Chapter 2

The Configuration Space of a Rigid Body

The basic problem to be considered in this chapter consists of a freely moving rigid body $\mathcal{B}$ surrounded by stationary rigid bodies $\mathcal{O}_1 \dots \mathcal{O}_k$. The stationary bodies represent fingertips, fixturing elements, or terrain segments supporting $\mathcal{B}$ against gravity. The body $\mathcal{B}$ represents the object begin grasped, a workpiece, or the rigidified multi-legged vehicle. This chapter introduces the notion of the rigid-body configuration space, or *c-space*, which is essential for analyzing the mobility and stability of $\mathcal{B}$ with respect to its surrounding bodies. The chapter begins with a parametrization of $\mathcal{B}$'s c-space in terms of hybrid coordinates. Configuration space obstacles (c-obstacles) are then introduced, and several of their properties are described. The chapter proceeds to describe the first and second-order geometry of the c-space obstacles, as this geometry plays a key role in subsequent chapters. Finally, the notion of generalized forces or wrenches is introduced in the context of configuration space. Rigid-Body

2.1 The Notion of Configuration Space

The points of the rigid body $\mathcal{B}$ retain their relative distance as the body moves in the environment, and $\mathcal{B}$'s *configuration* specifies the stationary state of the object in the environment. Equivalently, the position of each of $\mathcal{B}$'s constituent points can be determined from its configuration. The specification of $\mathcal{B}$'s configuration requires a selection of two frames, depicted in Figure 2.1. The first is a fixed *world frame*, denoted $\mathcal{F}_W$, which establishes a coordinate system for the environment, or *workspace*, in which $\mathcal{B}$ moves. We assume that workspace is modeled as an n-dimensional Euclidean space, $\mathbb{R}^n$, where $n = 2$ or 3. The second is a *body frame*, denoted $\mathcal{F}_B$, which is rigidly attached to $\mathcal{B}$. The configuration of $\mathcal{B}$ can be specified by a vector $d \in \mathbb{R}^n$ describing the position of $\mathcal{F}_B$'s origin with respect to the origin of $\mathcal{F}_W$, and an *rotation matrix*, $R \in \mathbb{R}^{n \times n}$, whose columns describe the relative orientation of the axes of $\mathcal{F}_B$ with respect to those of $\mathcal{F}_W$. The collection of $n \times n$ orientation matrices forms a group under matrix multiplication, termed the *special orthogonal group*, and denoted by the symbol $SO(n)$.

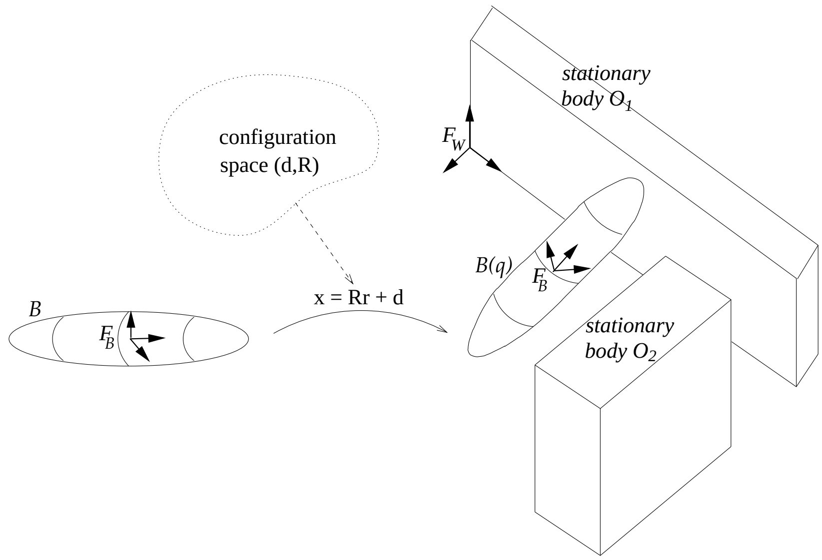

Figure 2.1: The physical geometry underlying the c-space representation of a 3D body $\mathcal{B}$. Think of $\mathcal{B}$'s configuration as a *placement* of $\mathcal{B}$ in its workspace.

Characterization of $\boldsymbol{SO(n)}$. The special orthogonal group of $n \times n$ orientation matrices is given by

$$SO(n) = \left\{ R \in I\!R^{n \times n} : R^T R = I \quad \text{and} \quad \det(R) = 1 \right\},$$

where I is an $n \times n$ identity matrix.

The characterization of $SO(n)$ provides two important insights. First, every rotation matrix acts on vectors $v \in I\!R^n$ so as to preserve their length, since $\|Rv\| = (v^T R^T R v^T)^{1/2} = \|v\|$. Second, $SO(n)$ is a compact smooth manifold of dimension $\frac{1}{2}n(n-1)$ in the space $I\!R^{n \times n}$. In particular, $SO(2)$ is a one-dimensional loop in the space of 2×2 matrices, while $SO(3)$ is a compact three-dimensional manifold in the space of 3×3 matrices.

Definition 1 (Configuration Space)**.** *The configuration space of $\mathcal{B}$, denoted $\mathcal{C}$, is the smooth manifold $\mathcal{C} = I\!R^n \times SO(n)$, consisting of pairs (d, R) such that $d \in I\!R^n$ and $R \in SO(n)$.*

The dimension of $\mathcal{C}$ is the sum: $m = n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$, giving $m = 3$ when $\mathcal{B}$ is a 2-dimensional (2D) body and $m = 6$ when $\mathcal{B}$ is a 3-dimensional (3D) body. We now introduce a parametrization of $\mathcal{C}$ in terms of *hybrid coordinates* [7]. This parametrization allows us to locally represente $\mathcal{C}$ as a Euclidean space $I\!R^m$, with some periodicity rules for the coordinates representing the orientation matrices.

We first introduce coordinates for $SO(n)$. The group $SO(n)$ is an important instance of a *Lie group*.[1] A standard means for parametrizing Lie groups is via *exponential coordinates*:

$$R(\boldsymbol{\theta}) \;=\; e^{[\boldsymbol{\theta}\times]}$$

where the matrix exponential can be formally defined via the series: $\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots$, and where $[\boldsymbol{\theta}\times]$ is a skew-symmetric matrix[2].

[1] Lie groups are matrix groups possessing a smooth manifold structure.

[2] These skew-symmetric matrices form the *Lie Algebra* of the Lie group.

while the exponential coordinates for $SO(3)$ are a vector $\boldsymbol{\theta} \in I\!R^3$ (since $SO(3)$ is a three-dimensional manifold). The exponential coordinates are constructed in two stages.

Exponential Coordinates for $\boldsymbol{SO(n)}$. The exponential coordinate for $SO(2)$ is a scalar θ (since $SO(2)$ is a one-dimensional manifold). The skew symmetric matrix in the matrix exponential representation of $SO(2)$ is the 2×2 matrix $[\theta\times] = \theta J$ where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Consequently, the $2{\times}2$ orientation rotation matrices are globally parametrized by the formula

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in I\!R,$$

where θ is the relative orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$, measured using the right-hand-rule (which measures angles in the counterclockwise direction around the upward-pointing normal to the plane).

For 3×3 rotation matrices in $SO(3)$, the skew symmetric matrix $[\boldsymbol{\theta}\times]$ has a physical interpretation as a *cross-product* matrix: $[\boldsymbol{\theta}\times]\vec{v} = \boldsymbol{\theta} \times \vec{v}$ for any vector $\vec{v} \in I\!R^3$. The direction of the vector $\boldsymbol{\theta}$ physically corresponds to the *axis of rotation*, and the norm of the vector, $||\boldsymbol{\theta}||$, corresponds to the angle of rotation[3] about the axis of rotation. For $SO(3)$, it can be shown that the matrix exponential formula reduces to *Rodriguez' Formula*:

$$R(\boldsymbol{\theta}) = I + \sin(\|\boldsymbol{\theta}\|)[\hat{\boldsymbol{\theta}}\times] + \big(1 - \cos(\|\boldsymbol{\theta}\|)\big)[\hat{\boldsymbol{\theta}}\times]^2 \quad \boldsymbol{\theta} \in I\!R^3,$$

where I is a 3×3 identity matrix and $[\hat{\boldsymbol{\theta}}\times]$ is the cross-product matrix of $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/\|\boldsymbol{\theta}\|$. In Rodrigez' formula $\hat{\boldsymbol{\theta}}$ and $\|\boldsymbol{\theta}\|$ are the axis and angle of rotation of $R(\boldsymbol{\theta})$, measured according to the right-hand rule.

The parametrization of $SO(2)$ is periodic in 2π, with each 2π interval parametrizing the entire $SO(2)$. The parametrization of $SO(3)$ in terms of $\boldsymbol{\theta}$ satisfies the following periodicity rule. The origin of $\boldsymbol{\theta}$-space is mapped by $R(\boldsymbol{\theta})$ to the identity matrix I. Similarly, all concentric spheres of radius $\|\boldsymbol{\theta}\| = 2\pi, 4\pi, \ldots$ are mapped to I. Each pair of antipodal points on the sphere of radius $\|\boldsymbol{\theta}\| = \pi$ is mapped to the same matrix R, since $R(\pi\hat{\boldsymbol{\theta}}) = R(-\pi\hat{\boldsymbol{\theta}})$ for all $\hat{\boldsymbol{\theta}}$. Similarly, antipodal points on the spheres of radius $\|\boldsymbol{\theta}\| = 3\pi, 5\pi, \ldots$ are identified. Consider now a path in $\boldsymbol{\theta}$-space from the origin to the sphere of radius π along a fixed direction $\hat{\boldsymbol{\theta}}$. This path represents a rotation of $\mathcal{B}$ about $\hat{\boldsymbol{\theta}}$ by an angle which increases from zero to π. Moving next to the antipodal point $-\pi\hat{\boldsymbol{\theta}}$, rotation of $\mathcal{B}$ from π to 2π continues on a path which moves along $-\hat{\boldsymbol{\theta}}$ back to the origin. Since $\hat{\boldsymbol{\theta}}$ can have any direction, the entire manifold $SO(3)$ is parametrized by the ball with center at the origin and radius π, with antipodal points on its bounding sphere identified.

Definition 2 (Hybrid Coordinates)**.** *When $\mathcal{B}$ is a 2D body the* hybrid coordinates *for its c-space are $q = (d, \theta) \in I\!R^2 \times I\!R$. When $\mathcal{B}$ is a 3D body, the* hybrid coordinates[4] *for its c-space are $q = (d, \boldsymbol{\theta}) \in I\!R^3 \times I\!R^3$.*

[3] *Euler's Theorem* states that every rigid body rotation corresponds is equivalent to a rotation about a fixed axis.

[4] Formally, the hybrid coordinates are $I\!R^n \times \mathfrak{se}(n)$, where $\mathfrak{se}(n)$ is the Lie algebra of $SO(n)$. However, $\mathfrak{se}(n)$ is isomorphic to $I\!R^n$, and so $I\!R^n$ is used for simplicity

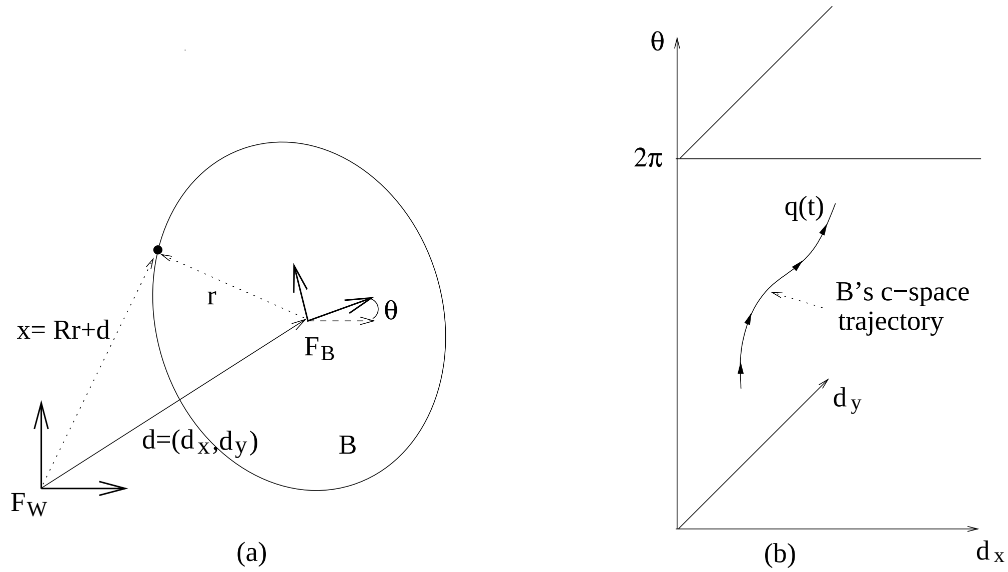

Figure 2.2: (a) Hybrid coordinates $q = (d_x, d_y, \theta)$ for $\mathcal{B}$'s c-space. (b) A c-space trajectory representing $\mathcal{B}$'s physical motion.

When $\mathcal{B}$ is a 2D body its c-space is simply $I\!R^3$ in hybrid coordinates, partitioned into 2π layers along the θ axis (see Figure 2.2). Each 2π layer provides a full parametrization of c-space. Hence a path $q(t)$ can freely move between layers, or it can remain in a particular layer by wrapping through its bounding planes. When $\mathcal{B}$ is a 3D body its c-space is simply $I\!R^6$ in hybrid coordinates, with the $\boldsymbol{\theta}$ coordinates partitioned into a central ball and concentric shells each having a radius/thickness of π. Here, too, a path $q(t)$ can freely move between neighboring shells, or it can remain in the inner ball by wrapping through antipodal points on its bounding sphere.

To summarize, c-space allows us to model the physical motions of $\mathcal{B}$ as trajectories, $q(t)$, of a point in $I\!R^m$, where $m = 3$ or 6. Before we proceed to fill this space with forbidden regions representing the stationary bodies, let us review the notion of rigid-body transformation.

The rigid-body transformation. As $\mathcal{B}$ moves along a c-space trajectory $q(t)$, the position of its points with respect to the world frame $\mathcal{F}_W$ is specified as follows. Let b denote points of $\mathcal{B}$ expressed in its body frame $\mathcal{F}_B$, and let x denote points expressed in $\mathcal{F}_W$ (Figure 2.2(a)). The *rigid-body transformation*, denoted $X(q, b)$, gives the world position of $\mathcal{B}$'s points at a configuration q,

$$x = X(q,b) \triangleq \begin{cases} R(\theta)b + d & q = (d,\theta) \in I\!R^3,\ b \in \mathcal{B} \quad \text{(2D case)} \\ R(\boldsymbol{\theta})b + d & q = (d,\boldsymbol{\theta}) \in I\!R^6,\ b \in \mathcal{B} \quad \text{(3D case).} \end{cases}$$

The notation $X_b(q)$ will specify the rigid-body transformation such that the point $b \in \mathcal{B}$ is held fixed. In this case $X_b(q)$ gives the world position of the fixed point b as a function of q.

Figure 2.3: The c-obstacle induced by a stationary disc, shown for two choices of $\mathcal{F}_B$'s origin: (a) at the ellipse's center, and (b) at the ellipse's tip.

2.2 Configuration Space Obstacles

From the perspective of $\mathcal{B}$, the rigid stationary bodies $\mathcal{O}_1 \ldots \mathcal{O}_k$ form obstacles which constrain its possible motions. Since it is physically impossible for two different rigid bodies to occupy the same space, the stationary bodies induce forbidden regions in $\mathcal{B}$'s c-space, called *c-obstacles*. Let $\mathcal{B}(q)$ denote the set of physical points occupied by $\mathcal{B}$ when it is at a configuration q, and let $\mathcal{O}$ be one of the stationary bodies. The *c-obstacle* induced by $\mathcal{O}$, denoted $\mathcal{CO}$, is the set of configurations q at which $\mathcal{B}(q)$ intersects $\mathcal{O}$,

$$\mathcal{CO} \stackrel{\triangle}{=} \{q \in I\!R^m : \mathcal{B}(q) \cap \mathcal{O} \neq \emptyset\} \quad \text{where } m = 3 \text{ or } 6.$$

When $\mathcal{B}$ is an n-dimensional body, the c-obstacle $\mathcal{CO}$ is an m-dimensional set in the ambient c-space $I\!R^m$, even when $\mathcal{O}$ is a point obstacle. The boundary of $\mathcal{CO}$ is an $(m-1)$-dimensional set, consisting of configurations at which $\mathcal{B}$ touches $\mathcal{O}$ from the outside. A curve on $\mathcal{CO}$'s boundary represents a motion of $\mathcal{B}$ which maintains continuous contact with $\mathcal{O}$. In planar environments one can conceptually construct the boundary of $\mathcal{CO}$ as follows. First one fixes the orientation of $\mathcal{B}$ to a particular orientation θ. Then one moves $\mathcal{B}$ along the perimeter of $\mathcal{O}$ with this fixed orientation, making sure that $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$. The trace of $\mathcal{B}$'s origin during this circumnavigation forms a closed curve which is precisely the boundary of the fixed-θ slice of $\mathcal{CO}$. When this process is repeated for all θ, the resulting stack of loops forms the c-obstacle boundary.

Example 1. Figure 2.3(a) shows an ellipse $\mathcal{B}$ moving in a planar environment populated by a stationary disc $\mathcal{O}$. The c-obstacle induced by $\mathcal{O}$ is depicted in Figure 2.3(b) for two choices of $\mathcal{F}_B$'s origin, at the ellipse's center and at the tip of its major axis. While the two c-obstacles differ in their geometric shape (i.e. surface normal and curvature), they are topologically equivalent. This observation holds true under any choice of $\mathcal{F}_W$ and $\mathcal{F}_B$.

The c-obstacle distance function. An analytic description of the c-obstacle can be constructed as follows. Let $\mathsf{dst}(x, \mathcal{O})$ denote the minimal distance of a point x from a fixed set $\mathcal{O}$, given by $\mathsf{dst}(x, \mathcal{O}) = \min_{y \in \mathcal{O}}\{\|x - y\|\}$. The minimal distance between $\mathcal{B}(q)$ and $\mathcal{O}$, denoted $d(q)$, is defined by

$$d(q) \stackrel{\triangle}{=} \min_{x \in \mathcal{B}(q)} \{\mathsf{dst}(x, \mathcal{O})\} = \min_{b \in \mathcal{B}} \left\{\mathsf{dst}\big(X(q, b), \mathcal{O}\big)\right\},$$

where $x = X(q, b)$ is the rigid-body transformation of the point $b \in \mathcal{B}$ when $\mathcal{B}$ lies at configuration q. Note that $d(q)$ is strictly positive outside $\mathcal{CO}$ and is identically zero inside $\mathcal{CO}$. Hence the c-obstacle $\mathcal{CO}$ is described by the inequality,

$$\mathcal{CO} = \{q \in I\!R^m : d(q) \leq 0\}.$$

One can equivalently write $\mathcal{CO} = \{q \in I\!R^m : d(q) = 0\}$, but the above formulation anticipates later chapters where c-space is used to analyze the motions of a quasi-rigid body.

A detailed discussion of the c-obstacles can be found in textbooks dedicated to robot motion planning [1, 2, 4, 5]. The following list summarizes some of their key properties [5].

1. **Compactness and connectivity propagate.** When $\mathcal{B}$ is compact and path connected, any compact and path connected obstacle $\mathcal{O}$ induces a compact and path connected c-obstacle $\mathcal{CO}$.

2. **Union propagates.** When an obstacle $\mathcal{O}$ is a union of two sets, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, its c-obstacle is a union of the c-obstacles corresponding to the individual sets, $\mathcal{CO} = \mathcal{CO}_1 \cup \mathcal{CO}_2$.

3. **Convexity propagates.** Recall that a set $\mathcal{S} \subseteq I\!R^n$ is *convex* if every pair of points in $\mathcal{S}$ can be connected by a line segment lying wholly in $\mathcal{S}$. When $\mathcal{O}$ and $\mathcal{B}$ are convex bodies, each fixed-orientation slice of $\mathcal{CO}$ is a convex set.

4. **Polygonality propagates.** When $\mathcal{B}$ and $\mathcal{O}$ are polygonal bodies, each fixed-orientation slice of $\mathcal{CO}$ is a two-dimensional polygonal set. When $\mathcal{B}$ and $\mathcal{O}$ are polyhedral bodies, each fixed-orientation slice of $\mathcal{CO}$ is a three-dimensional polyhedral set.

A popular method for computing the explicit shape of the c-obstacles for planar bodies can be summarized as follows. The method assumes that $\mathcal{B}$ and $\mathcal{O}$ are convex polygons. In this case each fixed-θ slice of $\mathcal{CO}$, denoted $\mathcal{CO}|_\theta$, is also a convex polygon. The vertices of $\mathcal{CO}|_\theta$ correspond to configurations at which a vertex of $\mathcal{B}$ (having a fixed orientation θ) touches a vertex of $\mathcal{O}$, such that the bodies' interiors are disjoint. The vertices on the boundary of $\mathcal{CO}|_\theta$ can be computed by a simple algorithm which merges the vertices of $\mathcal{B}$ and $\mathcal{O}$ on a common unit circle [2, 4].

When $\mathcal{B}$ is a smooth convex body and $\mathcal{O}$ is a disc, one can explicitly parametrize the boundary of $\mathcal{CO}$ as follows. First note that as $\mathcal{B}$ traces the perimeter of $\mathcal{O}$ with a fixed orientation, the contact point monotonically traces the entire perimeter of $\mathcal{B}$. Also note that having $\mathcal{B}$ trace with a fixed orientation the perimeter of $\mathcal{O}$ in $\mathcal{F}_W$ is equivalent to having $\mathcal{O}$ trace the perimeter of the stationary $\mathcal{B}$ in $\mathcal{F}_B$. Based on these observations, let $\beta(s)$ for $s \in I\!R$ be a parametrization of $\mathcal{B}$'s perimeter in $\mathcal{F}_B$, such that the tangent $\beta'(s)$ is a unit vector. Let $J\beta'(s)$ be the unit outward normal to $\mathcal{B}$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let r be the radius of disc $\mathcal{O}$, and let x_0 be the position of its center in $\mathcal{F}_W$. Then during a motion of $\mathcal{O}$ along $\mathcal{B}$'s perimeter, the curve traced by $\mathcal{O}$'s center in $\mathcal{F}_B$ is: $\beta(s) + rJ\beta'(s)$ for $s \in I\!R$. Based on a simple calculation (see Exercise 8), the curve traced by $\mathcal{B}$'s origin in $\mathcal{F}_W$ is: $d(s,\theta) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big)$, where $R(\theta)$ is $\mathcal{B}$'s fixed orientation matrix. When θ varies freely in $I\!R$, the function $\varphi(s,\theta) = \big(d(s,\theta),\theta\big) : I\!R^2 \to I\!R^3$ provides a parametrization of $\mathcal{CO}$'s boundary in term of s and θ. The c-obstacles depicted in Figure 2.3 were generated using this technique.

Example: `Obtain the c-obstacle parametrization for an ellipse obstacle, described by` $(x - x_0)^T P(x - x_0) \le 1$ `where` $P > 0$. `At the contact point` $x(s)$: $P(x(s) - x_0) =$

[5]The term "propagate in this list implies that the property in the n-dimensional Euclidean workspace propagates, or is conserved, under the mapping to configuration space.n

$-\lambda R(\theta_0)J\beta'(s)$ for some $\lambda > 0$. Multiplying both sides by $P^{-1/2}$ gives: $P^{1/2}(x(s)-x_0) = -\lambda P^{-1/2}R(\theta_0)J\beta'(s)$. Taking the norm of both sides gives:

$$1 = (x(s)-x_0)^T P(x(s)-x_0) = \lambda\|P^{-1/2}R(\theta_0)J\beta'(s)\| \Rightarrow \lambda(s) = \frac{1}{\|P^{-1/2}R(\theta_0)J\beta'(s)\|}.$$

Substituting for $\lambda(s)$ in the contact-normals equation gives

$$P(x(s)-x_0) = -\lambda(s)R(\theta_0)J\beta'(s) \Rightarrow x(s) = x_0 - \lambda(s)P^{-1}R(\theta_0)J\beta'(s).$$

On the other hand, $x(s) = R(\theta_0)b(s)+d(s)$. Substituting for $x(s)$ and solving for $d(s)$ gives

$$\begin{aligned} d(s,\theta) &= x(s) - R(\theta)b(s) = x_0 - \lambda(s)P^{-1}R(\theta)J\beta'(s) - R(\theta)b(s) \\ &= x_0 - R(\theta)\left(b(s) + \lambda(s)P^{-1}J\beta'(s)\right), \end{aligned}$$

where θ is now freely varying in $I\!R$. Note that $b(s)+\lambda(s)P^{-1}J\beta'(s)$ is the curve traced by $\mathcal{O}$'s center in $\mathcal{F}_B$ (what about $\mathcal{O}$'s orientation?)

The c-obstacle boundary is generally a piecewise smooth surface in the 2D case. For instance, when $\mathcal{B}$ is a convex polygon and $\mathcal{O}$ is a disc, $\mathcal{CO}$'s boundary consists of two types of smooth two-dimensional "patches" meeting along one-dimensional curves. An edge-patch generated by an edge of $\mathcal{B}$ sliding on $\mathcal{O}$, and a vertex-patch generated by a vertex of $\mathcal{B}$ sliding on $\mathcal{O}$. The boundary of $\mathcal{CO}$ is locally smooth at any configuration at which $\mathcal{B}$ touches $\mathcal{O}$ at a single point, such that the two bodies are smooth in the vicinity of the contact. In particular, the entire boundary of $\mathcal{CO}$ is smooth when $\mathcal{B}$ and $\mathcal{O}$ are smooth convex bodies (see exercise). Similar observations hold for the five-dimensional boundary of $\mathcal{CO}$ in the 3D case.

2.3 The C-Obstacles 1'st and 2'nd-Order Geometry

When $\mathcal{B}$ is contacted by stationary finger bodies $\mathcal{O}_1,\ldots,\mathcal{O}_k$, its configuration q lies on the boundary of each c-obstacle $\mathcal{CO}_i$ for $i = 1\ldots k$. We shall see in Chapter 4 that the free motions of $\mathcal{B}$ are determined in this case by the first and second-order geometry of the c-obstacle boundaries i.e., by the c-obstacles' normal and curvature. Let us now focus on a particular stationary body $\mathcal{O}$, and derive formulas for the normal and curvature of its c-obstacle boundary, denoted $\mathsf{bdy}(\mathcal{CO})$. We shall assume that $\mathcal{B}$ touches $\mathcal{O}$ at a single point, such that the two bodies have smooth boundaries in the vicinity of the contact. We first obtain a formula for the c-obstacle normal, then obtain a formula for its curvature.

2.3.1 The C-Obstacle Normal

By construction $\mathcal{CO} = \{q \in I\!R^m : d(q) \leq 0\}$. If $d(q)$ would have been differentiable at $q \in \mathsf{bdy}(\mathcal{CO})$, its gradient $\nabla d(q)$ would be collinear with the c-obstacle outward normal at q. But $d(q)$ is identically zero inside $\mathcal{CO}$ and is monotonically increasing away from $\mathcal{CO}$, implying that it is *non-differentiable* at $q \in \mathsf{bdy}(\mathcal{CO})$ However, $d(q)$ is Lipschitz continuous,