Computing the Screw Parameters of a Rigid Body Displacement

Problem Statement:

We wish to determine the screw displacement parameters for a spatial displacement by tracking the motion of three non-collinear points. These parameters consist of:

$$\begin{aligned} \phi &= \text{the angle of rotation about the screw axis} \\ d^{||} &= \text{the translation along the screw axis} \\ \vec{\omega} &= \text{A unit vector parallel to the screw axis} \\ \vec{\rho} &= \text{a vector to a point on the screw axis} \end{aligned}$$

Assume that we have a rigid body which contains three non-colinear points: P, Q, R. Let P_0, Q_0, and R_0 denote the positions of the points in the body before displacement. Let P_1, Q_1, and R_1 the position of these points after a screw displacement.

The Solution:

Let's first recall *Rodriguez' Displacement Equation.* Consider a point P located at position $\vec{x}_0$ in a fixed reference frame. A rigid body containing that point then undergoes a screw displacement (with screw displacement parameters ϕ, $d^{||}$, $\vec{\omega}$, $\vec{\rho}$). The point P is displaced to some new location P' whose position (to the fixed observer) is $\vec{x}_1$. The coordinates of the points and the screw displacement parameters are related by Rogdriguez' displacement equation:

$$\vec{x}_1 - \vec{x}_0 \;=\; \tan(\frac{\phi}{2})\vec{\omega} \times [\vec{x}_1 + \vec{x}_0 - 2\rho] + d^{||}\vec{\omega}\ . \tag{1}$$

To determine the screw parameters from the displacement of these three points, we will solve the following three simulataneous copies of Rodriguez' equation, which each equation modeling the displacment of a separate point in the same rigid body. I.e., we will track the displacements of three non-collinear points as they are affected by the screw displacements.

$$P_1 - P_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (P_1 + P_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{2}$$

$$Q_1 - Q_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (Q_1 + Q_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{3}$$

$$R_1 - R_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (R_1 + R_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{4}$$

where each equation is the **Rodriguez displacement equation** for the respective points P, Q, and R.

Step #1: Subtract Equation (4) from Equations (2) and (3):

$$(P_1 - P_0) - (R_1 - R_0) = \tan(\frac{\phi}{2})\ \vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)] \tag{5}$$

$$(Q_1 - Q_0) - (R_1 - R_0) = \tan(\frac{\phi}{2})\ \vec{\omega} \times [(Q_1 + Q_0) - (R_1 + R_0)] \tag{6}$$

Form the cross product of $[(Q_1 - Q_0) - (R_1 - R_0)]$ with Equation (6):

$$\begin{aligned}[(Q_1 - Q_0) \quad & -(R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)] \\ & = \tan(\tfrac{\phi}{2})[(Q_1 - Q_0) - (R_1 - R_0)] \times \{\vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)]\}\end{aligned} \tag{7}$$

Note: from Equation (6), we know that $[(Q_1 - Q_0) - (R_1 - R_0)]$ is perpendicular to $\vec{\omega}$, since it results from the cross produce of a vector with $\vec{\omega}$. Therefore, the right hand side of Equation (7) will be a vector proportional to $\vec{\omega}$.

We can use the vector identity $\vec{a} \times (\vec{b} \times c) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ to simplify Equation (7):

$$\begin{aligned}[(Q_1 - Q_0) - (R_1 - R_0)] \quad & \times [(P_1 - P_0) - (R_1 - R_0)] \\ & = \tan(\tfrac{\phi}{2})[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]\vec{\omega}\end{aligned} \tag{8}$$

We can solve Equation (8) for $\tan(\frac{\phi}{2})\vec{\omega}$:

$$\tan(\frac{\phi}{2})\vec{\omega} = \frac{[(Q_1 - Q_0) - (R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)]}{[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]} \tag{9}$$

Thus, the rotation angle, $\tan(\frac{\phi}{2})$ can be computed as the norm to the vector in Equation (9), while $\vec{\omega}$ is the normalized vector of Equation (9).

Step #2: Now take the cross product of $\vec{\omega}$ with equation (2) and use the aforementioned vector cross product identity:

$$\begin{aligned}\vec{\omega} \times (P_1 - P_0) & = \vec{\omega} \times [\tan(\tfrac{\phi}{2})\ \vec{\omega} \times (P_1 + P_0 - 2r\vec{ho}) + d^{||}\vec{\omega}] \\ & = \tan(\tfrac{\phi}{2})\ \ [[\vec{\omega} \cdot (P_1 + P_0)]\vec{\omega} - (P_0 + P_1) - 2(\vec{\omega} \cdot \vec{\rho})\vec{\omega} + 2\rho]\end{aligned} \tag{10}$$

Note that $\rho - (\vec{\omega} \cdot \vec{\rho})\vec{\omega} = \vec{\rho}_\perp$, where $\vec{\rho}_\perp$ is the component of $\vec{\rho}$ which is perpendicular to $\vec{\omega}$. That is, while $\vec{\rho}$ is a vector from the origin of the reference frame to *any* point on the screw axis, $\vec{\rho}_\perp$ is the shortest vector to the point on the screw axis closest to the origin of the reference frame. Equation (10) can then be solved for $\vec{\rho}_\perp$:

$$\vec{\rho}_\perp = \frac{1}{2}\left[\frac{\vec{\omega} \times (P_1 - P_0)}{\tan\frac{\phi}{2}} - (\vec{\omega} \cdot (P_1 + P_0))\vec{\omega} + P_0 + P_1\right] \tag{11}$$

Step 3: Finally, we can use Equation (2), (3), or (4) to find $d^{||}$:

$$d^{||} = \vec{\omega} \cdot (P_1 - P_0) = \vec{\omega} \cdot (Q_1 - Q_0) = \vec{\omega} \cdot (R_1 - R_0) \tag{12}$$