Chapter 11

Wrench Resistant Grasps

Chapter 6 introduced the notion of *wrench resistance*, a basic property which helps to ensure grasp security. This notion will be refined in this chapter for the case of rigid body grasps involving frictional contacts. The frictional grasps studied in this part of the book inherit all of the c-space geometry from the frictionless contact case, but have the added complexity of frictional mechanics. The Coulomb friction model constraints are imposed on the co-tangent space of contact wrenches, and not directly on the c-space geometry. Fortunately, these constraints can be modeled and analyzed using basic concepts from *convex analysis*, which are reviewed in this chapter.

As introduced in Chapter 6, a frictional grasp is *wrench resistant* when any perturbing wrench applied to the grasped object can be counterbalanced by feasible finger forces lying within the friction cones at the contacts. Because this concept is central to this chapter, Definition 4 of Chapter 6 is repeated here.

Definition 1. *Let a rigid body $\mathcal{B}$ be held at a configuration q_0 by k rigid finger bodies $\mathcal{O}_i, \ldots, \mathcal{O}_k$. The grasp is said to be* **wrench resistant** *if there exists feasible contact forces, $\vec{f} \in C_1 \times \cdots \times C_k$, such that:*

$$G\vec{f} \; + \; \mathbf{w}_{ext} \; = \; \vec{0} \tag{11.1}$$

for all $\mathbf{w}_{ext} \in T^*_{q_0}\mathcal{C}$, where G is the grasp map, C_i is the friction cone associated with the i^{th} contact, and $\vec{f} = \begin{pmatrix} f_1 & \cdots & f_k \end{pmatrix}^T$ is the vector of finger contact forces.

Definition 1 is intuitively appealing because it directly captures the notions of grasp safety discussed in Chapter 6. However, it does not lead to practical procedures to test if a given grasp is wrench resistant. Section 11.1 provides an alternative (but equivalent) theorem on wrench resistance in terms of internal grasp forces and *non-marginal* equilibrium grasps. This reformulation leads to practical tests for wrench resistance. Moreover, it allows the catalog of equilibrium grasps developed in Chapter 5 to be used to construct wrench resistant grasps. Section 11.2 develops a general computational test for arbitrary grasps based on the formulation of the wrench resistibility property in terms of *Linear Matrix Inequalities* (LMIs), for which there exist efficient computational algorithms. To illustrate the practical utility of this chapter's concepts, the problem of grasp force optimization is formulated as a convex optimization problem in Section 11.3. Finally, Section 11.4 analyzes a grasp from the control system point of view. It will be seen that a wrench resistant grasp is a controllable grasp.

11.1 Wrench Resistance and Internal Forces

We have seen in Chapter 6 that a wrench resistant grasp must be an equilibrium grasp–i.e., the equilibrium condition is a necessary condition for wrench resistance. This result will be extended below to show that a *non-marginal equilibrium grasp* is a necessary and sufficient condition for wrench resistance.

Definition 2. *Let a rigid body $\mathcal{B}$ be held in an equilibrium grasp by frictional rigid bodies $\mathcal{O}_i, \ldots, \mathcal{O}_k$ at configuration q_0. Let $f_1^0,\ f_2^0,\ \cdots,\ f_k^0$ denote the equilibrating finger contact forces when $\mathcal{B}$ is held at q_0. The equilibrium grasp is said to be* **marginal** *if one or more of the equilibrating finger contact forces lie on the boundary of their respective friction cones: $f_j^0 \in \text{bdy}(C_j)$ for one or more indices $j \in 1, \ldots, k$. If all equilibrium contact forces lie in the interior of their respective friction cones, $f_j^0 \in int(C_j)\ \forall\ j = 1, \ldots, k$, then the grasp is a* **non-marginal** *equilibrium grasp.*

The following theorem provides an alternative characterization of wrench resistance that leads to graphical procedures as well as computational algorithms for assessing a grasp's wrench resistance ability. Recall from Chapter 4 that an *internal* or *squeeze* force is a finger contact force, $\vec{f}_{int}$, lying in the null space of the grasp map, G: $G\vec{f}_{int} = \vec{0}$.

Theorem 1 (Wrench Resistance and Internal Forces). *Let a rigid body $\mathcal{B}$ be in frictional point contact with k rigid frictional finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$ at equilibrium configuration q_0. The grasp (G, C) is* **wrench resistant** *iff:*

i) The grasp map $G:\ C_1 \times \cdots \times C_k \to T^*_{q_0}I\!R^m$ is surjective (it has full rank m),

ii) There exists an **internal force** *$\vec{f}_{int} = [f_i, \ldots, f_k]^T$ that lies in the* interior *of the friction cone C: $\vec{f}_{int} \in Null(G) \cap int(C_1 \times \cdots \times C_k)$. I.e., the grasp is a non-marginal equilibrium grasp.*

Before tackling the proof of this theorem, let us first consider the physical interpretation of the theorem's two conditions. To see the necessity of Condition (i), assume that the grasp map G has rank l, where $l < m$. In that case, there is a set of net body wrenches that cannot be counterbalanced by any combination of finger forces, and thus the grasp fails to be wrench resistant. However, a full rank condition on G is not sufficient by itself to guarantee wrench resistibility. While it is always possible to find an $\vec{f}$ that solves Equation (11.1) when G is full rank, the solution is not guaranteed to lie in the set of feasible contact forces, $C = C_1 \times \cdots \times C_k$. Condition (ii) ensures the feasibility of the contact forces as follows. Since an internal force produces no net wrench on $\mathcal{B}$, its magnitude can be arbitrarily scaled without affecting the equilibrium grasp [1]. Assume that there exists a particular contact force vector, $\vec{f}_p$, that solves Equation (11.1) for a given $\vec{w}_{ext}$, but does not necessarily lie in C. If there also exists for the given grasp configuration an internal force in the interior of the friction cone, $\vec{f}_{int} \in Null(G) \cap int(C)$, then as the fingers squeeze along the internal force subspace, there is a corresponding increase in the normal contact force magnitudes, which will allow the grasp to support greater feasible tangential force components (see Exercises). Hence, with sufficient squeezing, the net contact force, $\vec{f}_{net} = \vec{f}_p + s^* \vec{f}_{int}$ (where s^* is a sufficiently large scaling of the feasible internal force) is guaranteed to lie in C.

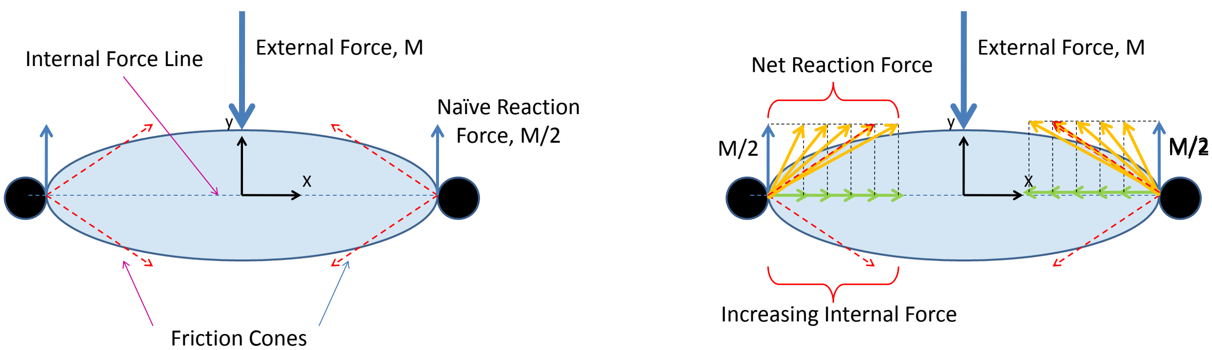

Figure 11.1: (a) A two-finger frictional grasp of an ellipse with physically unrealizable tangential reaction forces applied at each finger. (b) As the internal forces are increased, the net reaction force (combining internal force and tangential force) eventually lies withing the friction cone, and is therefore feasible.

Example: An example of this construction is depicted in Figure 11.1 for a frictional two-fingered antipodal grasp of an ellipse. Assume that the grasp is initially in equilibrium without any external forces applied to the ellipse. Next, a perturbing force of magnitude M is applied along the y-axis, which is orthogonal to the normals of both finger contacts. A particular solution to Equation (11.1) in this case is for each finger to apply a force of magnitude $M/2$ along the tangents to each contact, so as to oppose the perturbing force. However, a pure tangential force is *not* feasible with the Coulomb friction constraint. But, this grasp possesses a 1-dimensional space of internal forces that physically corresponds to equal magnitude squeezes by each finger along their respective normals. As the squeezing increases, more and more tangential forces, which are needed to counterbalance the perturbing wrench, can be supported by the frictional contact. At a sufficiently large normal force (i.e., $f_n \geq M/(2\mu)$), the contacts can support the tangential forces that are needed to counterbalance this particular perturbing wrench. Theorem 1 confirms that such a squeezing can be found for any perturbing wrench.

Based on this discussion, Theorem 1 is proven as follows.

[1] Clearly there are practical limits to the magnitude of forces that can be applied before mechanism actuator limits are reached or plastic material deformations are experienced. These practical issues are ignored for now.

Proof: To prove the sufficiency of Theorem 1 (i.e., if G is surjective and if there exists an internal force in $int(C)$, then the grasp is wrench resistant) , consider an arbitrary wrench, $\mathbf{w}_{ext} \in T^*_{q_0}\mathcal{C}$, perturbing the grasped object $\mathcal{B}$ when it is held by frictional contacts Let $\vec{f}_p$ be any particular contact force vector that solves Equation (11.1) for $\mathbf{w}_{ext}$; $\vec{f}_p$ need not be feasible. If G is surjective, then at least one such contact force, $\vec{f}_p$, always exists. If there exists $\vec{f}_{int} \in Null(G) \cap int(C)$, then the net contact force $\vec{f}_{net} = \vec{f}_p + s\vec{f}_{int}$ is also a solution to Equation (11.1) for any $s \in I\!R$. We now need to show that there exists a value of s such that $\vec{f}_{net} \in C$. First note that

$$\lim_{s\to\infty} \frac{\vec{f}_p + s\vec{f}_{int}}{s} = \vec{f}_{int} \in int(C) .$$

Thus, because $int(C)$ is an open set, there exists a sufficiently large value of the scalar s, denoted by s_{min}, such that

$$\frac{\vec{f}_p + s_{min}\vec{f}_{int}}{s_{min}} \in int(C) . \tag{11.2}$$

Since the set of feasible forces, C, is a *cone*[2], if $\vec{f}_p$ and $\vec{f}_{int}$ satisfy Equation (11.2), then it must be true that:

$$\vec{f}_p + s_{min}\vec{f}_{int} \in int(C) .$$

Hence, $\vec{f}_{net} = \vec{f}_p + s\vec{f}_{int}$ is a feasible solution of Equation (11.1) for all $s \geq s_{min}$.

To show the necessity of the two conditions of Theorem 1 (i.e., that if (G, FC) is wrench resistant, then G must be surjective and there must exist an internal force in $int(C)$), assume that (G, C) is a wrench resistant grasp. The necessity of G's surjectivity was discussed above: G must be surjective in order to find a solution to the equation $G\vec{f}_p + \mathbf{w}_{ext} = 0$. To show the necessity of Condition (ii), choose a finger contact force $\vec{f}_1 \in int(C)$. Let $\mathbf{w}^1_{net}$ be the net force acting on $\mathcal{B}$ due to the application of contact forces $\vec{f}_1$: $G\vec{f}_1 = \mathbf{w}^1_{net}$. Because (G, FC) is assumed to be force closure, then there exists a feasible contact force $\vec{f}_2$ such that $G\vec{f}_2 = \mathbf{w}_{net}$. Contact force $\vec{f}_2$ need not lie in $int(C)$, though it must be an element of C. Hence, $\vec{f}_{diff} = \vec{f}_2 - \vec{f}_1$ lies in $Null(G)$, since $G(\vec{f}_2 - \vec{f}_1) = \mathbf{w}_{ext} - \mathbf{w}_{ext} = 0$. Because $int(C)$ is a convex cone, $\vec{f}_{diff} \in int(C)$.[3] Thus, there must exist at least one internal force, $\vec{f}_N$, that lies in $int(C)$. □

Wrench resistance requires the grasp map G to be surjective. The following proposition shows that frictional grasps having a minimal number of fingers and a nondegenerate arrangement of the contacts will indeed have a surjective grasp map. The proof is derived in the Exercises.

Proposition 11.1.1. *Let $k \geq 2$ planar fingers bodies make frictional point contact with a planar rigid body $\mathcal{B}$. The grasp map G is full rank unless all finger contacts are coincident.*

[2] If $\vec{f}_1, \vec{f}_2 \in C_p$, where C_p is a cone, then $a_1\vec{f}_1 + a_2\vec{f}_2 \in C_p$ for all $a_1, a_2 \in I\!R^+$.

[3] There are only two possibilities regarding $\vec{f}_2$–it lies in $int(C)$, or it lies in $bdy(C)$. If $\vec{f}_2 \in int(C)$, then it should be immediate clear that $\vec{f}_{diff} = \vec{f}_2 - \vec{f}_1$ lies in $int(C)$. However, if $\vec{f}_2 \in bdy(C)$, then it is also true that $\vec{f}_{diff} = \vec{f}_2 - \vec{f}_1$ lies in $int(C)$ due to the fact that C is a cone.

Similarly, let $k \geq 3$ finger bodies make point with a 3-dimensional rigid body $\mathcal{B}$ at point contacts whose contact forces are governed by the Coulomb friction law. The grasp map G is full rank unless all finger contacts lie along a common line. Finally, let $k \geq 2$ soft fingers contact a 3-dimensional object. The grasp map is full rank kunless all of the finger contacts are coincident.

Examples of Frictional Wrench Resistant Grasps. Because every wrench resistant grasp must also be a non-marginal equilibrium grasp, the catalog of equilibrium grasps developed in Chapter 5 can be used to construct wrench resistant grasps for a given object.

11.2 Wrench-Resistance as a Linear Matrix Inequality

Using Theorem 1 and its corollary, the wrench resistibility of many simple grasps can be resolved with simple analyses. Moreover, many (but not all!) wrench resistant grasps of a given object can be constructed from the equilibrium grasp catalog in Chapter 5. However, a computational procedure is generally required to assess the wrench resistibility of an arbitrary frictional grasp. Any such algorithm must apply to general grasp geometries and also have computational efficiency. This section shows how the problem of assessing the wrench resistance of a given grasp can be converted into a *Linear Matrix Inequality* (LMI) problem, for which there are efficient computational procedures.

11.2.1 Reformulating Friction Constraints in terms of Positive Definite Symmetric Matrices

The practical computational difficulty of determining the resistibility of a complex frictional grasp arises from the cone-like nature of the feasible contact force constraints. Fortunately, these constraints can be reformulated to take advantage of advances in convex analysis. Consider a hard point contact with Coulomb friction. The contact force can be described as $f = \begin{pmatrix} f_x & f_y & f_n \end{pmatrix}^T$ where f_n is the normal contact force and f_x, f_y are the components of the contact force tangent to the surface. Recall that the contact forces are constrained:

$$f_n \geq 0; \qquad = \sqrt{f_x^2 + f_y^2} \leq \mu f_n \tag{11.3}$$

where μ is the Coulomb friction coefficient. These contraints can be equivalently expressed as follows.

Proposition 11.2.1. *Let a rigid finger body $\mathcal{O}$ be in frictional point contact with rigid object $\mathcal{B}$. Let $f = \begin{pmatrix} f_x & f_y & f_n \end{pmatrix}^T$ denote the finger contact force and let μ denote the Coulomb friction coefficient. Let $P(f)$ be the following symmetric matrix:*

$$P(\vec{f}) \;=\; \begin{bmatrix} \mu f_n & 0 & f_x \\ 0 & \mu f_n & f_y \\ f_x & f_y & \mu f_n \end{bmatrix} \;. \tag{11.4}$$

The feasible contact forces constraints in Equation (11.3) can be equivalently expressed as:

$$P(\vec{f}) \succeq 0 \tag{11.5}$$

where $A \succeq 0$ *denotes that the* $N_A \times N_A$ *matrix* A *is positive semi-definite* [4]*, i.e., that* $\vec{u}^T A \vec{u} \geq 0$ *for all* $\vec{u} \in I\!R^{N_A}$.

Proof: Because $P(\vec{f})$ is a real symmetric matrix, its eigenvalues are real. Let λ_1, λ_2, and λ_3 denote the three eigenvalues of $P(\vec{f})$. For the 3×3 matrix to be positive semidefinite, $\lambda_1, \lambda_2, \lambda_3 \geq 0$. The eigenvalues of the matrix $P(\vec{f})$ in Equation (11.4) are

$$\lambda_1 = \mu f_n \tag{11.6}$$

$$\lambda_2 = \mu f_n - \sqrt{f_x^2 + f_y^2} \tag{11.7}$$

$$\lambda_3 = \mu f_n + \sqrt{f_x^2 + f_y^2}\ . \tag{11.8}$$

Thus, a necessary conition for matrix $P(\vec{f})$ in Equation (11.4) to be positive semidefinite is that $\lambda_1 \geq 0$, which implies that $f_n \geq 0$. Since $f_n \geq 0$, the eigenvalue λ_3 is automatically positive semidefinite. Consequently, $P(\vec{f})$ will be positive semidefinite if $\lambda_2 \geq 0$, which implies that $\mu f_n \geq \sqrt{f_x^2 + f_y^2}$, which verifies the proposition. □

We will see below that the matrix inequality constraint (11.4) can be constructed in a principled way from the theory of Linear Matrix Inequalities. Further note that the constraints imposed by some other common frictional contact models can also be expressed as symmetric semi-definite matrices (see the Exercises for one example). Trivially, for frictionless point contact, $P(\vec{f})$, is described by a scalar, $P(\vec{f}) = f_n$.

The constraint matrix $P(\vec{f})$ in Equation (11.4) is linear in the finger contact force components, and hence it can be factored into a simple linear sum whose coefficients are constant symmetric matrices:

$$\begin{aligned} P(\vec{f}) &= \begin{bmatrix} \mu f_n & 0 & f_x \\ 0 & \mu f_n & f_y \\ f_x & f_y & \mu f_n \end{bmatrix} = f_x \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + f_y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f_n \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \\ &= f_x S_x + f_y S_y + f_n S_n\ . \end{aligned}$$

The symmetric constraint matrices for some other frictional models, such as the elliptic approximation of the soft contact model (see Exercises) are similarly linear in the contact forces, and therefore several of practical rigid body frictional contact models can be put in the form:

$$P(\vec{f}) = \sum_{j=1}^{m_i} f_j S_j \succeq 0 \tag{11.9}$$

where f_j is the j^{th} component of the m_i-dimensional finger force vector $\vec{f} = \begin{pmatrix} f_1 & f_2 & \cdots & f_{m_i} \end{pmatrix}^T$, and S_j is the *constant* symmetric matrix associated with f_j. For frictional point contacts on

[4] The relation $A \succ 0$ denotes that matrix A is strictly positive definite, i.e. $\vec{u}^T A \vec{u} > 0$ for all $\vec{u} \in I\!R^{N_A}$.

3-dimensional bodies, $m_i = 3$, and therefore the index j refers to the x-, y-i, and n- contact force components. Equation (11.9) is a *Linear Matrix Inequality* (LMI).

Definition 3. *A* **Linear Matrix Inequality** *(LMI) in the variable $\vec{x} \in I\!R^N$ takes the form:*

$$Q(\vec{x}) \triangleq Q_0 \ + \ \sum_{i=1}^{N} \ x_i Q_i \succeq 0 \tag{11.10}$$

where $Q_0, Q_1, \cdots, Q_N$ are constant symmetric $N \times N$ matrices and x_i is the i^{th} entry of the N-vector $\vec{x}$. The inequality (11.10) is said to be **non-strict***: $\vec{u}^T Q(x)\vec{u} \geq 0$ for all $\vec{u} \in I\!R^N$. Similarly, a* **strict** *LMI has the form $Q(\vec{x}) \succ 0$: $\vec{u}^T Q(x)\vec{u} > 0$ for all $\vec{u} \in I\!R^N$.*

Linear matrix inequalities can be used to represent a wide variety of constraints, including linear constraints, quadratic constraints, convex constraints, and matrix norm inequalities.

Example: A constraint of the form

$$||A\vec{x} + \vec{b}|| \ \leq \ \vec{c}^T\vec{x} + d \tag{11.11}$$

is termed a *second order cone constraint.* This constraint can equivalently be expressed as an LMI:

$$||A\vec{x} + \vec{b}|| \ \leq \ \vec{c}^T\vec{x} + \vec{d} \quad \Longleftrightarrow \begin{bmatrix} (\vec{c}^T\vec{x} + \vec{d})I & A\vec{x} + \vec{b} \\ (A\vec{x} + \vec{b})^T & (\vec{c}^T\vec{x} + \vec{d})I \end{bmatrix} \succeq 0 \tag{11.12}$$

where $\vec{x} \in I\!R^{N_x}$, $A \in I\!R^{N_A \times N_x}$, I is the $N_A \times N_A$ identity matrix, $\vec{c} \in I\!R^{N_x}$, and $d \in I\!R$. Note that constraints (11.3) can be reformulated to (11.4) using this correspondence (see Exercises).

From the computational point of view, there are two main types of LMI problems:

Definition 4. *The* **LMI feasibility problem** *tests for the existence of an $\vec{x}$ such that $Q(\vec{x}) \succ 0$, or $Q(\vec{x}) \succeq 0$. The* **LMI optimization problem** *minimizes a convex cost fuction $c(\vec{x})$ over all admissible $\vec{x}$ that satisfy $Q(\vec{x}) \succ 0$ or $Q(\vec{x}) \succeq 0$.*

In addition to the wrench resistance problems discussed in this chapter, many problems in control, estimation, and optimization can be formulated in terms of LMIs and their associated feasibility and optimization algorithms. Efficient algorithms (e.g., interior point methods) have been developed to solve several classes of LMI problems, and powerful software tools are ready available. The remainder of this section will show how the resistability of a grasp can be tested using an efficient LMI formulation.

The ensemble of friction cone constraints for all k fingers can be represented as a symmetric block-diagonal matrix, where the i^{th} matrix block is determined by the i^{th} friction cone constraint matrix:

$$\mathcal{P}(\vec{f}) = \text{Blockdiag}(P(f_1), P(f_2), \cdots, P(f_k)) \succeq 0 \ . \tag{11.13}$$

where $\vec{f}_i$ is the i^{th} finger force and $\vec{f}$ is the vector of all contact forces: $\vec{f} = \begin{pmatrix} \vec{f}_1 & \cdots & \vec{f}_k \end{pmatrix}^T$. Let $f_{i,j}$ denote the j^{th} component of the i^{th} finger force. Similarly, let $S_{i,j}$ denote the constant symmetric matrix associated with $f_{i,j}$ in Equation (11.9). Then (11.13) can be represented as an LMI:

$$\mathcal{P}(\vec{f}) = \sum_{l=1}^{N_c} f_l S_l \succeq 0 \tag{11.14}$$

where the index l is short-hand notation for the double index over fingers and contact force dimension:

$$l(i,j) = (i-1)n + j, \qquad i = 1,\ldots,k; \quad j = 1,\ldots,n$$

where n denotes the dimension of the finger contact force vector. Hence, $N_c = kn$. The matrix S_l denotes the block-diagonal constant symmetric matrix:

$$S_l = BlockDiag(0,\ldots,0,S_{i,j},0,\ldots,0).$$

Theorem 1 can be converted to standard *LMI feasibility* problem. The theorem has two basic conditions: (i) the grasp map G is surjective, (ii) there exists a grasp force $\vec{f}_{int} \in Null(G) \cap int(C)$. Condition (i) remains unchanged. Condition (ii) can be restated in the language of LMIs as: there exists $\vec{f} \in Null(G)$ such that $\mathcal{P}(\vec{f}) \succeq 0$. The second condition can be further adapted to the LMI framework. Let V denote a matrix whose columns are a basis for the $(N_c - m)$-dimensional null space of G. Thus, all null vectors can be expressed as:

$$Null(G) = V\vec{z}, \qquad \vec{z} \in I\!R^{(N_c-m)}.$$

Thus, Condition (ii) can be stated as a single LMI by reparametrizing Equation (11.14) in the basis V:

$$\tilde{\mathcal{P}}(\vec{z}) = \sum_{j=1}^{N_c-m} z_j \tilde{S}_j \succeq 0\ . \tag{11.15}$$

where

$$\tilde{S}_j = \sum_l V_{l,j} S_l. \tag{11.16}$$

and $V_{l,j}$ is the (l,j) element of matrix V.

Proposition 11.2.2 (LMI Wrench Resistance Test). *A rigid body grasp of $\mathcal{B}$ by k rigid finger bodies is* **wrench resistant** *if and only if*

(i) The grasp map G is surjective (or full rank),

(ii′) There exists $\vec{z} \in I\!R^{N_c-m}$ such that $\tilde{\mathcal{P}}(\vec{z}) \succeq 0$, where $\tilde{\mathcal{P}}(\vec{z})$ is given by Equation (11.15).

Proposition 11.2.2 shows that the problem of testing for wrench resistability leads to an *LMI feasibility* test, as one only seeks the existance of at least one vector $\vec{z}$ that satisfies Condition (ii′). The LMI feasibility problem can be reformulated as an *LMI optimization* *problem* using a standard approach. In this way, the powerful computational codes that have been developed for LMI optimization can be leveraged. First we make the observation that a non-strict symmetric LMI, $Q(\vec{x}) \succeq 0$, is satisfied if and only if there exists a negative semi-definite scalar, η, such that $Q(\vec{x}) + \eta I \succeq 0$ (see Exercises). I.e.,

$$Q(\vec{x}) \succeq 0 \quad \Longleftrightarrow \quad \exists\, \eta \leq 0 \quad \text{such that} \quad Q(\vec{x}) \;+\; \eta I \;\succeq 0\;. \tag{11.17}$$

Consequently, Condition (ii$'$) in Proposition 11.2.2 can be restated as:

$$\text{minimize} \;\; \{\; \eta \;\} \qquad \text{subject to}: \;\; \tilde{\mathcal{P}}(\vec{z}) \;+\; \eta\; I \succeq 0.$$

The LMI is feasible (and hence there exists an internal force satisfying Condition (ii) of Theorem 1) if and only if the optimal value, η^*, of the LMI minimization problem is negative.

11.3 Grasp Force Optimization

The techniques introduced so far in this chapter are useful for more than the problem of testing a given grasp's wrench resistability. This section formulates the *grasp force optimization* problem to show the practical utility of this chapter's brief review of convex analysis.

Suppose a multi-fingered robotic hand must apply a specified net wrench, $\mathbf{w}_{net}$ to a grasped object using a specific set of contact locations. Recall that the net wrench is the image of $\begin{pmatrix} f_1 & \cdots & f_k \end{pmatrix} \in C_1 \times \cdots \times C_k$ under the action of the grasp map G. Generally, there will not be a unique set of contact forces which are mapped through the grasp map to the desired object wrench. The goal of a grasp force optimization procedure is to select the contact forces which meet the task objective while simultaneously optimizing an additional criterion.

Definition 5 (Grasp Force Optimization Problem)**.** *Let (G, C) model the grasp of a rigid object $\mathcal{B}$ by k rigid frictional finger bodies at specific contact points $x_1, \ldots, x_k$. The* **grasp force optimization** *problem seeks to minimize a cost, $h(\vec{f})$, while satisfying the grasping constraints:*

$$\begin{aligned} \min \quad & \{\; h(\vec{f}) \;\} \\ \text{subject to}: \quad & \vec{f} \;\in\; C_1 \times \cdots \times C_k, \text{and} \\ & G\vec{f} + \vec{w}_{ext} = 0\;. \end{aligned}$$

In particular, if the cost-function is a convex function of the contact forces, then the grasp optimization problem becomes a *convex optimization problem*:

Definition 6 (Convex Optimization)**.** *Let $\vec{x} \in I\!R^n$ be an optimization variable. A convex minimization problem is of the form:*

$$\begin{aligned} \min_{\vec{x}} \quad & \{\; h(\vec{x}) \;\} \\ \text{subject to} \quad & g(\vec{x}) \geq \vec{0} \\ & A\vec{x} + \vec{b} = 0 \end{aligned}$$

where $h(\vec{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, the m_c inequality constraint functions $g(\vec{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ are assumed to be convex functions, and the matrix $A \in \mathbb{R}^{l \times p}$ and the vector $\vec{b} \in \mathbb{R}^l$ in the affine equality constraint are constants.

Convex optimization problems are efficiently solvable and yield a unique optimum solution. Powerful and general convex optimization toolkits are readily available. The general class of convex optimization problems includes many special versions that are relevant to grasp force optimization:

1. In a *Second Order Cone Program* (SOCP), the cost function takes the form $h(\vec{x}) = \vec{u}^T\vec{x}$, with $\vec{u}$ a constant vector. The m_c inequality constraints take the form $||A_i\vec{x} + \vec{b}_i|| \leq c_i^T\vec{x} + d_i$ for $i = 1, \ldots, m_c$, where $A_i \in \mathbb{R}^{(n_i-1)\times n}$, $\vec{b} \in \mathbb{R}^{n_i-1}$, $\vec{c}_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$. where the constraint $||A_i\vec{x} + \vec{b}_i|| \leq c_i^T\vec{x} + d_i$ defines a **second-order cone**. Friction constraints, when formulated in the manner of Equation (11.3), are second-order cone constraints.

2. In the *dual-form*, a **Semi-Definite Program** (SDP) takes the form of Definition 6 with the cost function taking the form $h(\vec{x}) = \vec{u}^T\vec{x}$, while the inequality constraints take the form of a linear matrix inequality: $S_0 + \sum_{i=1}^n x_iS_i \succeq 0$, where $S_0, \cdots, S_k$ are positive semi-definite constant symmetric matrices.

There are a variety of grasp optimization criteria that are both physically meaningful and convex. While this book attempts to provide the fundamental principles underlying all well-posed criteria, it is useful to put this formulation in a concrete context.

- **The gentlest grasp criterion.** The potential for the gripper to damage the grasped object is proportional to the magnitudes of the normal forces at the contacts. Hence, the *gentlest grasp* will minimize the worst case normal force applied at any of the active contacts. This goal can be expressed as the following optimization problem.

$$\begin{array}{rl} \min & \{\ \xi\ \} \\ \text{subject to :} & f_n^j \leq \xi; \quad \vec{f} \in C_1 \times \cdots \times C_k; \quad G\vec{f} + \vec{w}_{ext} = 0\ . \end{array}$$

  where $\xi \in \mathbb{R}$ and $\vec{f} \in C_1 \times \cdots \times C_k$ is short hand notation for the friction cone constraint, which may be formulated as a second-order cone constraint (and therefore may be solved using specialized SOCP algorithms), or as a linear matrix inequality constraint (which can be solved using SDP algorithms).

- **The robust grasp criterion.** The minimization of the contact normal forces in the gentlest grasp optimization solution may push the contact forces toward the edge of the friction cones. Such a solution is *non-robust* because it is not resiliant in the face of the realistically uncertain value of the friction coefficient. Alternatively, one would like to minimize the normal force without relying too much on the tangential frictional forces. This can be accomplished by adding a penality term (or *barrier function*) to the cost function in order to bias the contact force vectors toward the center of the friction cone, thereby minimizing the optimal grasp's reliance on friction. A natural barrier function in this case is $log(det(\mathcal{P}(\vec{f})))$, where $\mathcal{P}$ is the symmetric semi-definite matrix in Equation (11.14) modeling the friction cone constraints. The magnitude of this convex barrier function approaches infinity as the contact force vector veers toward the edge of the friction cone (see Exercises). Thus, this barrier function allows for a convex programming formulation:

$$\begin{array}{rl} \min\limits_{\xi} & \{\ \xi - log(det(\mathcal{P}(\vec{f})))\ \} \\ \text{subject to}: & \vec{f}\ \in\ C; \quad f_n^j \leq \xi; \quad G\vec{f} + \vec{w}_{ext} = 0\ . \end{array}$$

- **The smallest sum of normal forces criterion.** Instead of minimizing the worst case normal force, one may alternatively seek to minimize the *sum* of the normal forces. Note that this sum can be expressed as $\vec{u}^T\vec{f}$ where the vector $\vec{u}$ has the form $\vec{u} = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 1\end{pmatrix}^T$ for the example of a hard frictional point contact. Hence, the minimization of the sum of forces takes exactly the form of a Second Order Cone Program (SOCP). Similarly, the sum of squared normal forces, $\sum_{i=1}^k\ f_i^2$, or the squared Euclidean norm of the finger force vector, $\sum_{i=1}^k\ ||f_i||^2$, are both convex functions, and allow for a convex optimization framework.

11.4 Grasp Controllability

This section will analyze a multi-fingered robotic grasp of a rigid body as a control system, and particularly consider the notion of grasp *controllability.* Practically speaking, a grasp is *controllable* if a set of feasible finger forces (which constitute the system controls) can be used to move the grasped object from a starting configuration to an arbitrary desired final configuration. Importantly, when an equilibrium grasp is disturbed by some external action, a feedback control algorithm can restore the object to its original equilibrium state if the grasp is controllable. Not surprisingly, it will be shown that wrench resistability is a sufficient condition for grasp controllability. Interestingly, we will also see that controllability can be achieved without the wrench resistant grasp condition.

Dynamics of a Grasped Object. To model a grasp as a control system, we must develop the dynamical equations which govern the grasped object's motions under the influence of the finger contact forces. Assume that a rigid object $\mathcal{B}$ is grasped by k fingers, $\mathcal{O}_1, \cdots, \mathcal{O}_k$. For simplicity, the fingers are assumed to be massless point finger bodies that apply contact forces $f_1, \ldots, f_k$ to the object at given contact locations $\vec{x}_1, \ldots, \vec{x}_k$. Furthermore, we assume that as long as the fingers apply feasible forces, $f_i\ \in\ C_i$ for all $i = 1, \ldots, k$, then the finger contact points stay fixed on the object's surface, and in fact will move in concert with the object. In the following analysis, a *body fixed* reference frame, B, is located so that its origin lies at the object's center of mass. An inertially fixed *world* reference frame, denoted by W, is located so that it coincides with B when $\mathcal{B}$ lies at its initial equilibrium configuration.

Without loss of generality, assume that the z-axis of the inertial reference frame is antiparallel to the direction of gravity.

In the fixed inertial coordinates, the motion of rigid body $\mathcal{B}$ under the influence of external forces and torques is governed by the *Newton-Euler* dynamical equations:

$$m_{\mathcal{B}} \, \ddot{\vec{p}}_{cm} = \vec{F}_{ext} \tag{11.18}$$

$$\mathcal{I}^W \, \dot{\vec{\omega}}^W + \vec{\omega}^W \times \mathcal{I}^W \, \vec{\omega}^W = \vec{\tau}_{ext} \tag{11.19}$$

where $\vec{F}_{ext}$ and $\vec{\tau}$ are the external force and torque applied to $\mathcal{B}$ (with respect to its center of mass, as described in bases parallel to W), $m_{\mathcal{B}}$ is the mass of $\mathcal{B}$, $\vec{p}_{cm}$ is the location of $\mathcal{B}$'s center of mass with respect to the origin of frame W, $\vec{\omega}^W$ is $\mathcal{B}$'s angular velocity as measured in the interial frame W, and $\mathcal{I}^W$ is $\mathcal{B}$'s inertia tensor in the inertial frame: $\mathcal{I}^W = R_{WB} \, \mathcal{I}^B \, R_{WB}^T$, where $\mathcal{I}^B$ is the constant intertia tensor in the body fixed frame B, and $R_{WB} \in SO(3)$ is the relative orientation of frame B with respect to frame W.

For a rigid body grasp, the external forces on the object will consist of the finger contact forces as well as the force of gravity. Recall that net wrench on the object due to the finger forces is $\mathbf{w}_{ext} = G(q)\vec{f}$, where G is the grasp map and $\vec{f} = \begin{bmatrix} f_1 & f_2 & \cdots & f_k \end{bmatrix}^T$. Substituting the finger contact forces into Equations (11.18) and (11.19), and adding in the gravitational effects, the equations of motion are:

$$\begin{bmatrix} m_{\mathcal{B}} I & 0 \\ 0 & \mathcal{I}^W(q) \end{bmatrix} \begin{bmatrix} \ddot{\vec{p}} \\ \dot{\omega}^W \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\omega}^W \times \mathcal{I}^W(q) \, \vec{\omega}^W \end{bmatrix} = \begin{bmatrix} R_{WB}(q) & 0 \\ 0 & R_{WB}(q) \end{bmatrix} G(q)\vec{f} - m_{\mathcal{B}} g \nabla h(q) \tag{11.20}$$

where $h(q)$ is the height of $\mathcal{B}$'s center of mass relative to a reference plane and g is the gravitational constant.

Equation (11.20) represents a second order nonlinear mechanical system with the general form:

$$\mathfrak{M}(q)\ddot{q} + \mathfrak{B}(q, \dot{q}) + \mathfrak{G}(q) = G^W(q)\vec{f} \tag{11.21}$$

where

$$\mathfrak{M}(q) = \begin{bmatrix} m_{\mathcal{B}} I & 0 \\ 0 & \mathcal{I}^W(q) \end{bmatrix} \qquad \mathfrak{B}(q, \dot{q}) = \begin{bmatrix} 0 \\ \vec{\omega}^W \times \mathcal{I}^W(q) \, \vec{\omega}^W \end{bmatrix} \tag{11.22}$$

$$\mathfrak{G}(q) = m_{\mathcal{B}} \, g \, \nabla h(q) \qquad G^W(q) = \begin{bmatrix} R_{WB}(q) & 0 \\ 0 & R_{WB}(q) \end{bmatrix} G(q) \, . \tag{11.23}$$

The matrix $\mathfrak{M}(q)$ is the *mass matix*, while vectors $\mathfrak{B}(q, \dot{q})$ and $\mathfrak{G}(q)$ respectively represent the Coriolis and gravity forces acting on the system.

Controllability. For control analysis, it is useful to transform system (11.21) to first-order form by introducing the state $z = \begin{pmatrix} q & \dot{q} \end{pmatrix}^T$ and rearranging terms:

$$\dot{z} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} z - \begin{bmatrix} 0 \\ \mathfrak{M}^{-1}(z)(\mathfrak{B}(z) + \mathfrak{G}(z)) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathfrak{M}^{-1}(z) G^W(z) \end{bmatrix} \vec{f} \, . \tag{11.24}$$

Equation (11.24) takes the general form of a nonlinear affine control system:

$$\dot{z} \;=\; a(z) + b(z)\vec{u} \tag{11.25}$$

where $\vec{u} \in I\!R^p$ is the vector of p distinct control inputs, and $z \in I\!R^N$ (where $N = 2m$ for the grasping system under consideration). The control inputs may be restricted: $\vec{u} \in U$, for some $U \subset I\!R^p$.

The system (11.25) is *controllable* if for *any* $z_0, z_f \in I\!R^N$, there exists a time, $T > 0$, and a feasible control input $u : [0,T] \rightarrow U$ such that $z(t_0) = z_0$ and $z(t = T) = z_f$. I.e., the system can be steered in finite time from its initial state to a desired final state using only allowable controls in U. In general, controllability of the type described in this definition can be hard to assess for *all* initial and final configurations. Hence, we will be satisfied with *small time local controllability.*

Definition 7 (Reachable Set). *Let $V \subset I\!R^N$ be an open set. The* reachable set at time *T, $\mathcal{R}^V(z_0, T)$ of control system (11.25) is the set of all states $\{z(T)\}$ such that there exists an admissible control $u : [0,T] \rightarrow U$ that steers the control system (11.25) from $z(0) = z_0$ to $z(T) = z_f$ while satisfying $z(t) \in V$ for all $t \in [0,T]$. More generally, let the set of states reachable up to time T be defined as:*

$$\mathcal{R}^V(x)(z_0, \leq T) \;=\; \bigcup_{\tau \,\in\, [0,T]} \mathcal{R}^V(z_0, \tau)\ .$$

Definition 8 (Small Time Local Controllability, STLC). *The control system (11.25) is* **small-time locally controllable** *at z_0 if $\mathcal{R}^V(z_0, \leq T)$ contains a neighborhood of z_0 for all choices of V and for some $T > 0$.*

In other words, a control system will be small time locally controllable (STLC) at state z_0 if it can be steered in finite time to any nearby state in a neighborhood of z_0 with feasible controls.

Let us return to the grasping situation. If the grasp is wrench resistant, then recall that the grasp map G is a full rank matrix, and the image of the set of feasible fingers forces under the action of G spans the entire wrench space: $G(C) = T^*_{q_0}\mathcal{C} \simeq I\!R^m$. Thus, Equation (11.21) takes the form of a *fully actuated second order mechanical system*:

$$\mathfrak{M}(q)\ddot{q} + \mathfrak{B}(q,\dot{q}) + \mathfrak{G}(q) = \vec{u} \tag{11.26}$$

where the control inputs $\vec{u}$ represent the wrenches that can be generated by the finger contacts, and we know that the wrenches (control inputs) span $T^*_{q_0}\mathcal{C} \simeq I\!R^m$ in the wrench resistant case. We say that system (11.26) is *fully actuated* when $dim(q) = dim(\vec{u})$–i.e., when there are as many distinct control inputs as their are independent configuration variables. It is a well established fact that a second-order fully actuated mechanical system is STLC about an equilibrium configuration.

Proposition 11.4.1 (Controllability of Wrench Resistant Grasps). *Let a rigid object $\mathcal{B}$ be grasped by k rigid finger bodies. If the grasp is wrench resistant, then the grasp is small time locally controllable about its equilibrium configuration.*

Constructing a Feedback Controller. Unfortunately, controllability analysis only determines if grasp *can be* controlled, but does not specify a control feedback structure which will ensure that the controllability condition is exploited. There are several feedback procedures which can realize a practical grasp controller.

A natural choice is *feedback linearization* (known as the *computed torque* method in robotics). If a grasp is wrench resistant, the finger contact forces can be chosen so that

$$G^W(q)\vec{f} \;=\; \mathfrak{M}(q)(\dot{q}_d - K_v\dot{e} - K_p e) + \mathfrak{B}(q,\dot{q}) + \mathfrak{G}(q) \tag{11.27}$$

where q_d is the desired configuration of the grasped object (e.g., the equilibrium configuration), $e = q - q_d$, $\dot{e} - \dot{q} - \dot{q}_d$, and K_v and K_p are positive definite feedback gain matrices. Substituting (11.27) into Equation (11.21) results in the equation

$$\mathfrak{M}(q)(\ddot{e} \;+\; K_v\dot{e} \;+\; K_p e) \;=\; 0\;.$$

Since the mass matrix $\mathfrak{M}(q)$ is positive semi definite, this equation is equivalent to:

$$\ddot{e} \;+\; K_v\dot{e} \;+\; K_p e \;=\; 0\;.$$

Thus, the error dynamics take the form of a 2^{nd} order linear constant coefficient ordinary differential equation. The matrices K_p and K_v can be chosen to implement a desired feedback control behavior (e.g., a specified rise time or overshoot).

Controllable Grasps Need not be Wrench Resistant. While a wrench resistant grasp is a controllable, the converse need not be true. To illustrate this point, we must reanalyze Equation (11.24) in the case when the grasp is not wrench resistant, and therefore it can no longer be considered a fully actuated control system. In general, one must apply non-linear controllability theory to assess the general conditions under which system (11.24) is controllable. However, a linearized analysis will suffice for the current discussion. If the linearized system is *linearly controllable* at the point of linearization, then it is also nonlinearly controllable in a small neighborhood of that point.

Let us linearize (11.24) about the equilibrium point, $z_0 = (q_0, \vec{0})$. Without loss of generality, choose a c-space parametrization so that $q_0 = \vec{0}$. Generally, a non-linear control systems of the form $\dot{z} = p(z,u)$ can be linearized about the operating conditions (z_0,u_0) as follows:

$$p(z,u) = p(z_0,u_0) + \left.\frac{\partial p(z,u)}{\partial z}\right|_{z_0,u_0}(z - z_0) \;+\; \left.\frac{\partial p(z,u)}{\partial u}\right|_{z_0,u_0}(u - u_0)\;. \tag{11.28}$$

Substituting (11.25) and (11.24) into this equation yields the following linearized equations

$$\dot{\tilde{z}} \;=\begin{bmatrix} 0 & I \\ A_c & 0 \end{bmatrix}\tilde{z} + \begin{bmatrix} 0 \\ B_c \end{bmatrix}\tilde{f} \triangleq\; A\;z \;+\; B\;\tilde{f} \tag{11.29}$$

where $\tilde{f}$ is to be understood as the perturbation of the finger contact forces about the equilibrium force values, and $\tilde{z}$ is understood to be the deviation from the equilibrium state.

Note that since we are linearizing about the non-zero equilibrating finger forces, $\tilde{f}$ is not restricted to the friction cone. The matrices A_c and B_c have the form

$$A_c = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix} \qquad B_c = \mathfrak{M}^{-1}(q_0)G^W(q_0)$$

where matrix Q is a 3×3 skew symmetric matrix:

$$Q = \frac{\partial}{\partial \theta} R_{WB}(\theta)\vec{e}_z$$

with θ a parametrization of $SO(3)$, and $\vec{e}_z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.

A linear control system of the form (11.29) is *controllable* if the following *controllability matrix* has full rank

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix} . \tag{11.30}$$

where $A \in I\!\!R^{N \times N}$, and $N = 6$ for planar grasps and $N = 12$ for 3-dimensional grasps.. Substituting (11.29) into (11.30) yields (for a 3-dimensional grasp)

$$\mathfrak{C} = \begin{bmatrix} 0 & B_c & 0 & A_cB_c & 0 & A_c^2B_c & 0 & A_c^3B_c & 0 & A_c^4B_c & 0 & A_c^5B_c \\ B_c & 0 & A_cB_c & 0 & A_c^2B_c & 0 & A_c^3B_c & 0 & A_c^4B_c & 0 & A_c^5B_c & 0 \end{bmatrix} \tag{11.31}$$

The controllability matrix (11.31) is clearly full rank if B_c is full rank. Since matrix $\mathfrak{M}(q)$ is symmetric positive definite (and therefore full rank), the rank of B_c depends solely upon the rank of $G^W(q)$. If the grasp is wrench resistant, then G^W is full rank, and hence the grasp is controllable, confirming Proposition 11.4.1. However, if B_c is not full rank (and hence the grasp is not wrench resistant), the grasp may still be controllable. Inspection of matrix 11.31 shows that a sufficient condition for controllability in this case is

$$rank \begin{bmatrix} B_c & A_cB_c \end{bmatrix} = m. \tag{11.32}$$

Before proceeding to analyze (11.32) in detail, note that if G^W is not full rank (implying that B_c is not full rank), condition (11.32) may still satisfied if the subspace of $I\!\!R^m$ not spanned by G^W is spanned by terms in A_cB_c. Practically speaking, gravitational effects can contribute to grasp controllability, as the force of gravity in effect acts as an additional "virtual" finger force on the object.

To analyze (11.32) in more detail, we must expand the matrices A_c and B_c. Note that:

$$G^W(q_0) = \begin{bmatrix} R_{WB}(\theta_0) & 0 \\ 0 & R_{WB}(\theta_0) \end{bmatrix} G \triangleq \begin{bmatrix} R_{WB}(\theta_0) & 0 \\ 0 & R_{WB}(\theta_0) \end{bmatrix} \begin{bmatrix} \leftarrow & G_F & \rightarrow \\ \cdots & \cdots & \cdots \\ \leftarrow & G_\tau & \rightarrow \end{bmatrix} = \begin{bmatrix} R_{WB}(\theta_0)G_F \\ R_{WB}(\theta_0)G_\tau \end{bmatrix} .$$

I.e., the grasp map G has been divided into sub-matrices G_F and G_τ that respectively govern the net force and net torque on the grasped object. Using this division, the expressions for A_c and B_c in (11.32) can be refined:

$$\begin{bmatrix} B_c & A_cB_c \end{bmatrix} = \begin{bmatrix} m_{\mathcal{B}}^{-1} R_{WB}(\theta_0)G_f & QI^W(\theta_0)R_{WB}(\theta_0)G_\tau \\ I^W(\theta_0)R_{WB}(\theta_0)G_\tau & 0 \end{bmatrix} . \tag{11.33}$$

Because R_{WB} and I^W are full rank matrices, grasp controllability requires that G_τ have rank 3, and that the matrix $\left[m_{\mathcal{B}}^{-1} R_{WB}(\theta_0) G_f \quad Q I^W(\theta_0) R_{WB}(\theta_0) G_\tau\right]$ have rank 3. The full rank condition will be satisified except for the following cases:

- the grasp consists of two hard point contacts whose connecting line passes through the center of mass
- the grasp consists of three or more hard point contacts which all lie on a line passing through the center of mass
- the grasp consists of three or more hard point contacts, and the vectors from the center of mass to each contact point is orthogonal to the direction of gravity.

11.5 Exercises

Problem #1: Using Equation (11.11), show that the hard point contact Coulomb friction contact model of Equation (11.3) is equivalent to the Linear Matrix Inequality of Equations (11.4) and (11.5).

Problem #2: A "soft" contact model between a compliant finger tip and a compliant or rigid grasped object can be approximated within the rigid body point contact framework. Such a deformable contact can support not only tangential contact forces due to Coulomb friction, but also a torque about the normal to the contact. One particular model that captures such a finger contact is the *elliptical approximation of the soft contact* model. This model is defined by the following contact force constraints

$$f_n \geq 0; \quad \frac{1}{\mu^2}(f_x^2 + f_y^2) + \frac{1}{\gamma_i^2}\tau^2 \leq f_n^2. \tag{11.34}$$

where f_n is the contact normal force, f_x and f_y are the tangental contact forces, τ_n is the torque about the contact normal. In addition to the standard Coulomb friction coefficient, μ, the *Coulomb torsional coefficient*, γ, describes the limit on the torsional forces that can be supported by the contact.

Show that the constraints in Equation (11.34) are equivalent to the following matrix constraint:

$$P(\vec{f_i}) = \begin{bmatrix} f_n & 0 & 0 & \mu^{-1} f_x \\ 0 & f_n & 0 & \mu^{-1} f_y \\ 0 & 0 & f_n & \gamma^{-1}\tau_n \\ \mu^{-1} f_x & \mu^{-1} f_y & \gamma^{-1}\tau_n & f_n \end{bmatrix} \succeq 0\ . \tag{11.35}$$

Problem #3: Provide the missing proof of Proposition 11.1.1, which states that the Grasp Map associated to a frictional hard finger contact grasp is full rank unless:

- all planar fingers contacts lie at the same point

- three or more 3-dimensional finger contact points lie on the same line.

Problem #4: Prove that Equation (11.17) equivalently formulates the LMI feasibility test.

Problem #5: For the case of frictional point contact, show that the barrier function $log(det(\mathcal{P}(\vec{f})))$ is convex, and that its magnitude tends to infinity as the contact force approaches the friction cone boundary.

Problem #6: Prove the following proposition.

Proposition 11.5.1. *Let a rigid object $\mathcal{B}$ be grasped by k rigid frictional finger bodies $\mathcal{O}_1, \cdots, \mathcal{O}_k$ at a configuration q_0. Let $\vec{f}_{int} \in Null(G) \cap int(C)$ be an internal force, where G is the grasp map and $C = C_1 \times \cdots \times C_k$ is the set of friction cones. Let $\vec{f}_{net} = \vec{f}_p + s\vec{f}_{int}$, where $\vec{f}_p$ is a particular solution to $G\vec{f} + \mathbf{w}_{ext} = 0, and where\ s \in I\!R^+$. The magnitude of the normal force at each active contact increases in proportion to the squeezing effort s.*

11.6 Bibliographical Notes

As discussed in the bibliographical notes of Chapter 6, the notion of wrench resistability (or force closure) has a long history. The key results in this chapter were developed in the 1990's. Theorem 1 and its proof is due to Murray, Li, and Sastry [3]. The first formal definition of grasp analysis in terms of convex programs dates to Kerr and Roth [8] who formulated the force closure feasibility problem using approximated friction cone constraints in order to get a Linear Programming formulation. Using similar approximations, and Cheng and Orin [7] developed a Linear Programming formulation of the grasp optimization problem. The reformulation of the friction cone constraints as matrix inequalities started with Martin Buss and coworkers [5, 6], who were interested in the grasp force optimization problem. Han, Li and Trinkle [2] were the first to recognize that problems of force closure could be addressed within the Linear Matrix Inequality framework. The formulation of the grasp optimization problem as a second order cone programming problem is due to Lobo et. al [4], who were themselves motivated by Buss' earlier work. Han, Li and Trinkle extended their formulation to include more complex cost functions and barrier functions. Brook, Shoham, and Dayan [9] were the first to develop the relationship between grasp controllability and wrench resistibility.

11.7 Solutions to the Exercises

Solution #1: The terms in Equation (11.11) can be equated to the terms of (11.3):

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \vec{b} = \vec{0} \qquad \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad d = 0 \ .$$

Substitution of these terms in 11.11 yields the formula (11.4).

Solution #2: The determinant of the matrix in Equation (11.35) is:

$$det \begin{bmatrix} f_n & 0 & 0 & \mu^{-1}f_x \\ 0 & f_n & 0 & \mu^{-1}f_y \\ 0 & 0 & f_n & \gamma^{-1}\tau_n \\ \mu^{-1}f_x & \mu^{-1}f_y & \gamma^{-1}\tau_n & f_n \end{bmatrix} = f_n^2 \left[f_n^2 - \frac{\tau_n^2}{\gamma^2} - \frac{1}{\mu^2}(f_x^2 + f_y^2) \right] . \tag{11.36}$$

For this determinant to be positive semidefinite,

$$f_n^2 \geq 0 \qquad \text{and} \qquad \frac{\tau_n^2}{\gamma^2} + \frac{1}{\mu^2}(f_x^2 + f_y^2) \leq f_n^2,$$

which accurately recovers the constraint between the normal force and the tangential and torsional contact forces. The eignenvalues of matrix (11.36) are the roots of the characteristic equation:

$$(\lambda - f_n)^2(\lambda^2 - 2\lambda f_n + (f_n^2 - \psi^2))$$

where $\psi^2 = \tau^2/\gamma^2 + (f_x^2 + f_y^2)/\mu^2$. For the eigenvalues to be positive semi-definite, $f_n \geq 0$, which completes the verification.

Solution #3: To show that a k-fingered planar grasp involving 2 or more non-coincident frictional point contacts has a full rank grasp map, let z_i and z_j denote the locations of the contacts of fingers i and j, where $i, j \in 1, \ldots, k$. Without loss of generality, assign a fixed reference frame with origin at point z_i and with the unit x-axis pointing from z_i to z_j. In this coordinate system, the portion of the grasp map associate with these two contacts is:

$$\begin{bmatrix} G_i G_j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & x_j \end{bmatrix} \tag{11.37}$$

where G_i and G_u are the portions of the grasp map associated with contacts i and j. Clearly, this matrix is full rank as long as $x_i \neq x_j$, i.e., as long as the contacts are not concident. Adding additional contacts into the grasp map analysis will not change the rank of G.

The proof for a k-fingered frictional spatial grasp is similar. Let p_i and Let p_j be the locations of two point contacts. Assigned a reference frame with origin at contact point p_i, and with x-basis vector pointing from p_i to p_j. Orient the y- and z- basis vectors of the reference frame to satisfy the right hand rule. The basis vectors of the j^{th} finger's contact frame do not necessarily align with this reference frame. However, since the contact forces allowed under the hard frictional point contact model span all three Cartesian directions, without loss of generality one can assign a basis for these forces which aligns with the reference frame. Hence, with these choices of bases, the portion of the grasp map associated to these two contacts is:

$$G = \begin{bmatrix} \cdots & G_i & \cdots & G_j & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -x_j & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & x_j & 0 & \cdots \end{bmatrix} \tag{11.38}$$

Thus, the grasp map has at most rank=5. Hence, for the frictional hard contact model, two contact points are not sufficient to realize wrench resistability, as no torques can be generated about the axis passing through the two contact points. Consider an m^{th} finger placed at $\begin{bmatrix} x_m & y_m & z_m \end{bmatrix}^T$. Again, the basis vectors of the contact forces can be aligned with the bases of the reference frame. The grasp map is:

$$\begin{aligned} G &= \begin{bmatrix} \cdots & G_i & \cdots & G_j & \cdots & G_m & \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & z_m & y_m & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -x_j & \cdots & z_m & 0 & -x_m & \cdots \\ \cdots & 0 & 0 & 0 & \cdots & 0 & x_j & 0 & \cdots & y_m & -x_m & 0 & \cdots \end{bmatrix} \end{aligned}$$

If the m^{th} contact lies along the line connecting contacts i and j, then $y_m = z_m = 0$, and the grasp map is not full rank. Otherwise, the grasp map is full rank.

Solution #4: In order to prove that the condition $Q(\vec{x}) \succeq 0$ is the same as the condition

$$\exists\, \eta \leq 0 \quad \text{such that} \quad Q(\vec{x}) \;+\; \eta I \;\succeq 0\,. \tag{11.39}$$

note that Equation (11.39) is only true if there exists a scalar η such that $\eta \leq -\lambda_{min}(Q)$, where $\lambda_{min}(Q)$ is the smallest eigenvalue of Q. However, in order for η to be a non-positive number, it must be true that $\lambda_{min}(Q) \geq 0$. Hence, eigenvalues of Q must be non-negative, which implies that Q is a positive-semi-definite matrix.

Solution #5:

For the hard frictional point contact model constraint matrix in Equation (11.4), the optimization barrier function: $\Phi(\vec{f}) = log(det(\mathcal{P}(\vec{f})))$, takes the form:

$$\Phi(\vec{f}) = \log(det(\mathcal{P}(\vec{f}))) = log\left[\mu f_n(\mu^2 f_n^2 - (f_x^2 + f_y^2))\right]. \tag{11.40}$$

where $\mathcal{P}(\vec{f}) \succeq 0$. As the contact force approaches the friction cone boundary, $(f_x^2 + f_y^2) \to \mu^2 f_n^2$. Hence

$$\lim_{(f_x^2+f_y^2)\to\mu^2 f_n^2} \Phi(\vec{f}) = \lim_{z\to 0^+} log(z) = -\infty\,.$$

From Solution #1 we also know that the constraint $\mathcal{P} \succeq 0$ implies that $f_n \geq 0$, and hence the term $\left[\mu^2 f_n^2 - (f_x^2 + f_y^2)\right]$ in Equation (11.40) is a paraboloid, which is a convex function. The monotonicity of the $log(\cdot)$ function does not change the convex nature of the function. For a more general proof of the convexity of $\Phi(\vec{f})$ which holds for other contact models, see [2].

Solution #6: Proposition 11.5.1 can be proved as follows.

For a hard point contact with friction, the i^{th} finger contact force can be expressed as $\vec{f_i} = f_t^i \vec{t_i} + f_n^i \vec{n}_i$ where $\vec{t_i}$ is the unit vector pointing in the direction of the tangential frictional force and $\vec{n}_i$ is the inward pointing unit normal vector. The i^{th} finger's net contact force can be expressed as $\vec{f}_{i,net} = \vec{f}_{i,p} + s\vec{f}_{i,int}$, where $\vec{f}_{i,p}$ is a particular solution to balance the external wrench. The internal contact force can be expressed as $\vec{f}_{i,int} = \xi_{i,int}\vec{t}_{i,int} + \eta_{i,int}\vec{n}_i$, where the constants $\xi_{i,int}$ and $\eta_{i,int}$ denote the components of the internal force along the i^{th} finger's tangential and normal directions, and $\eta_{i,int} > 0$ for every active contact. Since $\vec{f}_{i,net} = \vec{f}_{i,p} + s(\xi_{i,int}\vec{t}_{i,int} + \eta_{i,int}\vec{n}_i)$, as s increases, the magnitude of the normal force is $s\eta_{i,int}$, which varies linearly with increasing value of s.

Bibliography

[1] M.S. Lobo, L. Vanderberghe, S. Boyd, and H. Lebret, "Applications of Second-order Cone Programming," *Linear Algebra and Its Applications*, vol. 284, pp. 193-228, 1998.

[2] L. Han, J. C. Trinkle, and Z.X. Li, "Grasp Analysis as Linear Matrix Inequality Problems," *IEEE Trans. Robotics and Automation*, vol. 16, no. 6, pp. 663-674, Dec. 2000.

[3] R.M. Murray, Z. Li, S.S. Sastry, A Mathematical Introduction to Robotics, CRC Press, 1994.

[4] M.S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of Second-Order Cone Programming," *Linear Algebra and its Applications*, vol. 284, pp. 193-228, 1998.

[5] M. Buss, H. Hashimoto, and J.B. Moore, "Recursive Algorithms for Real-Time Grasping Force Optimization," *IEEE Trans. Robotics and Automation*, vol 12, no. 3, pp406-418, 1996.

[6] M. Buss, L. Faybusovich, and J.B. Moore, "Dikin-type algorithms for dextrous grasping force optimization," *Int. J. Robotics Research*, vol. 17, no. 8, pp. 831-839, 1998.

[7] F.-T. Cheng and D.E. Orin, "Efficient Algorithms for Optimal Force Distribution–the compact dual LP method, *IEEE Trans. Robotics and Automation*, vol. 6, no. 2, pp-178-187, 1990.

[8] J. Kerr and B. Roth, "Analysis of Multifingered Hands," *Int. J. Robotics Research*, vol. 4, no. 4, pp. 3-17, 1986.

[9] N. Brook, M. Shoham, and J. Dayan, "Controllability of Grasps and Manipulations in Multi-Fingered Hands, *IEEE. Trans. on Robotics and Automation*, vol. 14, no. 1, pp. 185-192, Feb. 1998.