The Classical Matrix Groups
CDs 270, Spring 2010/2011

The notes provide a brief review of matrix groups. The primary goal is to motivate the language and symbols used to represent rotations ($SO(2)$ and $SO(3)$) and spatial displacements ($SE(2)$ and $SE(3)$).

1 Groups

A group, $G$, is a mathematical structure with the following characteristics and properties:

i. the group consists of a set of elements $\{g_j\}$ which can be indexed. The indices $j$ may form a finite, countably infinite, or continuous (uncountably infinite) set.

ii. An associative binary group operation, denoted by $\ast$, termed the group product. The product of two group elements is also a group element:

$$\forall g_i, g_j \in G \quad g_i \ast g_j = g_k, \quad \text{where } g_k \in G.$$  

iii. A unique group identify element, $e$, with the property that: $e \ast g_j = g_j$ for all $g_j \in G$.

iv. For every $g_j \in G$, there must exist an inverse element, $g_j^{-1}$, such that

$$g_j \ast g_j^{-1} = e.$$

Simple examples of groups include the integers, $\mathbb{Z}$, with addition as the group operation, and the real numbers mod zero, $\mathbb{R} - \{0\}$, with multiplication as the group operation.

1.1 The General Linear Group, $\mathcal{GL}(N)$

The set of all $N \times N$ invertible matrices with the group operation of matrix multiplication forms the General Linear Group of dimension $N$. This group is denoted by the symbol $GL(N)$, or $\mathcal{GL}(N, K)$ where $K$ is a field, such as $\mathbb{R}$, $\mathbb{C}$, etc. Generally, we will only consider the cases where $K = \mathbb{R}$ or $K = \mathbb{C}$, which are respectively denoted by $\mathcal{GL}(N, \mathbb{R})$ and $\mathcal{GL}(N, \mathbb{C})$. By default, the notation $GL(N)$ refers to real matrices; i.e., $\mathcal{GL}(N) = \mathcal{GL}(N, \mathbb{R})$.

The identity element of $\mathcal{GL}(N)$ is the identify matrix, and the inverse elements are clearly just the matrix inverses. Note that the product of invertible matrices is necessarily invertible. If $A, B \in \mathcal{GL}(N)$, then $\det(A) \neq 0$ and $\det(B) \neq 0$. Hence, $\det(AB) = \det(A) \det(B) \neq 0$. Similarly, $\det((AB)^{-1}) = \det(A^{-1}) \det(B^{-1}) = (1/\det(A)) (1/\det(B)) \neq 0$. 

1
2 Subgroups

A subgroup, \( H \), of \( G \) (denoted \( H \subseteq G \)) is a subset of \( G \) which is itself a group under the group operation of \( G \). Note that this subgroup must contain the identity element.

The General Linear Group has several important subgroups, which as a family make up the Classical Matrix Subgroups.

2.1 The Classical Matrix Subgroups

The Special Linear Group, \( \text{SL}(N) \), consists of all members of \( \text{GL}(N) \) which have determinant 1. To see that this set of matrices forms a group, note that if \( A, B \in \text{SL}(N) \), then to show that \( A \ast B \in \text{SL}(N) \), note that \( \det(AB) = \det(A) \cdot \det(B) = 1 \cdot 1 = 1 \). Also, for any \( A \in \text{SL}(N) \), \( \det(A^{-1}) = \left[\det(A)\right]^{-1} = [1]^{-1} = 1 \), so that every inverse is a member of \( \text{SL}(N) \).

The Orthogonal Group, \( \text{O}(N) \), consists of all real \( N \times N \) matrices with the property that:

\[
A^T A = I \quad \text{for all } A \in \text{O}(N)
\]

(Note that this relationship and the group properties also implies that for any \( A \in \text{O}(N) \), \( A A^T = I \) as well). As described in class, the group \( \text{O}(N) \) can represent spherical displacements in \( N \)-dimensional Euclidean space. To check that \( \text{O}(N) \) forms a group, note that:

- The product of two orthogonal matrices is an orthogonal matrix. Let \( A, B \in \text{O}(N) \). To check if the product \( AB \) is orthogonal, note that: \((AB)^T(AB) = B^T A^T A B = B^T B = I \), and thus the product \( AB \) is orthogonal.

- Recall that the inverse of an orthogonal matrix is the same as its transpose: \( A^T = A^{-1} \) for all \( A \in \text{O}(N) \). Thus, since \( A^T A = I \) for orthogonal matrices, it is also true that the inverse of \( A \), \( A^{-1} \), is an orthogonal matrix: \([A^{-1}]^T A^{-1} = [A^T]^T A^T = A A^T = I \).

The Special Orthogonal Group, \( \text{SO}(N) \), consists of all orthogonal matrices whose determinants have value +1. To show that these matrices form a group, we can immediately apply the results from the analyses of \( \text{O}(N) \) and \( \text{SL}(N) \) above to further show that the product of matrices in \( \text{SO}(N) \) has determinant +1, and that the inverses of all matrices in \( \text{SO}(N) \) have determinant +1.

The Unitary Group, \( \text{U}(N) \), consists of orthogonal matrices with complex matrix entries: \( \text{U}(N) = \text{O}(N, \mathbb{C}) \). Note that in this case of complex valued matrices, the matrix transpose operation is replaced by the Hermitian operation (transpose and complex conjugation): \( A^* A = I \) for all \( A \in \text{U}(N) \), where \( A^* \) is the transposed complex conjugate of \( A \).

The Special Unitary Group, \( \text{SU}(N) \), consists of those unitary matrices with determinant having value +1.
The Special Euclidean Group, $\mathcal{SE}(N)$, consists of all rigid body transformations of $N$-dimensional Euclidean space which preserve the length of vectors (i.e., distances between points). Matrices in $\mathcal{SE}(2)$ describe planar rigid body displacements, while matrices in $\mathcal{SE}(3)$ describe spatial rigid body displacements. Matrices in $\mathcal{SE}(N)$ take the form:

\[
\begin{bmatrix}
R & \vec{d} \\
\vec{0}^T & 1
\end{bmatrix}
\]

where $R \in SO(N)$, $\vec{d} \in \mathbb{R}^N$, and the vector $\vec{0}$ is an $N$-vector whose elements are identically zero.

### 2.2 Some Simple Examples

- $GL(1) = \mathbb{R} - \{0\}$.
- $GL(1, \mathbb{C}) = \mathbb{C} - \{0\}$.
- $\mathcal{O}(1) = \{1, -1\}$.
- $SO(1) = \{1\}$.
- $SU(1) = \{e^{i\theta}\}$, for all $\theta \in \mathbb{R}$.
- $SO(2) = 2 \times 2$ matrices of the form:

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

Note, we say that $SO(2)$ and $SU(1)$ are isomorphic because there is a one-to-one correspondence between every element in the two groups.