CDS 270: Solution to Homework #1

Solution to Problem 1: (5 points) Problem #2 in Chapter 2 (page 73) of the MLS text.

Let $g_1 \in SE(n)$ and $g_2 \in SE(n)$ $(n = 2, 3)$ take the form

$$g_1 = \begin{bmatrix} R_1 & \vec{d_1} \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{d_2} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $R_1, R_2 \in SO(n)$ and $\vec{d_1}, \vec{d_2} \in \mathbb{R}^n$. The product of g_1 and g_2 takes the form:

$$g_1 \cdot g_2 = \begin{bmatrix} R_1 R_2 & \vec{d_1} + R_1 \vec{d_2} \\ \vec{0}^T & 1 \end{bmatrix}$$

Since $SO(n)$ forms a group, the product $R_1 R_2 \in SO(n)$. Since $R_1 \vec{d_2} \in \mathbb{R}^n$, $\vec{d_1} + R_1 \vec{d_2} \in \mathbb{R}^n$. Thus, $g_1 g_2 \in SE(n)$. The identity matrix is the identity element of $SE(n)$. The inverse of matrix g_1 is:

$$\begin{bmatrix} R_1^T & -R_1^T \vec{d_1} \\ \vec{0}^T & 1 \end{bmatrix} .$$

Since $R_1 \in SO(n)$, $R_1^T \in SO(n)$. Similarly, $R_1^T \vec{d_1} \in \mathbb{R}^n$. Hence, $g_1^{-1} \in SE(n)$. Thus, $SE(n)$ forms a group.

Solution to Problem 2: (10 points) Problem #4 in Chapter 2 (page 73) of the MLS text.

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w} \times (R^T\vec{v})) \\
&= (R\vec{w}) \times (RR^T\vec{v}) \\
&= (R\vec{w}) \times \vec{v} \\
&= \widehat{(R\vec{w})}\vec{v}
\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= \widehat{(R\vec{v})}(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^T R\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}$$

Solution Problem 3: (10 points) Problem #10(b, c) in Chapter 2 (page 75) of the MLS text. It is not necessary to answer the question about surjectivity.

Note that

$$\hat{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then:

$$\hat{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2 I; \quad \hat{\omega}^3 = -w^3 J$$

Hence the exponetial of $\hat{\omega}$ can be computed as:

$$\begin{aligned}
\exp(\theta\hat{\omega}) &= \left(I + \frac{\theta}{1!}\hat{\omega} + + \frac{\theta^2}{2!}\hat{\omega}^2 + \cdots \right) \\
&= \left(I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots \right) \\
&= \left(1 - \frac{w^2\theta^2}{2!} + \cdots \right) I + \left(\frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots \right) J \\
&= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
\end{aligned}$$

Clearly, the exponential map from $so(2)$ to $SO(3)$ can not be surjective, as every point in $SO(3)$ can not be covered by every point in $so(2)$. This map is not injective since $\exp(\theta\hat{\omega}) = \exp((\theta + 2\pi)\hat{\omega})$.

Problem 4: (10 points) Prove (or show) that a body undergoing spherical motion (where one point is fixed through the motion) has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of N particles. Let P_1 denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since P_1 does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle P_2 in the body. Particle P_2 has 3 DOF as a particle. However, it is constrained to lie a fixed distance, d_{12} from particle P_1 due to the fact that P_1 and P_2 are part of the same rigid body. The fixed distance relationship imposes one constraint on P_2. Next consider a point P_3, which lie a fixed distance from P_1 and P_2. Therefore, there are two constraints on its location. Now, consider a particle P_4. Since its must lie a fixed distance from P_1, P_2, and P_3, there are three constraints on its motion. Particles $P_5, \ldots, P_N$ similarly have 3 constraints.

The total number of degrees of freedom of the N particles are: $3(N-1)+0 = 3N-3$. The total number of constraints on these particles are: $1 + 2 + 3(N-3) = 3N-6$. Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: $(3N-3) - (3N-6) = 3$.

Problem 5: (20 points) Problem #11(a, b, e) in Chapter 2 (page 76) of the MLS text.

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = ge^{\phi\hat{\xi}'}g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi^{'} = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi^{'} = (h\hat{\xi}^{'} h^{-1})^{\vee} \tag{1}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}^{'} = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix} .$$

Expanding Eq. (1) gives:

$$\xi = (h\hat{\xi}^{'} h^{-1})^{\vee} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\vec{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T\vec{p} + R\vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \vec{v}^s = R\vec{v}^b - R\hat{\omega}^b R^T\vec{p} = R\vec{v}^b - \hat{\omega}^s\vec{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\vec{v}^s = R\vec{v}^b - \hat{\omega}^s\vec{p} = R\vec{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \end{bmatrix} \begin{bmatrix} \vec{v}^b \\ \omega^b \end{bmatrix}$$

Therefore:

$$V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \vec{0}^T & 1 \end{bmatrix} V^b$$

Solution Problem 6: (10 points) Problem #14(a, b) in Chapter 2 (page 77) of the MLS text. It is not necessary to answer the question about surjectivity.

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \qquad Ad_{g^{-1}} = \begin{bmatrix} R^T & -\widehat{(R^T\vec{p})}R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $\widehat{(R^T\vec{p})} = R^T\hat{p}R$. Let's now compute $Ad_g Ad_{g^{-1}}$:

$$Ad_g Ad_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $Ad_{g^{-1}}$ must equal $(Ad_g)^{-1}$ since $Ad_g Ad_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1R_2 & \vec{p}_1 + R_1\vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\begin{aligned} Ad_{g_1g_2} &= \begin{bmatrix} R_1R_2 & (\vec{p}_1 + R_1\vec{p}_2)\hat{}\,R_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_1^TR_1R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 & \hat{p}_1R_1R_2 + R_1\hat{p}_2R_2 \\ 0 & R_1R_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1R_1 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2R_2 \\ 0 & R_2 \end{bmatrix} = Ad_{g_1}Ad_{g_2} \end{aligned}$$