT';
  ```

- Create a system whose simulation demonstrates the observer

  ```matlab
  LC=L*C;
  Aobs=A-LC;
  sysObs=ss(Aobs,[L B],eye(2),zeros(2,2));

  xh=lsim(sysObs,[yout,usquare],tout);
  allplot=[yout xh(:,1)];  \( \text{or allplot=[yout xh(:,1) xh(:,2)];} \)
  plot(allplot);
  ```
Position Estimate

Estimator poles at $-1 \pm i$

Position and Velocity Estimate
Create a system which simulates estimated state feedback

\[ \dot{x} = Ax + Bu, \quad \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \quad u = -K\hat{x} + k_r r \]

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A & -BK \\
LC & A - LC - BK
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} +
\begin{bmatrix}
Bk_r \\
Bk_r r
\end{bmatrix} r
\]

In MATLAB:

```matlab
ALCBK = A - LC - BK;
Atot = [A(1,1) A(1,2) -BK(1,1) -BK(1,2); A(2,1) A(2,2) -BK(2,1) -BK(2,2);
        LC(1,1) LC(1,2) ALCBK(1,1) ALCBK(1,2); LC(2,1) LC(2,2) ALCBK(2,1) ALCBK(2,2)];
Btot = [Bref(1,1); Bref(2,1); Bref(1,1); Bref(2,1)];
Ctot = [1 0 0 0]
Dtot = [0];
systot = ss(Atot, Btot, Ctot, Dtot);
[Ytot, Ttot, Xtot] = lsim(systot, [usquare(:,1)], tsquare);

graphtot = [Ytot Xtot(:,1) Xtot(:,2) Xtot(:,3) Xtot(:,4) usquare(:,1)];
plot(graphtot);
```
Estimator poles at $-10 \pm 10i$
Error in Initial condition of $\hat{x}_1 = 0.5$

Estimator poles at $-10 \pm 10i$

Estimator poles at $-1 \pm i$
**Frequency Domain Modeling**

**Defn.** The frequency response of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.

\[
u = A \sin(\omega t)\]

\[
y = B \sin(\omega t + \phi)\]

**Bode plot (1940; Henrik Bode)**
- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)
Transmission of Exponential Signals

Exponential signal: \( e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t} (\cos \omega t + i \sin \omega t) \)

- Construct constant inputs + sines/cosines by linear combinations
  - Constant: \( u(t) = c = ce^{0t} \)
  - Sinusoid: \( u(t) = A \sin(\omega t) = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t}) \)
  - Decaying sinusoid: \( u(t) = Ae^{-\sigma t} \sin(\omega t) \)

- Exponential response can be computed via the convolution equation

\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Be^{s\tau} d\tau
\]

\[
= e^{At}x(0) + e^{At} (sI - A)^{-1} e^{(sI-A)\tau} \bigg|_{\tau=0}^t B
\]

\[
= e^{At}x(0) + e^{At} (sI - A)^{-1} \left( e^{(sI-A)t} - I \right) B
\]

\[
= e^{At} \left( x(0) - (sI - A)^{-1} B \right) + (sI - A)^{-1} Be^{st}
\]

\[
y(t) = Cx(t) + Du(t)
\]

\[
= Ce^{At} \left( x(0) - (sI - A)^{-1} B \right) + \left( C(sI - A)^{-1} B + D \right) e^{st}
\]
Exponential response of a linear state space system

\[ y(t) = Ce^{At} \left( x(0) - (sI - A)^{-1}B \right) + \left( C(sI - A)^{-1}B + D \right)e^{st} \]

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio \( y(s)/u(s) \)
- \( G(s) = C(sI - A)^{-1}B + D \) is the transfer function between input and output
- Note response at eigenvalues of \( A \)

Frequency response

\[ u(t) = A \sin \omega t = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t}) \]

\[ y_{ss}(t) = \frac{A}{2i} \left( G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t} \right) \]

\[ = A \cdot |G(i\omega)| \sin(\omega t + \text{arg} G(i\omega)) \]

Common transfer functions

| \( \dot{y} = u \) | \( \frac{1}{s} \) |
| \( y = \ddot{u} \) | \( s \) |
| \( \dot{y} + ay = u \) | \( \frac{1}{s+a} \) |
| \( \dot{y} = u \) | \( \frac{1}{s^2} \) |
| \( \ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = u \) | \( \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \) |
| \( y = k_p u + k_d \dot{u} + k_i \int u \) | \( k_p + k_ds + \frac{k_i}{s} \) |
| \( y(t) = u(t - \tau) e^{-\tau s} \) | \( \tau \) |
Example: Electrical Circuits

Circuit dynamics (Kirchoff’s laws):

\[
\frac{v_1 - v}{R_1} = \frac{v - v_2}{R_2}, \quad \Rightarrow \quad v = \frac{R_2 v_1 + R_1 v_2}{R_1 + R_2}
\]

\[
v_2 = G(s)v = -\frac{ak}{s + a} \left( \frac{R_2 v_1 + R_1 v_2}{R_1 + R_2} \right)
\]

\[
\frac{v_2}{v_1} = -\frac{R_2 ak}{R_1 ak + (R_1 + R_2)(s + a)}
\]

- Algebraic manipulation can be used as long as we assume exponential signals and all of the components (blocks) are linear.
- Transfer function between input and output show gain-bandwidth tradeoff.
Transfer Function Properties

**Theorem.** The transfer function for a linear system \( Σ = (A, B, C, D) \) is given by

\[
G(s) = C(sI - A)^{-1} + D \quad s \in \mathbb{C}
\]

**Theorem.** The transfer function \( G(s) \) has the following properties (for SISO systems):

- \( G(s) \) is a ratio of polynomials \( n(s)/d(s) \) where \( d(s) \) is the characteristic equation for the matrix \( A \) and \( n(s) \) has order less than or equal to \( d(s) \).
- The steady state frequency response of \( Σ \) has gain \( |G(jω)| \) and phase \( \arg G(jω) \):
  \[
  u = Msin(ωt) \\
  y = |G(iω)|Msin(ωt + \arg G(iω)) + \text{transients}
  \]

**Remarks**

- Formally, \( G(s) \) is the Laplace transform of the impulse response of \( Σ \)
- Typically we write “\( y = G(s)u \)” for \( Y(s) = G(s)U(s) \), where \( Y(s) \) & \( U(s) \) are Laplace transforms of \( y(t) \) and \( u(t) \). (Multiplication in Laplace domain corresponds to convolution.)
- MATLAB: \( G = \text{ss2tf}(A, B, C, D) \)