Computing the Screw Parameters of a Rigid Body Displacement

Problem Statement:
We wish to determine the screw displacement parameters for a spatial displacement by tracking the motion of three non-collinear points. These parameters consist of:

\begin{align*}
\phi &= \text{the angle of rotation about the screw axis} \\
d || &= \text{the translation along the screw axis} \\
\vec{\omega} &= \text{A unit vector parallel to the screw axis} \\
\vec{\rho} &= \text{a vector to a point on the screw axis}
\end{align*}

Assume that we have a rigid body which contains three non-colinear points: P, Q, R. Let \(P_0, Q_0,\) and \(R_0\) denote the positions of the points in the body before displacement. Let \(P_1, Q_1,\) and \(R_1\) the position of these points after a screw displacement.

The Solution:
Let’s first recall Rodriguez’ Displacement Equation. Consider a point \(P\) located at position \(\vec{x}_0\) in a fixed reference frame. A rigid body containing that point then undergoes a screw displacement (with screw displacement parameters \(\phi, d ||, \vec{\omega}, \vec{\rho}\)). The point \(P\) is displaced to some new location \(P'\) whose position (to the fixed observer) is \(\vec{x}_1\). The coordinates of the points and the screw displacement parameters are related by Roddriguez’ displacement equation:

\[
\vec{x}_1 - \vec{x}_0 = \tan\left(\frac{\phi}{2}\right)\vec{\omega} \times \left[\vec{x}_1 + \vec{x}_0 - 2\vec{\rho}\right] + d || \vec{\omega}.
\]  

(1)

To determine the screw parameters from the displacement of these three points, we will solve the following three simultaneous copies of Rodriguez’ equation, which each equation modeling the displacement of a separate point in the same rigid body. I.e., we will track the displacements of three non-collinear points as they are affected by the screw displacements.

\[
P_1 - P_0 = \tan\left(\frac{\phi}{2}\right)\vec{\omega} \times (P_1 + P_0 - 2\vec{\rho}) + d || \vec{\omega}
\]

(2)

\[
Q_1 - Q_0 = \tan\left(\frac{\phi}{2}\right)\vec{\omega} \times (Q_1 + Q_0 - 2\vec{\rho}) + d || \vec{\omega}
\]

(3)

\[
R_1 - R_0 = \tan\left(\frac{\phi}{2}\right)\vec{\omega} \times (R_1 + R_0 - 2\vec{\rho}) + d || \vec{\omega}
\]

(4)

where each equation is the Rodriguez displacement equation for the respective points P, Q, and R.

**Step #1:** Subtract Equation (4) from Equations (2) and (3):

\[
(P_1 - P_0) - (R_1 - R_0) = \tan\left(\frac{\phi}{2}\right)\vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)]
\]

(5)
\[(Q_1 - Q_0) - (R_1 - R_0) = \tan\left(\frac{\phi}{2}\right) \vec{\omega} \times [(Q_1 + Q_0) - (R_1 + R_0)] \quad (6)\]

Form the cross product of \([(Q_1 - Q_0) - (R_1 - R_0)]\) with Equation (6):

\[[(Q_1 - Q_0) - (R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)] = \tan\left(\frac{\phi}{2}\right) [(Q_1 - Q_0) - (R_1 - R_0)] \times \{\vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)]\} \quad (7)\]

Note: from Equation (6), we know that \([\vec{\omega} \times [\vec{P} + \vec{Q}]]\) is perpendicular to \(\vec{\omega}\), since it results from the cross product of a vector with \(\vec{\omega}\). Therefore, the right hand side of Equation (7) will be a vector proportional to \(\vec{\omega}\).

We can solve Equation (8) for \(\tan(\phi/2)\omega\):

\[\tan(\phi/2)\omega = \frac{[(Q_1 - Q_0) - (R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)]}{[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]} \quad (8)\]

Thus, the rotation angle, \(\tan(\phi/2)\omega\), can be computed as the norm to the vector in Equation (9), while \(\vec{\omega}\) is the normalized vector of Equation (9).

\textbf{Step #2:} Now take the cross product of \(\vec{\omega}\) with equation (2) and use the aforementioned vector cross product identity:

\[\vec{\omega} \times (P_1 - P_0) = \vec{\omega} \times [\tan(\phi/2)\omega \times (P_1 + P_0 - 2\rho\vec{h}) + d\parallel \omega] = \tan(\phi/2)\omega \times [(\vec{\omega} \cdot (P_1 + P_0))\omega - (P_0 + P_1) - 2(\vec{\omega} \cdot \vec{\rho})\omega + 2\rho] \quad (10)\]

Note that \(\rho - (\vec{\omega} \cdot \vec{\rho})\omega = \vec{\rho} \perp\), where \(\vec{\rho} \perp\) is the component of \(\vec{\rho}\) which is perpendicular to \(\vec{\omega}\). That is, while \(\vec{\rho}\) is a vector from the origin of the reference frame to \textit{any} point on the screw axis, \(\vec{\rho} \perp\) is the shortest vector to the point on the screw axis closest to the origin of the reference frame. Equation (10) can then be solved for \(\vec{\rho} \perp\):

\[\vec{\rho} \perp = \frac{1}{2} \left[\frac{\vec{\omega} \times (P_1 - P_0)}{\tan(\phi/2)} - (\vec{\omega} \cdot (P_1 + P_0))\omega + P_0 + P_1\right] \quad (11)\]

\textbf{Step 3:} Finally, we can use Equation (2), (3), or (4) to find \(d\parallel\):

\[d\parallel = \vec{\omega} \cdot (P_1 - P_0) = \vec{\omega} \cdot (Q_1 - Q_0) = \vec{\omega} \cdot (R_1 - R_0) \quad (12)\]