The Classical Matrix Groups

CDs 270, Spring 2010/2011

The notes provide a brief review of *matrix groups.* The primary goal is to motivate the language and symbols used to represent rotations ($\mathcal{SO}(2)$ and $\mathcal{SO}(3)$) and spatial displacements ($\mathcal{SE}(2)$ and $\mathcal{SE}(3)$).

1 Groups

A group, G, is a mathematical structure with the following characteristics and properties:

i. the group consists of a set of elements $\{g_j\}$ which can be indexed. The indices j may form a finite, countably infinite, or continous (uncountably infinite) set.

ii. An associative binary group operation, denoted by $'*'$, termed the *group product.* The product of two group elements is also a group element:

$$\forall\ g_i, g_j\ \in\ G \qquad g_i\ *\ g_j\ =\ g_k, \quad where\ g_k\ \in\ G.$$

iii. A unique group identify element, e, with the property that: $e\ *\ g_j\ =\ g_j$ for all $g_j\ \in G$.

iv. For every $g_j\ \in\ G$, there must exist an inverse element, g_j^{-1}, such that

$$g_j\ *\ g_j^{-1}\ =\ e.$$

Simple examples of groups include the integers, $\mathbb{Z}$, with addition as the group operation, and the real numbers mod zero, $\mathbb{R} - \{0\}$, with multiplication as the group operation.

1.1 The General Linear Group, $\mathcal{GL}(N)$

The set of all $N \times N$ invertible matrices with the group operation of matrix multiplication forms the *General Linear Group* of dimension N. This group is denoted by the symbol $GL(N)$, or $\mathcal{GL}(N, \mathbb{K})$ where K is a field, such as $\mathbb{R}$, $\mathbb{C}$, etc. Generally, we will only consider the cases where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, which are respectively denoted by $\mathcal{GL}(N, \mathbb{R})$ and $\mathcal{GL}(N, \mathbb{C})$. By default, the notation $GL(N)$ refers to real matrices; i.e., $\mathcal{GL}(N)\ =\ \mathcal{GL}(N, \mathbb{R})$.

The identity element of $\mathcal{GL}(N)$is the identify matrix, and the inverse elements are clearly just the matrix inverses. Note that the product of invertible matrices is necessarily invertible. If $A, B\ \in\ \mathcal{GL}(N)$, then $det(A) \neq 0$ and $det(B) \neq 0$. Hence, $det(AB)\ =\ det(A)\ det(B)\ \neq\ 0$. Similarly, $det[(AB)^{-1}]\ =\ det[A^{-1}]\ det[B^{-1}]\ =\ (1/det(A))\ (1/det(B))\ \neq 0$.

2 Subgroups

A subgroup, H, of G (denoted $H \subseteq G$) is a subset of G which is itself a group under the group operation of G. Note that this subgroup must contain the identity element.

The General Linear Group has several important subgroups, which as a family make up the *Classical Matrix Subgroups.*

2.1 The Classical Matrix Subgroups

The Special Linear Group, $\mathcal{SL}(N)$, consists of all members of $GL(N)$ which have determinant 1. To see that this set of matrices forms a group, note that if $A, B \in SL(N)$, then to show that $A * B \in SL(N)$, note that $det(AB) = det(A) \cdot det(B) = 1 \cdot 1 = 1$. Also, for any $A \in SL(N)$, $det(A^{-1}) = [det(A)]^{-1} = [1]^{-1} = 1$, so that every inverse is a member of $\mathcal{SL}(N)$.

The Orthogonal Group, $\mathcal{O}(N)$, consists of all real $N \times N$ matrices with the property that:

$$A^T\, A \;=\; I \qquad \text{for all } A \in \mathcal{O}(N)$$

(Note that this relationship and the group properties also implies that for any $A \in \mathcal{O}(N)$, $A\, A^T = I$ as well). As described in class, the group $\mathcal{O}(N)$ can represent spherical displacements in N-dimensional Euclidean space. To check that $\mathcal{O}(N)$ forms a group, note that:

- The product of two orthogonal matrices is an orthogonal matrix. Let $A, B \in \mathcal{O}(N)$. To check if the product AB is orthogonal, note that: $(AB)^T(AB) = B^T\, A^T\, A\, B = B^T\, B = I$, and thus the product AB is orthogonal.
- Recall that the inverse of an orthogonal matrix is the same as its transpose: $A^T = A^{-1}$ for all $A \in \mathcal{O}(N)$. Thus, since $A^T A = I$ for othogonal matrices, it is also true that the inverse of A, A^{-1}, is an orthogonal matrix: $[A^{-1}]^T\, A^{-1} = [A^T]^T\, A^T = A\, A^T = I$.

The Special Orthogonal Group, $\mathcal{SO}(N)$, consists of all orthogonal matrices whose determinants have value $+1$. To show that these matrices form a group, we can immediately apply the results from the analyses of $\mathcal{O}(N)$ and $\mathcal{SL}(N)$ above to further show that the product of matrices in $\mathcal{SO}(N)$ has determinant $+1$, and that the inverses of all matrices in $\mathcal{SO}(N)$ have determinant $+1$.

The Unitary Group, $\mathcal{U}(N)$, consists of orthogonal matrices with complex matrix entries: $\mathcal{U}(N) = \mathcal{O}(N, \mathbb{C})$. Note that in this case of complex valued matrices, the matrix transpose operation is replaced by the Hermitian operation (transpose and complex conjugation): $A^*\, A = I$ for all $A \in \mathcal{U}(N)$, where A^* is the transposed complex conjugate of A.

The Special Unitary Group, $\mathcal{SU}(N)$, consists of those unitary matrices with determinant having value $+1$.

The Special Euclindean Group, $\mathcal{SE}(N)$, consists of all rigid body transformations of N-dimensional Euclidean space which preserve the length of vectors (i.e., distances between points). Matrices in $\mathcal{SE}(2)$ describe planar rigid body displacements, while matrices in $\mathcal{SE}(3)$ describe spatial rigid body displacements. Matrices in $\mathcal{SE}(N)$ take the form:

$$\begin{bmatrix} R & \vec{d} \\ \vec{0}^T & 1 \end{bmatrix}$$

where $R \in \mathcal{SO}(N)$, $\vec{d} \in \mathbb{R}^N$, and the vector $\vec{0}$ is an N-vector whose elements are identically zero.

2.2 Some Simple Examples

- $GL(1) \;=\; \mathbb{R} - \{0\}$.
- $GL(1, \mathbb{C}) \;=\; \mathbb{C} - \{0\}$.
- $\mathcal{O}(1) \;=\; \{1, -1\}$.
- $\mathcal{SO}(1) \;=\; \{1\}$.
- $\mathcal{SU}(1) \;=\; \{e^{i\theta}\}$, for all $\theta \;\in \mathbb{R}$.
- $\mathcal{SO}(2) \;=\; 2 \times 2$ matrices of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note, we say that $\mathcal{SO}(2)$ and $SU(1)$ are *isomorphic* because there is a one-to-one correspondence between every element in the two groups.