CDS 270: Solution to Homework #1

Solution to Problem 1: (5 points) Problem #2 in Chapter 2 (page 73) of the MLS text.

Let \( g_1 \in SE(n) \) and \( g_2 \in SE(n) \) \((n = 2, 3)\) take the form

\[
 g_1 = \begin{bmatrix} R_1 & \vec{d}_1 \\ \vec{0}^T & 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} R_2 & \vec{d}_2 \\ \vec{0}^T & 1 \end{bmatrix}
\]

where \( R_1, R_2 \in SO(n) \) and \( \vec{d}_1, \vec{d}_2 \in \mathbb{R}^n \). The product of \( g_1 \) and \( g_2 \) takes the form:

\[
 g_1 \cdot g_2 = \begin{bmatrix} R_1 R_2 & \vec{d}_1 + R_1 \vec{d}_2 \\ \vec{0}^T & 1 \end{bmatrix}
\]

Since \( SO(n) \) forms a group, the product \( R_1 R_2 \in SO(n) \). Since \( R_1 \vec{d}_2 \in \mathbb{R}^n, \vec{d}_1 + R_1 \vec{d}_2 \in \mathbb{R}^n \). Thus, \( g_1 g_2 \in SE(n) \). The identity matrix is the identity element of \( SE(n) \). The inverse of matrix \( g_1 \) is:

\[
 \begin{bmatrix} R_1^T & -R_1^T \vec{d}_1 \\ \vec{0}^T & 1 \end{bmatrix}
\]

Since \( R_1 \in SO(n) \), \( R_1^T \in SO(n) \). Similarly, \( R_1^T \vec{d}_1 \in \mathbb{R}^n \). Hence, \( g_1^{-1} \in SE(n) \). Thus, \( SE(n) \) forms a group.

Solution to Problem 2: (10 points) Problem #4 in Chapter 2 (page 73) of the MLS text.

Part (a): Let’s assume that the statement in part (b) of the problem is true. Let \( \vec{w} \) be a \( 3 \times 1 \) vector and let \( \vec{v} \) be any \( 3 \times 1 \) vector. Then:

\[
 (R\vec{w}R^T)\vec{v} = R\vec{w}(R^T\vec{v}) \\
 = R(\vec{w} \times (R^T\vec{v})) \\
 = (R\vec{w}) \times (RR^T\vec{v}) \\
 = (R\vec{w}) \times \vec{v} \\
 = (R\vec{w})\vec{v}
\]

Since this must be true for any vector \( \vec{v} \), then \( R\vec{w}R^T = (R\vec{w})\).  

Part (b): We can now assume that part (a) holds.

\[
 (R\vec{v}) \times (R\vec{w}) = (R\vec{v})(R\vec{w}) \\
 = (R\vec{v}R^T)(R\vec{w}) \\
 = R\vec{v}R^TR\vec{w} \\
 = R(\vec{v}\vec{w}) \\
 = R(\vec{v} \times \vec{w})
\]
Solution Problem 3: (10 points) Problem #10(b, c) in Chapter 2 (page 75) of the MLS text. It is not necessary to answer the question about surjectivity.

Note that
\[ \dot{\omega} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} = wJ \text{ where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

Then:
\[ \dot{\omega}^2 = w^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -w^2I; \quad \dot{\omega}^3 = -w^3J \]

Hence the exponential of \( \dot{\omega} \) can be computed as:
\[
\exp(\theta \dot{\omega}) = \left( I + \frac{\theta}{1!} \dot{\omega} + \frac{\theta^2}{2!} \dot{\omega}^2 + \cdots \right) = \left( I + \frac{w\theta}{1!}J - \frac{w^2\theta^2}{2!}I - \frac{w^3\theta^3}{3!}J + \cdots \right) = \left( 1 - \frac{w^2\theta^2}{2!} + \cdots \right) I + \left( \frac{w\theta}{1!} - \frac{w^3\theta^3}{3!} + \cdots \right) J
\]
\[
= \begin{bmatrix} \cos(w\theta) & -\sin(w\theta) \\ \sin(w\theta) & \cos(w\theta) \end{bmatrix}
\]

Clearly, the exponential map from \( \text{so}(2) \) to \( \text{SO}(3) \) can not be surjective, as every point in \( \text{SO}(3) \) can not be covered by every point in \( \text{so}(2) \). This map is not injective since \( \exp(\theta \dot{\omega}) = \exp((\theta + 2\pi)\dot{\omega}) \).

Problem 4: (10 points) Prove (or show) that a body undergoing spherical motion (where one point is fixed through the motion) has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of \( N \) particles. Let \( P_1 \) denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since \( P_1 \) does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle \( P_2 \) in the body. Particle \( P_2 \) has 3 DOF as a particle. However, it is constrained to lie a fixed distance, \( d_{12} \) from particle \( P_1 \) due to the fact that \( P_1 \) and \( P_2 \) are part of the same rigid body. The fixed distance relationship imposes one constraint on \( P_2 \). Next consider a point \( P_3 \), which lie a fixed distance from \( P_1 \) and \( P_2 \). Therefore, there are two constraints on its location. Now, consider a particle \( P_4 \). Since its must lie a fixed distance from \( P_1, P_2, \) and \( P_3 \), there are three constraints on its motion. Particles \( P_5, \ldots, P_N \) similarly have 3 constraints.

The total number of degrees of freedom of the \( N \) particles are: \( 3(N-1) + 0 = 3N - 3 \). The total number of constraints on these particles are: \( 1 + 2 + 3(N-3) = 3N - 6 \). Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: \( (3N - 3) - (3N - 6) = 3 \).

Problem 5: (20 points) Problem #11(a, b, e) in Chapter 2 (page 76) of the MLS text.
**Part (a):** Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$ e^{\phi \hat{\xi}} = I + \frac{\phi}{1!} \hat{\xi} + \frac{\phi^2}{2!} \hat{\xi}^2 + \frac{\phi^3}{3!} \hat{\xi}^3 + \ldots $$

First, let’s consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$ \hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} $$

then $\hat{\xi}^2 = 0$. Thus

$$ e^{\phi \hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v} \phi \\ 0 & 1 \end{bmatrix} $$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where $I$ is the $2 \times 2$ identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$ \hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 1 \end{bmatrix} $$

Let

$$ g = \begin{bmatrix} I & \hat{\omega} \vec{v} \\ 0 & 1 \end{bmatrix} $$

Let is define a new twist, $\hat{\xi}'$:

$$ \hat{\xi}' = g^{-1} \hat{\xi} g 
= \begin{bmatrix} I & \hat{\omega} \vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 1 \end{bmatrix} 
= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2 \vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} 
= \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} $$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$ e^{\phi \hat{\xi}'} = \begin{bmatrix} e^{\phi \hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}. $$

Using Eq. (2.35) on page 42 of the MLS text:

$$ e^{\phi \hat{\xi}} = g e^{\phi \hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi \hat{\omega}} & (I - e^{\phi \hat{\omega}}) \hat{\omega} \vec{v} \phi' \\ 0 & 1 \end{bmatrix} $$

which is clearly an element of $SE(2)$. 

3
Part (b): It is easy to see from part (a) that the twist \( \xi = (v_x, v_y, 0)^T \) maps directly to the planar translation \((v_x, v_y)\).

The twist corresponding to pure rotation about a point \( \vec{q} = (q_x, q_y) \) can be thought of as the Ad-transformation of a twist, \( \xi' = (0, 0, \omega)^\vee \), which is pure rotation, by a transformation, \( g \), which is pure translation by \( \vec{q} \):

\[
\xi = \text{Ad}_h \xi' = (h \xi' h^{-1})^\vee
\]

where

\[
h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{x} = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.
\]

Expanding Eq. (1) gives:

\[
\xi = (h \xi' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega} \vec{q}^T \\ \vec{0}^T & 0 \end{bmatrix} = \begin{bmatrix} q_y & -q_x \\ 0 & 1 \end{bmatrix}
\]

assuming \( \omega = 1 \).

Part (e): Let \( \check{V}^b \) denote the planar body velocity:

\[
\check{V}^b = \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}
\]

where \( \hat{\omega}^b \in \text{so}(2), \vec{v}^b \in \mathbb{R}^2 \). Then the planar spatial velocity is:

\[
\check{V}^s = \text{Ad}_g \check{V}^b = g \check{V}^b g^{-1} = \begin{bmatrix} R & \vec{p}^b \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T \vec{p}^b \\ \vec{0}^T & 0 \end{bmatrix} = \begin{bmatrix} R \hat{\omega}^b R^T - R \hat{\omega}^b R^T \vec{p}^b + R \vec{v}^b \\ \vec{0}^T & 0 \end{bmatrix}
\]

Therefore:

\[
\hat{\omega}^s = R \hat{\omega}^b R^T \quad \vec{v}^s = R \vec{v}^b - R \hat{\omega}^b R^T \vec{p}^b = R \vec{v}^b - \hat{\omega}^s \vec{p}
\]

The spatial angular velocity can be simplified as follows:

\[
\hat{\omega}^s = R \hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b
\]

Using this result:

\[
\vec{v}^s = R \vec{v}^b - \hat{\omega}^s \vec{p} = R \vec{v}^b + \hat{\omega}^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} = \begin{bmatrix} R & \vec{p}^b \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \vec{v}^b \\ \omega^b \end{bmatrix}
\]

Therefore:

\[
V^s = \begin{bmatrix} \vec{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \vec{p}^b \\ \vec{0}^T & 1 \end{bmatrix} V^b
\]
Solution Problem 6: (10 points) Problem #14(a, b) in Chapter 2 (page 77) of the MLS text. It is not necessary to answer the question about surjectivity.

Part (a): Let $g \in SE(3)$ denote a homogeneous transformation matrix:

$$g = \begin{bmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{bmatrix} \quad \text{Ad}_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Then:

$$g^{-1} = \begin{bmatrix} R^T & -R^T \hat{p} \\ \vec{0}^T & 1 \end{bmatrix} \quad \text{Ad}_{g^{-1}} = \begin{bmatrix} R^T & -(R^T \hat{p})R^T \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T \hat{p} \\ 0 & R^T \end{bmatrix}$$

where we have made use of the identity $(R^T \hat{p}) = R^T \hat{p}R$. Let’s now compute $\text{Ad}_g \text{Ad}_{g^{-1}}$:

$$\text{Ad}_g \text{Ad}_{g^{-1}} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T \hat{p} \\ \vec{0}^T & R^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Hence, $\text{Ad}_{g^{-1}}$ must equal $(\text{Ad}_g)^{-1}$ since $\text{Ad}_g \text{Ad}_{g^{-1}} = I$.

Part (b): If

$$g_1 = \begin{bmatrix} R_1 & \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} R_2 & \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Then

$$g_1 g_2 = \begin{bmatrix} R_1 R_2 & \vec{p}_1 + R_1 \vec{p}_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence:

$$\text{Ad}_{g_1 g_2} = \begin{bmatrix} R_1 R_2 & (\vec{p}_1 + R_1 \vec{p}_2)R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & \hat{p}_1 R_1 R_2 + R_1 \hat{p}_2 R_1 R_2 \\ 0 & R_1 R_2 \end{bmatrix} = \text{Ad}_{g_1} \text{Ad}_{g_2}$$