Chapter 3

Configuration Space Tangent and Cotangent Vectors

The rigid-body configuration space provides a geometric framework for describing the constraints imposed on the object's motions by the grasping fingers. An important component of this framework is the representation of the object's velocities as c-space tangent vectors, and the representation of the finger forces as c-space cotangent vectors. This chapter consists of three parts. Section 3.1 describes the relation of the c-space tangent vectors to the rigid-body linear and angular velocities. Based on the virtual work principle, Section 3.2 describes how the rigid body forces are represented as c-space cotangent vectors. Section 3.3 introduces the line geometry of the rigid-body tangent and cotangent vectors. Using this theory, we obtain a graphical depiction of the c-obstacle tangent space, a representation which will prove useful in subsequent chapters.

3.1 C-Space Tangent Vectors

Let a rigid body $\mathcal{B}$ move along a c-space trajectory $q(t)$. The tangent vector $\dot{q}(t) = \frac{d}{dt}q(t)$ represents an instantaneous motion of $\mathcal{B}$ in the physical environment. The geometric interpretation of $\dot{q}(t)$ is straightforward in the 2D case. In this case $\dot{q}(t) = (\dot{d}(t), \dot{\theta}(t))$ is simply $\mathcal{B}$'s linear and rotational velocity with respect to the world frame $\mathcal{F}_W$. In the 3D case $\dot{q}(t) = (\dot{d}(t), \dot{\boldsymbol{\theta}}(t))$. However, in this case only $\dot{d}(t)$ retains the interpretation of being $\mathcal{B}$'s linear velocity with respect to $\mathcal{F}_W$. In order to assign an intuitive meaning to $\dot{\boldsymbol{\theta}}(t)$, we describe under what condition it can be interpreted as $\mathcal{B}$'s angular velocity vector.

Let us first summarize the notion of rigid-body angular velocity vector. Let $R(t)$ be a curve in $SO(3)$ such that $R(0) = R_0$. The tangent to $R(t)$ at R_0 is given by $\dot{R} = \frac{d}{dt}\big|_{t=0} R(t)$. Since the matrices of $SO(3)$ satisfy the identity $R(t)R^T(t) = I$, the derivative $\dot{R}$ satisfies the identity $\dot{R}R_0^T + R_0\dot{R}^T = 0$, implying that $\dot{R}R_0^T$ is a skew-symmetric matrix. The *angular velocity vector,* denoted ω, parametrizes the skew-symmetric matrices $\dot{R}R_0^T$ as cross-product matrices, $\dot{R}R_0^T = [\omega\times]$ for $\omega \in I\!R^3$. Based on this definition, the derivative of $R(t)$ at R_0 is given by

$$\frac{d}{dt}\bigg|_{t=0} R(t) = [\omega\times]R_0 \quad \omega \in I\!R^3. \tag{3.1}$$

The angular velocity vector provides an interpretation of $\dot{R}$ as an instantaneous rotation of $\mathcal{B}$ about an axis collinear with ω passing through $\mathcal{B}$'s origin, with $\|\omega\|$ being $\mathcal{B}$'s rotational velocity about this axis.

In order to interpret $\dot{\boldsymbol{\theta}}$ as an angular velocity vector, we shall assume that $\mathcal{B}$ has a distinguished configuration, denoted (d_0, R_0), which will later be its nominal equilibrium configuration. The orientation matrices $SO(3)$ are parametrized by exponential coordinates centered at R_0,

$$R(\boldsymbol{\theta}) = \exp\big([\boldsymbol{\theta}\times]\big)R_0 \qquad \boldsymbol{\theta} \in \mathbb{R}^3.$$

The c-space coordinates are still $(d, \boldsymbol{\theta}) \in \mathbb{R}^3 \times \mathbb{R}^3$, but now $\mathcal{B}$'s orientation R_0 is parametrized by $\boldsymbol{\theta} = \vec{0}$. The following lemma asserts that the tangent vectors $\dot{\boldsymbol{\theta}}$ at $\boldsymbol{\theta} = \vec{0}$ are $\mathcal{B}$'s angular velocity vectors at the orientation R_0.

Lemma 3.1.1. *Let $SO(3)$ be parametrized by $R(\boldsymbol{\theta}) = \exp([\boldsymbol{\theta}\times])R_0$ for $\boldsymbol{\theta} \in \mathbb{R}^3$. Let $\boldsymbol{\theta}(t)$ be a curve in $\boldsymbol{\theta}$-space such that $\boldsymbol{\theta}(0) = \vec{0}$ and $\frac{d}{dt}\big|_{t=0}\boldsymbol{\theta}(t) = \dot{\boldsymbol{\theta}}$. Then*

$$\frac{d}{dt}\bigg|_{t=0} R(\boldsymbol{\theta}(t)) = [\dot{\boldsymbol{\theta}}\times]R_0 = [\omega\times]R_0,$$

where ω is $\mathcal{B}$'s angular velocity vector at R_0.

Proof: Using the expansion $\exp([\boldsymbol{\theta}\times]) = \big(I + [\boldsymbol{\theta}\times] + \frac{1}{2}[\boldsymbol{\theta}\times]^2 + \cdots\big)$, the derivative of $R(\boldsymbol{\theta})$ is: $\frac{d}{dt}\big|_{t=0} R(\boldsymbol{\theta}(t)) = \big([\dot{\boldsymbol{\theta}}\times] + \frac{1}{2}\big([\dot{\boldsymbol{\theta}}\times][\boldsymbol{\theta}\times] + [\boldsymbol{\theta}\times][\dot{\boldsymbol{\theta}}\times]\big) + \cdots\big)R_0 = [\dot{\boldsymbol{\theta}}\times]R_0$, where we substituted $\boldsymbol{\theta}(0) = \vec{0}$. Observing that $\dot{\boldsymbol{\theta}}$ plays the role of ω in (3.1), we conclude that $\dot{\boldsymbol{\theta}} = \omega$. □

Remark: The lemma implies that the collection of skew-symmetric matrices $[\omega\times]$ for $\omega \in \mathbb{R}^3$ spans the tangent space to $SO(3)$ at R_0. Since $SO(3)$ is a three-dimensional manifold, the skew-symmetric matrices form a three-dimensional vector space with respect to matrix addition and scalar multiplication. The skew-symmetric matrices also possess a *Lie algebra* structure which has been used in the kinematic modeling of serial chains [2, 4].

We shall denote the tangent vectors at the configuration q_0 as pairs $\dot{q} = (v, \omega)$, where $v = \dot{d}$ and $\omega = \dot{\boldsymbol{\theta}}$ are $\mathcal{B}$'s linear and angular velocity vectors at q_0. For the sake of uniformity, let us also parametrize the orientation matrices $SO(2)$ by exponential coordinates centered at R_0,

$$R(\theta) = \exp(\theta J)R_0 \qquad \theta \in \mathbb{R},\ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The c-space coordinates in the 2D case are $(d, \theta) \in \mathbb{R}^2 \times \mathbb{R}$, with the scalar $\theta = 0$ parametrizing the orientation R_0. The rotational velocity $\dot{\theta}$ at $\theta = 0$ is the angular velocity of $\mathcal{B}$ about an axis perpendicular to the planar environment. We denote this angular velocity by the same symbol ω, where the distinction between $\omega \in \mathbb{R}$ (2D case) and $\omega \in \mathbb{R}^3$ (3D case) will be clear from the context. The derivative of $R(\theta)$ at $\theta = 0$ satisfies a formula analogous to (3.1),

$$\frac{d}{dt}\bigg|_{t=0} R(\theta(t)) = \omega J R_0 \quad \omega \in \mathbb{R},\ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.2}$$

Note that $JR_0 = R_0 J$ in the above formula, since the matrices of $SO(2)$ commute and R_0 and J both belong to $SO(2)$.

Tangent space notation. Let $q_0 \in I\!R^m$ correspond to the nominal configuration (d_0, R_0), where $m=3$ or 6. The tangent space to $I\!R^m$ at q_0, denoted $T_{q_0} I\!R^m$, is the linear m-dimensional space spanned by all tangent vectors $\dot{q}$ based at q_0. Recall now that $\mathcal{S}$ denotes the boundary of the c-obstacle $\mathcal{CO}$, and that $T_{q_0}\mathcal{S}$ denotes the tangent space to $\mathcal{S}$ at $q_0 \in \mathcal{S}$. The latter tangent space is an $(m-1)$-dimensional subspace of all vectors $\dot{q} \in T_{q_0} I\!R^m$ tangent to $\mathcal{S}$ at q_0. It consists of those instantaneous motions of $\mathcal{B}$ along which the contact point moves tangentially with respect to the stationary body $\mathcal{O}$ (see exercise). In the 2D case $T_{q_0}\mathcal{S}$ is simply a plane tangent to $\mathcal{CO}$'s boundary at q_0. In the 3D case it is a five-dimensional linear space tangent to $\mathcal{CO}$'s boundary at q_0. The tangent space $T_{q_0}\mathcal{S}$ is depicted in the third part of this chapter using line geometry.

Exercise: The tangent space to $\mathcal{CO}$'s boundary at q_0 is given by $T_{q_0}\mathcal{S} = \{\dot{q} \in T_{q_0} I\!R^m : \eta(q_0) \cdot \dot{q} = 0\}$, where $\eta(q_0)$ is the c-obstacle normal at q_0. Verify that $\dot{q} \in T_{q_0}\mathcal{S}$ is the collection of instantaneous motions along which $\mathcal{B}$'s contact point moves tangentially with respect to the stationary body $\mathcal{O}$.

Solution: Since $\eta(q_0) = DX_b(q_0)^T n(x)$ and $\dot{x} = DX_b(q_0)\dot{q}$ by the chain rule, $T_{q_0}\mathcal{S} = \{\dot{q} : \eta(q_0) \cdot \dot{q} = n(x) \cdot \dot{x} = 0\}$.

Example: Recall the parametrization $\varphi(s,\theta)$ of $\mathcal{CO}$'s boundary associated with an ellipse $\mathcal{B}$ and a stationary disc $\mathcal{O}$. We have already verified as an exercise that the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial \theta}\varphi(s,\theta)$ are linearly independent and therefore span the tangent plane $T_q\mathcal{S}$ where $q = \varphi(s,\theta)$. The two tangent vectors are given by

$$\frac{\partial}{\partial s}\varphi(s,\theta) = \begin{pmatrix} -R(\theta)\beta'(s) \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial \theta}\varphi(s,\theta) = \begin{pmatrix} -JR(\theta)\big(\beta(s) + rJ\beta'(s)\big) \\ 1 \end{pmatrix},$$

where we omitted a scalar factor preceding $\frac{\partial}{\partial s}\varphi(s,\theta)$. In these expressions r is the radius of $\mathcal{O}$, $\beta(s)$ is the contact point expressed in $\mathcal{F}_B$, and $\beta'(s)$ is the unit tangent to $\mathcal{B}$'s boundary. Retaining $\frac{\partial}{\partial s}\varphi(s,\theta)$ and taking the sum $r\frac{\partial}{\partial s}\varphi(s,\theta) + \frac{\partial}{\partial \theta}\varphi(s,\theta)$ gives the basis vectors,

$$\dot{q}_1 = \begin{pmatrix} -Jn(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \dot{q}_2 = \begin{pmatrix} -JR(\theta)b \\ 1 \end{pmatrix},$$

where we substituted $b = \beta(s)$ and $n(x) = -R(\theta)J\beta'(s)$. (In the latter expression $J\beta'(s)$ is the outward unit normal to $\mathcal{B}$ expressed in $\mathcal{F}_B$, so that the inward unit normal to $\mathcal{B}$ expressed in $\mathcal{F}_W$ is $n(x) = -R(\theta)J\beta'(s)$.) The tangent vector $\dot{q}_1$ represents an instantaneous translation of $\mathcal{B}$ such that $\dot{x}$ is tangent to $\mathcal{O}$ at x. Since $DX_b(q) = [I \; JRb]$, the tangent vector $\dot{q}_2$ satisfies $\dot{x} = DX_b(q)\dot{q}_2 = [I \; JRb]\dot{q}_2 = 0$, implying that $\dot{q}_2$ represents an instantaneous rolling of $\mathcal{B}$ on $\mathcal{O}$.

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Make a c-obstacle figure with tangent planes showing the above basis
vectors.
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ToDo

We conclude this section with a derivation of the Jacobian of the rigid-body transformation at the configuration q_0. This Jacobian already appeared in the c-obstacle normal formula without a formal derivation, and it plays an important role in the subsequent representation of forces as cotangent vectors.

Lemma 3.1.2. *Let $x = X_b(q)$ be the rigid-body transformation such that b is held fixed on $\mathcal{B}$, and let q_0 be the nominal configuration described above. In the 2D case, the Jacobian of $X_b(q)$ is the 2×3 matrix*

$$DX_b(q) = [I \;\; JRb],$$

where I is a 2×2 identity matrix, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and R is $\mathcal{B}$'s orientation matrix.

In the 3D case, the Jacobian of $X_b(q)$ at $q = q_0$ is the 3×6 matrix

$$DX_b(q_0) = \big[I \;\; -[R_0 b\times]\big],$$

where I is a 3×3 identity matrix and R_0 is $\mathcal{B}$'s orientation matrix at q_0.

Proof: Let $q(t)$ be a c-space curve such that $q(0) = q$ and $\dot{q}(0) = \dot{q} = (v, \omega)$. The Jacobian of $X_b(q)$ satisfies the chain rule: $\frac{d}{dt}\big|_{t=0} X_b(q(t)) = DX_b(q)\dot{q}$. In the 2D case $X_b(q(t)) = R(\theta(t))b + d(t)$. Using (3.2), $\frac{d}{dt}\big|_{t=0} X_b(q(t)) = \dot{R}b + \dot{d} = \omega JRb + v = [I \;\; JRb]\binom{v}{\omega}$, implying that $DX_b(q) = [I \;\; JRb]$. In the 3D case, $X_b(q(t)) = R(\boldsymbol{\theta}(t))b + d(t)$ such that $q(0) = q_0$. Using (3.1), $\frac{d}{dt}\big|_{t=0} X_b(q(t)) = \dot{R}b + \dot{d} = [\omega\times]R_0 b + v = -[R_0 b\times]\omega + v = \big[I \;\; -[R_0 b\times]\big]\binom{v}{\omega}$, implying that $DX_b(q_0) = \big[I \;\; -[R_0 b\times]\big]$. □

3.2 C-Space Cotangent Vectors

The c-space cotangent vectors represent the action of physical forces on $\mathcal{B}$, based on the following principle. Let a force f act on $\mathcal{B}$ at a fixed point b, such that $x = X_b(q)$ is the world position of the force action point. In our setting f can be a contact force generated by a stationary body $\mathcal{O}$, or it can be a gravitational force acting at $\mathcal{B}$'s center of mass. As $\mathcal{B}$ moves along a c-space trajectory $q(t)$, the point x moves along a trajectory $x(t) = X_b(q(t))$. Now let us think of x as a point mass attached to $\mathcal{B}$. The instantaneous *work* done by f on the point mass x (i.e. the change in the point-mass kinetic energy measured in Joules per seconds) is given by the inner product $f \cdot \dot{x}(t)$. Since the point masss is rigidly attached to $\mathcal{B}$, the force f must induce an identical change in $\mathcal{B}$'s kinetic energy. This physical fact is the basis for the representation of forces as c-space covectors.

Let us first summarize the notion of a covector in $\mathbb{R}^m$. The *cotangent space* of $\mathbb{R}^m$ at q_0, denoted $T^*_{q_0}\mathbb{R}^m$, consists of all real-valued functions $h : T_{q_0}\mathbb{R}^m \to \mathbb{R}$ that act linearly on the tangent vectors $\dot{q} \in T_{q_0}\mathbb{R}^m$. The elements of $T^*_{q_0}\mathbb{R}^m$ are *cotangent vectors.* A cotangent vector can be represented by a fixed tangent vector acting on the tangent vectors $\dot{q} \in T_{q_0}\mathbb{R}^m$ as follows. Let $e_1 \ldots e_m$ be the standard basis for $T_{q_0}\mathbb{R}^m$, induced from the standard basis for the ambient space $\mathbb{R}^m$. The components of h with respect to this basis are the scalars $h(e_1) \ldots h(e_m)$. Now let $u_1 \ldots u_m$ be the components of $\dot{q}$ with respect to the same basis, so that $\dot{q} = \sum_{i=1}^m u_i e_i$. By the linearity of h,

$$h(\dot{q}) = \sum_{i=1}^{m} u_i h(e_i) = \big(h(e_1) \ldots h(e_m)\big) \cdot \big(u_1 \ldots u_m\big) = \big(h(e_1) \ldots h(e_m)\big) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \qquad \dot{q} \in T_{q_0}\mathbb{R}^m,$$

where '$\cdot$' denotes the Euclidean inner product in $T_{q_0}I\!R^m$. The action of h on $\dot{q}$ is thus represented by the inner product of the fixed row vector $(h(e_1)\dots h(e_m))$ with column vectors $(u_1\dots u_m)$. In the following discussion we shall treat cotangent vectors as fixed tangent vectors appearing on the left side of the Euclidean inner product.

The virtual work principle. Let $\mathcal{B}$ move along a c-space trajectory $q(t)$ such that $q(0) = q_0$ and $\dot{q} = \dot{q}(0)$. Let a force f act on $\mathcal{B}$ at a point $x = X_b(q(t))$ during this motion. The covector representing the action of f at $q = q_0$ will be denoted $\boldsymbol{w}$. The formula for $\boldsymbol{w}$ is based on the following virtual work principle. For all instantaneous motions $\dot{q} \in T_{q_0}I\!R^m$, *the work done by $\boldsymbol{w}$ on $\mathcal{B}$ along $\dot{q}$ must be equal to the work done by f on the point mass x along $\dot{x}$.* The work done by f on x is the inner product $f \cdot \dot{x}$, while the work done by $\boldsymbol{w}$ on $\mathcal{B}$ is the inner product $\boldsymbol{w} \cdot \dot{q}$, where $\boldsymbol{w}$ is yet unknown. Since $\dot{x} = DX_b(q_0)\dot{q}$ by the chain rule, the virtual work principle gives the relation

$$\boldsymbol{w} \cdot \dot{q} = f \cdot DX_b(q_0)\dot{q} = f^T(x)DX_b(q_0)\dot{q} \quad \text{for all } \dot{q} \in T_{q_0}I\!R^m.$$

Adopting the convention that $\boldsymbol{w}$ is written as a *column* vector, the linear function corresponding to $\boldsymbol{w}$ satisfies the formula

$$\boldsymbol{w} = DX_b^T(q_0)f \quad \text{where } f \text{ acts at } x = X_b(q_0).$$

Exercise: Let a force f act on $\mathcal{B}$ at a point x. Interpret f as a covector acting on the velocities $\dot{x}$ of a point mass at x. Interpret the wrench formula as a transformation from $T_x^*I\!R^n$ to $T_{q_0}^*I\!R^m$.

A concrete formula for $\boldsymbol{w}$ is obtained by substituting for $DX_b(q_0)$ according to Lemma 3.1.2.

Lemma 3.2.1. *Let a force f act on $\mathcal{B}$ at the point $x = X_b(q_0)$, where q_0 is $\mathcal{B}$'s nominal configuration. The wrench generated on $\mathcal{B}$ by f is given by*

$$\boldsymbol{w} = \begin{pmatrix} f \\ \tau \end{pmatrix} = \begin{cases} \begin{pmatrix} f \\ f \cdot JR_0 b \end{pmatrix} & \text{2D case} \\ \begin{pmatrix} f \\ R_0 b \times f \end{pmatrix} & \text{3D case,} \end{cases}$$

where R_0 is $\mathcal{B}$'s orientation at q_0, and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in the 2D case.

The resulting covector $\boldsymbol{w}$, called a *wrench,* consists of a force f and a torque τ. Note that the torque is denoted by the same symbol in the 2D and 3D cases. In the 2D case τ is a scalar acting about an axis perpendicular to the planar environment. In the 3D case τ is a vector in $I\!R^3$ satisfying the classical formula $\tau = p \times f$, where $p = R_0 b$ is the vector from $\mathcal{B}$'s origin to the force's action point, expressed in $\mathcal{F}_W$. The distinction between $\tau \in I\!R$ (2D case) and $\tau \in I\!R^3$ (3D case) will be clear from the context.

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Joel, can you insert a short sentence after the term wrench, alluding to
the source of this term?  I inserted a pointer to B. Roth paper at the
end of this chapter.  Perhaps 'wrenches act on screws?"
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ToDo

Example: Let $\mathcal{B}$ move while maintaining contact with a stationary body $\mathcal{O}$, and let q_0 be any configuration of $\mathcal{B}$ along this motion. In the case of a *frictionless* contact the force acting on $\mathcal{B}$ is collinear with $\mathcal{B}$'s inward unit normal at x, $f = \|f\|n(x)$. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$, the contact point velocity is tangent to $\mathcal{O}$ at x, implying that $f \cdot \dot{x} = \|f\|(n(x) \cdot \dot{x}) = 0$ during this motion. It follows from the virtual work principle that $\boldsymbol{w} \cdot \dot{q} = 0$. Since this argument holds for *all* $\dot{q} \in T_{q_0}\mathcal{S}$, the wrench $\boldsymbol{w} = \|f\| DX_b(q_0)^T n(x)$, seen as a fixed tangent vector at q_0, is orthogonal to the tangent space $T_{q_0}\mathcal{S}$. Indeed, the c-obstacle normal formula, which was derived based on purely geometric considerations, is given by $\eta(q_0) = DX_b(q_0)^T n(x)$.

As we shall see in the next chapter, the rigid-body contact models assume that two bodies touch at a single isolated point. However, in certain applications two rigid bodies are purposely designed to make contact along several points and even a continuum of points. The net wrench acting on $\mathcal{B}$ in such cases is the *sum* (integral in the case of a continuum) of the wrenches generated by the individual contacts. As a concrete example, consider the wrench generated by a drill acting on a workpiece $\mathcal{B}$. To a rough approximation, the drill applies a normal penetrating force along its axis, together with a matched pair of tangential forces at its two tips, called flutes. The tangential forces generate a net torque about the drill's axis as follows. Let b be the position of the drill's center, and let $b \pm \boldsymbol{r}$ be the position of the drill's flutes in $\mathcal{B}$'s frame ($i = 1, 2$). The net wrench generated by tangential forces $\pm f$ acting on $\mathcal{B}$ at the drill's flutes is given by $\boldsymbol{w} = \begin{pmatrix} -f \\ R_0(b - \boldsymbol{r}) \times (-f) \end{pmatrix} + \begin{pmatrix} f \\ R_0(b + \boldsymbol{r}) \times (f) \end{pmatrix} = \begin{pmatrix} \vec{0} \\ 2R_0\boldsymbol{r} \times f \end{pmatrix}$, which is a pure torque about the drill's axis.

Exercise—frame invariance of wrench formula. Let $(\mathcal{F}_W, \mathcal{F}_B)$ and $(\bar{\mathcal{F}}_W, \bar{\mathcal{F}}_B)$ be two choices of world and body frames. Let $q \in I\!R^m$ and $\bar{q} \in \overline{I\!R}^m$ be the c-space coordinates associated with the two choices of frames, such that q_0 and $\bar{q}_0$ are $\mathcal{B}$'s nominal configuration in the two c-space parametrizations. The wrench formula is $\boldsymbol{w} = DX_b(q_0)^T f$ and $\bar{\boldsymbol{w}} = DX_{\bar{b}}(\bar{q}_0)^T \bar{f}$ with respect to the two choices of frames, where (x, f) and $(\bar{x}, \bar{f})$ are the action point and force in $\mathcal{F}_W$ and $\bar{\mathcal{F}}_W$. Verify that the wrench formula is independent on the choice of world and body frames.

Solution: Since $\boldsymbol{w} \cdot \dot{q} = f^T DX_b(q_0)\dot{q} = f \cdot \dot{x}$ and $\bar{\boldsymbol{w}} \cdot \dot{\bar{q}} = \bar{f}^T DX_{\bar{b}}(\bar{q}_0)\dot{\bar{q}} = \bar{f} \cdot \dot{\bar{x}}$, the wrench formula is frame invariant if $f \cdot \dot{x} = \bar{f} \cdot \dot{\bar{x}}$. Let $\mathcal{F}_W$ and $\bar{\mathcal{F}}_W$ be related by a fixed translation and rotation (d_W, R_W), so that points in the two frames are related by the transformation $x = R_W \bar{x} + d_W$. The Jacobian of this transformation is the constant matrix R_W, and by the chain rule $\dot{x} = R_W \dot{\bar{x}}$. Similarly, f and $\bar{f}$ are related by the transformation $f = R_W \bar{f}$. Thus $f \cdot \dot{x} = \bar{f}^T R_W^T R_W \dot{\bar{x}} = \bar{f} \cdot \dot{\bar{x}}$, implying that the wrench formula is independent on the choice of frames.

Coordinate transformation of tangent and cotangent vectors. Let $q \in I\!R^m$ and $\bar{q} \in \overline{I\!R}^m$ be two c-space parametrizations associated with two choices of world and body frames, such that q_0 and $\bar{q}_0$ are $\mathcal{B}$'s nominal configuration in the two parametrizations. It can be verified that q and $\bar{q}$ are related by a coordinate transformation (a diffeomorphism) $q = F(\bar{q})$, such that $q_0 = F(\bar{q}_0)$. Since the c-space trajectories are related by the transformation $q(t) = F(\bar{q}(t))$, it follows from the chain rule that $\dot{q}(t) = F(\dot{\bar{q}}(t))$. The Jacobian $DF(\bar{q}_0)$ thus maps tangent vectors in $T_{\bar{q}_0}\bar{I\!\!R}^m$ to tangent vectors in $T_{q_0}I\!\!R^m$,

$$\dot{q} = DF(\bar{q}_0)\dot{\bar{q}} \qquad \dot{q} \in T_{q_0}I\!\!R^m,\ \dot{\bar{q}} \in T_{\bar{q}_0}\bar{I\!\!R}^m.$$

Now let $\boldsymbol{w}$ be a covector acting on the tangent space $T_{q_0}I\!\!R^m$. Using the tangent vector transformation rule, $\boldsymbol{w}\cdot\dot{q} = \boldsymbol{w}^T\big(DF(\bar{q}_0)\dot{\bar{q}}\big) = \big(\boldsymbol{w}^T DF(\bar{q}_0)\big)\dot{\bar{q}}$, where $\dot{q} \in T_{q_0}I\!\!R^m$ and $\dot{\bar{q}} \in T_{\bar{q}_0}\bar{I\!\!R}^m$. It follows that the covector $\bar{\boldsymbol{w}}^T = \boldsymbol{w}^T DF(\bar{q}_0) \in T^*_{\bar{q}_0}\bar{I\!\!R}^m$ corresponds to the covector $\boldsymbol{w} \in T^*_{q_0}I\!\!R^m$. Writing $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ as column vectors, the covector transformation rule is given by

$$\bar{\boldsymbol{w}} = DF^T(\bar{q}_0)\boldsymbol{w} \qquad \boldsymbol{w} \in T^*_{q_0}I\!\!R^m,\ \bar{\boldsymbol{w}} \in T^*_{\bar{q}_0}\bar{I\!\!R}^m.$$

Note that $DF(\bar{q}_0)$ maps tangent vectors "forward" from $T_{\bar{q}_0}\bar{I\!\!R}^m$ to $T_{q_0}I\!\!R^m$, while $DF^T(\bar{q}_0)$ maps cotangent vectors "backward" from $T^*_{q_0}I\!\!R^m$ to $T^*_{\bar{q}_0}\bar{I\!\!R}^m$.

Exercise: Let $q \in I\!\!R^m$ and $\bar{q} \in \bar{I\!\!R}^m$ be two c-space parametrizations associated with two choices of world and body frames, related by fixed rigid-body transformations (d_w, R_w) and $(d_{\mathcal{B}}, R_{\mathcal{B}})$. Derive the coordinate transformation $q = F(\bar{q})$ between the two parametrizations.

Exercise: Verify that $\bar{\boldsymbol{w}}$ represents the same linear function as $\boldsymbol{w}$.

Solution: Based on the tangent and cotangent vector transformation rules, Let $\dot{q} = DF(\bar{q}_0)\dot{\bar{q}}$ and $\bar{\boldsymbol{w}} = DF^T(\bar{q}_0)\boldsymbol{w}$. Hence $\boldsymbol{w}\cdot\dot{q} = \boldsymbol{w}^T\big(DF(\bar{q}_0)\dot{\bar{q}}\big) = \big(\boldsymbol{w}^T DF(\bar{q}_0)\big)\dot{\bar{q}} = \bar{\boldsymbol{w}}\cdot\dot{\bar{q}}$.

3.3 Line Geometry of Tangent and Cotangent Vectors

Line geometry provides an intuitive means for depicting the c-space tangent and cotangent vectors as directed lines in physical space. This section describes three useful facts concerning the line geometry of the c-space tangent and cotangent vectors. First, every tangent vector $\dot{q} = (v, \omega)$ can be represented as an instantaneous screw motion about a spatial axis. Second, every wrench $\boldsymbol{w} = (f, \tau)$ can be represented by a screw-like application of force and torque about a spatial axis. Third, the inner product $\boldsymbol{w}\cdot\dot{q}$ can be expressed as a geometric relation between the directed lines representing $\boldsymbol{w}$ and $\dot{q}$. This relation will allow us to graphically depict the c-obstacle tangent space. The tools developed here will also serve in subsequent chapters to depict equilibrium grasps and postures.

The set of directed spatial lines is parametrized by the following Plücker coordinates.

Definition 1. *The* **Plücker coordinates** *of the directed spatial lines are vectors* $l = (\hat{l}, p \times \hat{l}) \in I\!\!R^6$, *where* p *is any point on the line and* $\hat{l}$ *is the unit direction of the line.*

We will use the symbol l both for the physical line and for the vector representing the line in Plücker coordinates. Note that any point on l can serve as a reference point, since $(p + s\hat{l}) \times \hat{l} = p \times \hat{l}$ for $s \in I\!\!R$. As verified in exercise 3.x, the Plücker coordinates of all spatial lines span a smooth *four*-dimensional manifold in $I\!\!R^6$. This means that the spatial lines depends on four parameters—the line direction, $\hat{l}$, and the line base point, p, specified

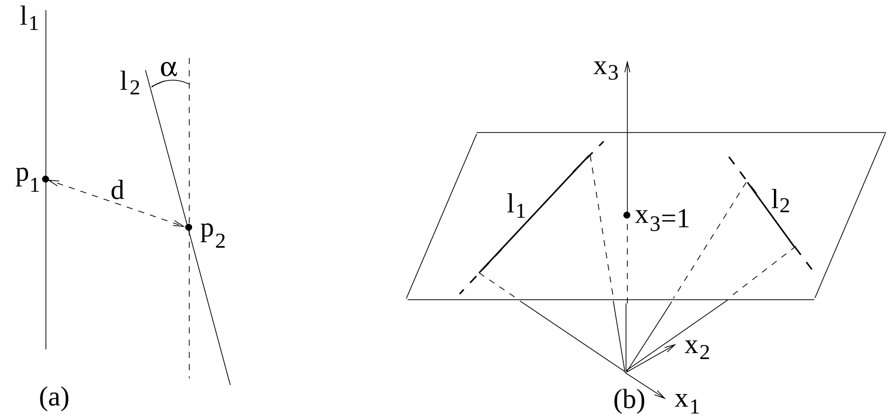

Figure 3.1: (a) The parameters specifying the relative position and orientation of l_1 and l_2. (b) The correspondence between the embedded planar lines l_1 and l_2 and the planes P_1 and P_2.

within a plane orthogonal to $\hat{l}$.[1] An important relation between spatial lines is the reciprocal product of their Plücker coordinates.

Definition 2. *The* **reciprocal product** *of two spatial lines, $l_1 = (\hat{l}_1, p_1 \times \hat{l}_1)$ and $l_2 = (\hat{l}_2, p_2 \times \hat{l}_2)$, is the "swaped" inner product,*

$$(\hat{l}_1, p_1 \times \hat{l}_1) \cdot (p_2 \times \hat{l}_2, \hat{l}_2) = (\hat{l}_1, p_1 \times \hat{l}_1) \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{pmatrix} \hat{l}_2 \\ p_2 \times \hat{l}_2 \end{pmatrix},$$

where O is a 3×3 zero matrix and I is a 3×3 identity matrix.

The reciprocal product of l_1 and l_2 is determined by the relative position and orientation of the two lines. Let $d \geq 0$ be the minimal distance between l_1 and l_2, and let $0 \leq \alpha \leq \pi$ be the angle between l_1 and l_2, measured about the lines' common normal as depicted in Figure 3.1(a). A geometric formula for the reciprocal product is specified in the following proposition.

Proposition 3.3.1. *The reciprocal product of two spatial lines l_1 and l_2 satisfies the formula*

$$(\hat{l}_1, p_1 \times \hat{l}_1) \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{pmatrix} \hat{l}_2 \\ p_2 \times \hat{l}_2 \end{pmatrix} = -s \cdot d \sin\alpha, \tag{3.3}$$

where d and α are the minimal distance and angle between the two lines, and $s = \pm 1$ according to the sign of the expression $(p_2 - p_1) \cdot (\hat{l}_1 \times \hat{l}_2)$.

When the reciprocal product of two spatial lines happens to be zero, the two lines must intersect at a common point. This key property is summarized in the following corollary.

[1] The four-dimensional manifold is topologically equivalent to the tangent bundle of the unit sphere, TS^2. Every point $\hat{l} \in S^2$ specifies a particular direction, and the sphere's tangent plane at $\hat{l}$ specifies the base points, p, of all spatial lines having this direction.

Corollary 3.3.2. *The reciprocal product of two spatial lines is zero iff the two lines intersect in $I\!R^3$, with the understanding that parallel lines intersect "at infinity."*

Proof: The reciprocal product vanishes when $d \sin\alpha = 0$. When $d = 0$, l_1 and l_2 intersect at a common point in $I\!R^3$. When $\sin\alpha = 0$, l_1 and l_2 are parallel and therefore intersect at infinity. □

The geometric intuition behind the reciprocal product is as follows. The spatial lines can be embedded in the $\chi_4 = 1$ hyperplane of $I\!R^4$, where $(\chi_1, \ldots, \chi_4)$ are the coordinates of $I\!R^4$. Every embedded line l determines a unique two-dimensional plane passing through the origin of $I\!R^4$. (An analogous situation arises when the planar lines are embedded in the $\chi_3 = 1$ plane of $I\!R^3$; in this case every embedded line determines a plane passing through the origin of $I\!R^3$, as shown in Figure 3.1(b).) Now let $l_1 = (\hat{l}_1, p_1 \times \hat{l}_1)$ and $l_2 = (\hat{l}_2, p_2 \times \hat{l}_2)$ be two spatial lines, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the planes associated with the embedded lines in $I\!R^4$. The two lines intersect at a common point in $I\!R^3$ iff $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect along a line passing through the origin of $I\!R^4$. The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect along a line in $I\!R^4$ iff their basis vectors are linearly dependent. Each embedded line l_i passes through the point $(p_i, 1) \in I\!R^4$ along the direction $(\hat{l}_i, 0) \in I\!R^4$. Hence the pair $\{(p_i, 1), (\hat{l}_i, 0)\}$ forms a basis for $\mathcal{P}_i$ $(i = 1, 2)$. The linear dependence of $\mathcal{P}_1$ and $\mathcal{P}_2$ is thus equivalent to the condition $\det \begin{bmatrix} p_1 & \hat{l}_1 & p_2 & \hat{l}_2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, which is precisely the reciprocal product of l_1 and l_2.

Our next step is to represent the c-space tangent vectors, $\dot{q} = (v, \omega)$, as instantaneous screw-like motions about a spatial axis, called *instantaneous twists*. The following statement is the classical Chasles' Theorem.

Theorem 1 (Instantaneous Twist). *Every tangent vectors $\dot{q} = (v, \omega) \in T_{q_0} I\!R^m$ can be written as*

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} p \times \omega \\ \omega \end{pmatrix} + \begin{pmatrix} z\omega \\ \vec{0} \end{pmatrix} = \|\omega\| \begin{pmatrix} p \times \hat{\omega} \\ \hat{\omega} \end{pmatrix} + \begin{pmatrix} z\omega \\ \vec{0} \end{pmatrix}, \tag{3.4}$$

where $p = \omega \times v / \|\omega\|^2$ and $z v \cdot \omega / \|\omega\|^2$. Every tangent vector thus corresponds to an instantaneous rotation of $\mathcal{B}$ about the line $l = (\hat{\omega}, p \times \hat{\omega})$ coupled with an instantaneous translation of $\mathcal{B}$ along this line, where $\hat{\omega} = \omega / \|\omega\|$.

Proof: The linear velocity v can be expressed as the sum: $v = (v \cdot \hat{\omega})\hat{\omega} + (I - \hat{\omega}\hat{\omega}^T)v$. Using the identity $[\hat{\omega}\times]^2 = \hat{\omega}\hat{\omega}^T - I$, $v = (v \cdot \hat{\omega})\hat{\omega} - [\hat{\omega}\times]^2 v = (v \cdot \hat{\omega})\hat{\omega} - \hat{\omega} \times (\hat{\omega} \times v)$. Substituting $z = (v \cdot \hat{\omega}) / \|\omega\| = (v \cdot \omega) / \|\omega\|^2$ and $p = (\hat{\omega} \times v) / \|\omega\| = (\omega \times v) / \|\omega\|^2$ gives the result. □

Note that the Plücker coordinates of the line representing the instantaneous rotation axis, $l = (\hat{\omega}, p \times \hat{\omega})$, are swapped in (3.4). The parameter z is called the *pitch* of the instantaneous twist. It specifies the amount of translation per unit rotation along the twist axis l. In particular, $z = 0$ corresponds to pure instantaneous rotation about l, while $z = \infty$ corresponds to pure instantaneous translation along l.

The wrenches acting on $\mathcal{B}$ can be analogously represented as a screw-like application of force and torque. The following statement is the classical Poinsot's theorem.

Theorem 2 (Wrench Screw). *Any cotangent vector* $\boldsymbol{w} = (f, \tau) \in T^*_{q_0} I\!R^m$ *can be written as*

$$\begin{pmatrix} f \\ \tau \end{pmatrix} = \begin{pmatrix} f \\ p \times f \end{pmatrix} + \begin{pmatrix} \vec{0} \\ zf \end{pmatrix} = \|f\| \begin{pmatrix} \hat{f} \\ p \times \hat{f} \end{pmatrix} + \begin{pmatrix} \vec{0} \\ zf \end{pmatrix}, \tag{3.5}$$

where $p = \tau \times f / \|f\|^2$ *and* $z = f \cdot \tau / \|f\|^2$*. Every wrench* $\boldsymbol{w}$ *thus applies a force along the line* $l = (\hat{f}, p \times \hat{f})$ *together with a torque about this line, where* $\hat{f} = f / \|f\|$.

Proof: The proof is similar to the proof of Chasles' Theorem. The force f can be expressed as the sum: $f = (f \cdot \hat{\tau})\hat{\tau} + (I - \hat{\tau}\hat{\tau}^T)f$. Using the identity $[\hat{\tau}\times]^2 = \hat{\tau}\hat{\tau}^T - I$, $f = (f \cdot \hat{\tau})\hat{\tau} - [\hat{\tau}\times]^2 f = (f \cdot \hat{\tau})\hat{\tau} - \hat{\tau} \times (\hat{\tau} \times f)$. Substituting $z = (f \cdot \hat{\tau})/\|\tau\| = (f \cdot \tau)/\|\tau\|^2$ and $p = (\hat{\tau} \times f)/\|\tau\| = (\tau \times f)/\|\tau\|^2$ gives the result. □

The pitch parameter, z, specifies the amount of torque applied by the wrench about l per unit force applied along this line. In particular, a zero-pitch wrench screw corresponds to a wrench generated by a pure force f acting along l (see exercise).

The following lemma specifies a formula for the inner product $\boldsymbol{w} \cdot \dot{q}$ as a geometric relation between the screw lines representing the wrench $\boldsymbol{w}$ and the tangent vector $\dot{q}$.

Lemma 3.3.3 (Geometric Formula for $\boldsymbol{w} \cdot \dot{q}$). *Let the wrench* $\boldsymbol{w} = (f, \tau) \in T^*_{q_0} I\!R^m$ *have a screw axis* l_1 *and pitch* z_1*. Let the tangent vector* $\dot{q} = (v, \omega) \in T_{q_0} I\!R^m$ *have a screw axis* l_2 *and pitch* z_2*. The inner product* $\boldsymbol{w} \cdot \dot{q}$ *satisfies the geometric formula*

$$\boldsymbol{w} \cdot \dot{q} = \|f\|\|\omega\|\big(-s \cdot d \sin\alpha + (z_1 + z_2)\cos\alpha\big),$$

where d *and is the minimum distance between* l_1 *and* l_2*,* α *is the angle between* l_1 *and* l_2*, and* $s = \pm 1$ *as specified above.*

Proof: Substituting $\boldsymbol{w} = (f, p_1 \times f) + (\vec{0}, z_1 f)$ and $\dot{q} = (p_2 \times \omega, \omega) + (z_2\omega, \vec{0})$ in $\boldsymbol{w} \cdot \dot{q}$ gives

$$\begin{aligned} \boldsymbol{w} \cdot \dot{q} &= (f, p_1 \times f)\begin{pmatrix} p_2 \times \omega \\ \omega \end{pmatrix} + (f, p_1 \times f)\begin{pmatrix} z_2\omega \\ \vec{0} \end{pmatrix} + (\vec{0}, z_1 f)\begin{pmatrix} p_2 \times \omega \\ \omega \end{pmatrix} \\ &= \|f\|\|\omega\| \left((\hat{f}, p_1 \times \hat{f}) \begin{pmatrix} p_2 \times \hat{\omega} \\ \hat{\omega} \end{pmatrix} + (z_1 + z_2)\hat{f} \cdot \hat{\omega} \right), \end{aligned}$$

where $\hat{f} = f/\|f\|$ and $\hat{\omega} = \omega/\|\omega\|$. The first summand is equal to $-s \cdot d \sin\alpha$ according to Proposition 3.3.1. The second summand is equal to $(z_1 + z_2)\cos\alpha$. □

A remark on reciprocal screws: Since the Plücker coordinates of an instantaneous twist are written in a swapped order, the inner product $\boldsymbol{w} \cdot \dot{q}$ is equivalent to the reciprocal product of the screws associated with $\boldsymbol{w}$ and $\dot{q}$,

$$\boldsymbol{w} \cdot \dot{q} = (f, p_1 \times f + z_1 f) \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{pmatrix} \omega \\ p_2 \times \omega + z_2\omega \end{pmatrix}.$$

When the reciprocal product of two screws is zero, the two screws are said to be *reciprocal.* This situation has a special significance when the screws are associated with $\boldsymbol{w}$ and $\dot{q}$. It

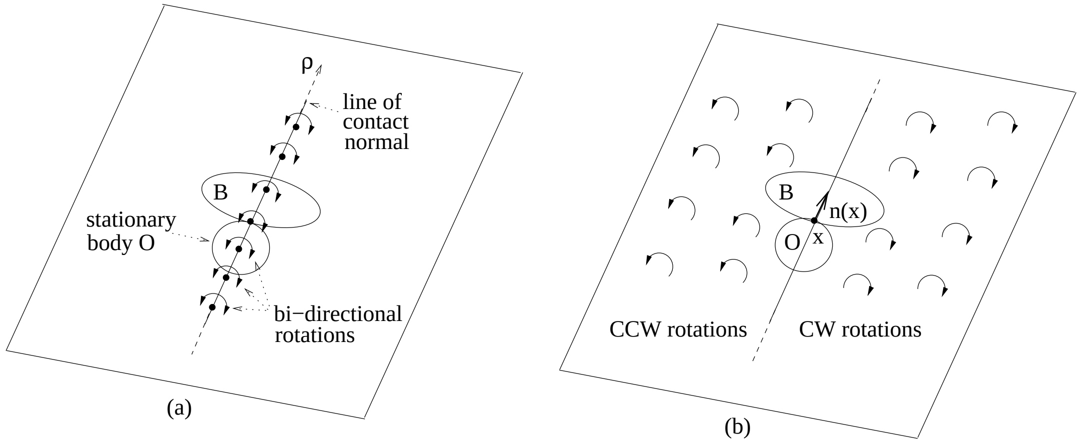

Figure 3.2: (a) $T_{q_0}\mathcal{S}$ consists of instantaneous rotations of $\mathcal{B}$ about points on the contact normal line. (b) Tangent vectors pointing away from $\mathcal{CO}$ at q_0 are CCW rotations on the left halfplane and CW rotations in the right halfplane, measured with respect to $n(x)$.

means that $\boldsymbol{w}\cdot\dot{q} = 0$, or in physical terms, that the wrench $\boldsymbol{w}$ *cannot impede* the instantaneous motion of $\mathcal{B}$ along $\dot{q}$. This fact will be key for characterizing the mobility of $\mathcal{B}$ with respect to the surrounding fingers $\mathcal{O}_1 \ldots \mathcal{O}_k$.

Graphical depiction of the c-obstacle tangent space. The formula for $\boldsymbol{w} \cdot \dot{q}$ can be used to characterize the c-obstacle tangent space. When $\boldsymbol{w}$ is generated by a force aligned with $\mathcal{B}$'s contact normal at x, $n(x)$, the collection of tangent vectors satisfying $\boldsymbol{w}\cdot\dot{q} = 0$ spans the c-obstacle tangent space. Formally speaking, the c-obstacle tangent space is given by $T_{q_0}\mathcal{S} = \{\dot{q} \in T_{q_0}I\!\!R^m : \eta(q_0)\cdot\dot{q} = 0\}$, where $\eta(q_0)$ is the c-obstacle normal at q_0. As discussed in the previous section, $\eta(q_0)$ can be interpreted as the wrench generated by a unit-magnitude normal force acting on $\mathcal{B}$ at x, so that $\boldsymbol{w} = \eta(q_0)$. Hence $T_{q_0}\mathcal{S}$ consists of tangent vectors satisfying $\boldsymbol{w}\cdot\dot{q} = 0$, where $\boldsymbol{w} = \eta(q_0)$. Since the wrench screw of a pure force has a vanishing pitch, $z_1 = 0$ for the wrench screw of $\eta(q_0)$. Assuming that the frames $\mathcal{F}_W$ and $\mathcal{F}_B$ share a common origin at the configuration q_0, the screw axis of the wrench $\boldsymbol{w} = \eta(q_0)$ coincides with the normal line at the contact point x (see exercise). Substituting $f = n(x)$ and $z_1 = 0$ in the formula for $\boldsymbol{w}\cdot\dot{q}$ gives

$$T_{q_0}\mathcal{S} = \{\dot{q} \in T_{q_0}I\!\!R^m : s\cdot\|\omega\| d\sin(\alpha) = \|\omega\| z_2\cos(\alpha)\}, \tag{3.6}$$

where (d, α, z_2) are the parameters of the instantaneous twist representing $\dot{q}$, expressed relative to the contact normal line. Note that $\|\omega\|$ appears on both sides of (3.6). Hence when an instantaneous screw motion satisfies the equation, it does so for all magnitudes $\|\omega\| \in I\!\!R$. We now discuss the implication of this formula for the finger c-obstacles associated with planar grasps.

Example—the tangent plane of a finger c-obstacle in 2D case. Let us embed the planar environment as the (x, y) plane in $I\!\!R^3$. Let $\mathcal{P}$ denote the horizontal plane containing the environment, and let $\boldsymbol{e} = (0, 0, 1)$ denote a unit vector perpendicular to $\mathcal{P}$. The linear velocity of $\mathcal{B}$ is now the vector $(v, 0)$ tangent to $\mathcal{P}$. The rotational velocity of $\mathcal{B}$ is now the vector $\omega\boldsymbol{e}$, where $\omega \in I\!\!R$ is $\mathcal{B}$'s angular velocity within $\mathcal{P}$. The instantaneous twists of $\mathcal{B}$ have zero pitch, since $z_2 = \omega((v, 0)\cdot\boldsymbol{e}) = 0$. All tangent vectors in $T_{q_0}I\!\!R^3$ are therefore pure instantaneous rotations about points of $\mathcal{P}$. Let us now determine which instantaneous rotations correspond to $T_{q_0}\mathcal{S}$. Since $\boldsymbol{e}$ is orthogonal to the contact normal line, $\sin(\alpha) = 1$. Substituting $z_2 = 0$ and $\sin(\alpha) = 1$ in (3.6) gives

$$T_{q_0}\mathcal{S} = \{\dot{q} \in T_{q_0}I\!R^3 : d = 0\},$$

where we cancelled out the sign parameter $s = \pm 1$. It follows that $T_{q_0}\mathcal{S}$ consists of instantaneous rotations of $\mathcal{B}$ about all points along the contact normal line,

$$T_{q_0}\mathcal{S} = \left\{ \dot{q} = \|\omega\| \begin{pmatrix} p \times \boldsymbol{e} \\ \boldsymbol{e} \end{pmatrix} : \omega \in I\!R,\ p = x + \rho n(x) \text{ for } -\infty \leq \rho \leq \infty \right\},$$

where p parametrizes the contact normal line in terms of a scalar ρ. This characterization of $T_{q_0}\mathcal{S}$ is depicted in Figure 3.2(a). Note that instantaneous rotations of $\mathcal{B}$ about points at infinity are translations along the tangent at the contact point with $\mathcal{O}$. Also note that $T_{q_0}\mathcal{S}$ is parametrized by two scalars, ρ and $\|\omega\|$, which is consistent with the fact that $T_{q_0}\mathcal{S}$ is a two-dimensional plane.

Remark: The halfspace of $T_{q_0}I\!R^m$ bounded by $T_{q_0}\mathcal{S}$ and pointing away from $\mathcal{CO}$ at q_0 consists of all tangent vectors $\dot{q} \in T_{q_0}I\!R^m$ satisfying the inequality $\eta(q_0) \cdot \dot{q} \geq 0$. In the 2D case these are instantaneous clockwise rotations about points on the right side of the contact normal line, and instantaneous counterclockwise rotations about points on the left side of the contact normal line, as depicted in Figure 3.2(b). Note that instantaneous rotations about points on the contact normal line are bi-directional, as these rotations correspond to tangent vectors in $T_{q_0}\mathcal{S}$.

Exercise: Consider the halfspace of tangent vectors pointing away from $\mathcal{CO}$ at q_0, given by $\{\dot{q} \in T_{q_0}I\!R^m : \eta(q_0) \cdot \dot{q} \geq 0\}$. Justify the characterization of this halfspace as clockwise rotations of $\mathcal{B}$ about points on the right side of the contact normal line, and as counterclockwise rotations of $\mathcal{B}$ about points on the left side of the contact normal line (see in Figure 3.2(b)).

Solution: Based on Lemma 3.3.3, $\eta(q_0) \cdot \dot{q} = -s \|\omega\| d$, where s is the sign of the inner product $(p_2 - p_1) \cdot (\hat{l}_1 \times \hat{l}_2)$. In our case $p_1 = x$, $\hat{l}_1 = (n(x), 0)$, and $\hat{l}_2 = \mathsf{sgn}(\omega)\boldsymbol{e}$. Hence $\hat{l}_1 \times \hat{l}_2$ is thus $\hat{l}_1 \times \hat{l}_2 = \mathsf{sgn}(\omega)(t(x), 0)$, where $t(x)$ is the unit tangent at x such that $\{(t(x), 0), (n(x), 0), \boldsymbol{e}\}$ is a right-handed triplet. The inequality $\eta(q_0) \cdot \dot{q} \geq 0$ is thus equivalent to the inequality $-\mathsf{sgn}(\omega)(p_2 - x) \cdot t(x) \geq 0$. The latter inequality is linear in p_2, and therefore defines two halfplanes according to the sign of ω. When $\mathsf{sgn}(\omega) = +1$ the instantaneous rotations are counterclockwise, and in this case p_2 lies in the halfplane $(p_2 - x) \cdot t(x) \leq 0$. When $\mathsf{sgn}(\omega) = -1$ the instantaneous rotations are clockwise, and in this case p_2 lies in the halfplane $(p_2 - x) \cdot t(x) \geq 0$.

C-Obstacle Tangent Space—3D case. In order to depict the c-obstacle tangent space in the 3D case, we need to become acquainted with the notion of linear subspaces of lines. These are collections of spatial lines which correspond to linear subspaces in Plücker coordinates. In particular, a two-dimensional linear subspace spanned by the lines $(\hat{l}_1, p \times \hat{l}_1)$ and $(\hat{l}_2, p \times \hat{l}_2)$ is a *flat pencil.* It is the collection of all spatial lines passing through p and embedded in a plane spanned by $(\hat{l}_1, \hat{l}_2)$. A three-dimensional linear subspace spanned by the lines $(\hat{l}_1, p \times \hat{l}_1)$,$(\hat{l}_2, p \times \hat{l}_2)$,$(\hat{l}_3, p \times \hat{l}_3)$ is a *solid pencil.* It is the collection of all lines passing

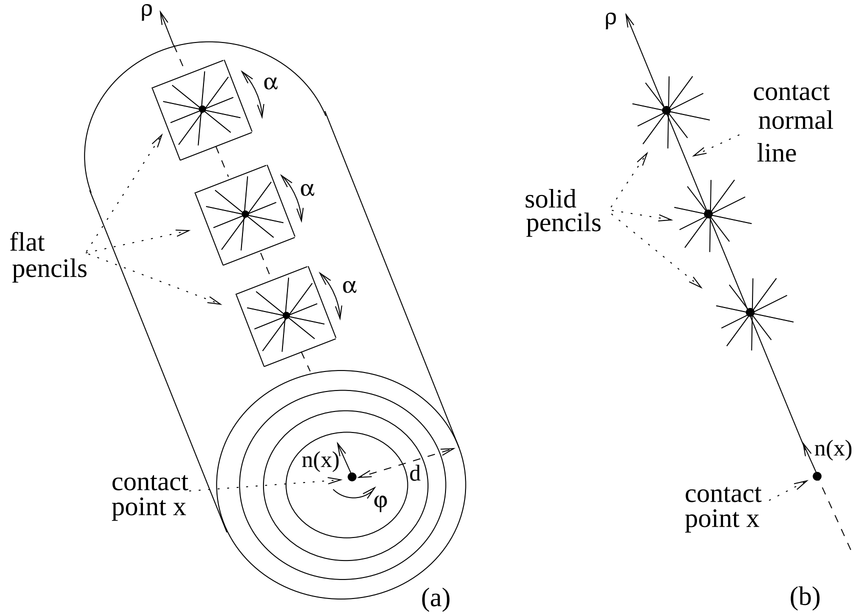

Figure 3.3: (a) The twist lines of $T_{q_0}\mathcal{S}$ are flat pencils tangent to concentric cylinders centered on the contact normal line, together with (b) solid pencils along the contact normal line.

through p along all spatial directions. The two pencils are the simplest line complexes. Other linear subspaces of lines will appear in subsequent chapters as a means for characterizing equilibrium grasps and postures.

Let $\boldsymbol{l}$ denote the contact normal line. The c-obstacle tangent space in the 3D case consists of all instantaneous twists satisfying the equation $s \cdot d \sin(\alpha) = z_2 \cos(\alpha)$. The variables in this equation are the distance d and angle α of the twist lines with respect to $\boldsymbol{l}$, together with their pitch z_2. We shall interpret the equation as specifying the pitch via the formula $z_2 = s \cdot d \tan(\alpha)$, where d and α are free variables. Let us now consider the collection of twist lines parametrized by the distance parameter d. The set of points x having a fixed positive distance from $\boldsymbol{l}$ is a cylinder of radius d centered on the contact normal line (the case of $d = 0$ is considered below). Let x be a point on the cylinder of radius $d > 0$. The collection of spatial lines passing through x and having a minimal distance d from $\boldsymbol{l}$ is a flat pencil with base point x, such that the pencil is tangent to the radius-d cylinder at x, as depicted in Figure 3.3(a). The lines of the flat pencil are parametrized by their angle α with respect to the line $\boldsymbol{l}$. Note that each twist line has a specific pitch determined by the formula $z_2 = s \cdot d \tan(\alpha)$.

The special case of $d = 0$ corresponds to points x which lie on the line $\boldsymbol{l}$. The twist lines at such points are the union of the flat pencils on neighboring cylinders whose radius d approaches zero. Since the flat pencils completely surround each point x on $\boldsymbol{l}$, their union is a solid pencil based at x. Since $z_2 = d \tan(\alpha) = 0$ along $\boldsymbol{l}$, all lines of the solid pencil have zero pitch and are therefore pure instantaneous rotations; see Figure 3.3(b). Note that instantaneous rotations about lines parallel to the tangent plane at the contact point become instantaneous translations as x moves to infinity along the line $\boldsymbol{l}$. To summarize, the instantaneous twists corresponding to $T_{q_0}\mathcal{S}$ consist of flat pencils tangent to the cylinders centered on the line $\boldsymbol{l}$, as well as solid pencils strung along $\boldsymbol{l}$, as depicted in Figure 3.3(a)-(b).

Finally, let (ρ, φ) parameterize each cylinder and let (t_1, t_2) be orthogonal unit tangent vectors at x, as depicted in Figure 3.3(a). The collection of radius-d cylinders is parameterized by $p(\rho, d, \varphi) = x + \rho n(x) + d[t_1\, t_2]\binom{cos\varphi}{\sin\varphi}$. The direction of the lines comprising each flat pencil is parametrized by $\hat{l}(\varphi, \alpha) = \cos(\alpha) n(x) + \sin(\alpha)[t_1\, t_2]\binom{-sin\varphi}{\cos\varphi}$. The parametrization of $T_{q_0}\mathcal{S}$ in terms of instantaneous twists is thus given by

$$T_{q_0}\mathcal{S} = \left\{ \dot{q} = \|\omega\| \begin{pmatrix} \hat{l}(\varphi, \alpha) \\ p(\rho, d, \varphi) \times \hat{l}(\varphi, \alpha) \end{pmatrix} : \ d, \rho \in I\!R_+,\ \omega, \varphi, \alpha \in I\!R \right\},$$

where $I\!R_+$ denotes the positive reals. We see that $T_{q_0}\mathcal{S}$ is spanned by five twist parameters, which is consistent with the fact that it is a five-dimensional linear space.

Do

`The pitch` z_2 `need not be zero when` $\cos\alpha = 0$`. I.e, twist lines parallel to the tangent plane at` x `can have an arbitrary` z_2`.`

Bibliographical Notes

The terms wrench and twist are discussed in a paper by B. Roth [3], titled "screws, motors, and wrenches that cannot be bought in a hardware store." Line geometry is the main tool used to analyze the kinematics of open and closed chain mechanisms e.g., [**?**, 5]. Recommended sources for line geometry in the context of mechanism theory is McCarthy's [1] and Selig's [4] books.

Exercises

Exercise: The *Plücker coordinates* of a spatial line l are defined as the vector $(\hat{l}, p \times \hat{l}) \in I\!R^6$. Provide a geometric interpretation for the term $p \times \hat{l}$.

Solution: The term $p \times \hat{l}$, traditionally called the line's "arm," is the vector orthogonal to l and having its tip on l.

Exercise 3.recip: Prove by direct computation the reciprocal product formula specified in Proposition 3.3.1.

Solution: By definition of the reciprocal product, $(\hat{l}_1, p_1 \times \hat{l}_1) \cdot (p_2 \times \hat{l}_2, \hat{l}_2) = \hat{l}_1 \cdot (p_2 \times \hat{l}_2) + (p_1 \times \hat{l}_1) \cdot \hat{l}_2$. Since $\hat{l}_1 \cdot (p_2 \times \hat{l}_2) = -p_2 \cdot (\hat{l}_1 \times \hat{l}_2)$ and $(p_1 \times \hat{l}_1) \cdot \hat{l}_2 = p_1 \cdot (\hat{l}_1 \times \hat{l}_2)$ by the triple scalar product identity, $(\hat{l}_1, p_1 \times \hat{l}_1) \cdot (p_2 \times \hat{l}_2, \hat{l}_2) = -(p_2 - p_1) \cdot (\hat{l}_1 \times \hat{l}_2)$. In the latter inner product $\hat{l}_1 \times \hat{l}_2$ is orthogonal to l_1 and l_2. Since p_1 and p_2 may freely vary along l_1 and l_2 without affecting the inner product, we may assume that the segment $p_2 - p_1$ is collinear with $\hat{l}_1 \times \hat{l}_2$, so that $\|p_2 - p_1\| = d$. Based on this argument, $-(p_2 - p_1) \cdot (\hat{l}_1 \times \hat{l}_2) = -s\|p_2 - p_1\|\|\hat{l}_1 \times \hat{l}_2\| =$

$-s \cdot d\sin(\alpha)$, where $s = \pm 1$ is the sign of $(p_2 - p_1) \cdot (\hat{l}_1 \times \hat{l}_2)$ and α is the angle between $\hat{l}_1$ and $\hat{l}_2$.

Exercise: The collection of all spatial lines in $I\!R^3$ forms a smooth four-dimensional manifold in $I\!R^6$. Justify this statement by locally parametrizing the spatial lines in terms of four scalar parameters.

Solution: Since the collection of spatial lines forms a four-dimensional manifold, every line l is surrounded by a neighborhood of lines parametrized by four scalar parameters. Let l have a direction $\hat{l} \in S^2$ (S^2 being the unit sphere), and let L be the plane orthogonal to $\hat{l}$ through the origin of $I\!R^3$. The local neighborhood of l in the four-dimensional manifold can be parametrized by the lines' intersection point with L in the vicinity of the origin, together with a local neighborhood of directions surrounding $\hat{l}$ on S^2.

Exercise: The *Plücker coordinates* of l are given by $(\hat{l}, p \times \hat{l}) \in I\!R^6$, where p is a point on l and $\hat{l}$ is the unit direction of l. Since the collection of spatial lines forms a four-dimensional manifold, the Plücker coordinates contain redundancies. Identify the two scalar constraints which are always satisfied by the *Plücker coordinates* in $I\!R^6$.

Solution: Let $x_1, \ldots, x_6$ be the coordinates of $I\!R^6$. The first three coordinated represent a unit magnitude direction and hence must lie on the unit sphere. The first scalar constraint is thus $x_1^2 + x_2^2 + x_3^2 = 0$. Since $\hat{l} \cdot (p \times \hat{l}) = 0$, the second scalar constraint is given by $x_1x_4 + x_2x_5 + x_3x_6 = 0$.

Exercise: Let l_1, l_2, l_3 be three planar lines having 2D *Plücker coordinates* $l_i = (\hat{l}_i, p_i \times \hat{l}_i)$ for $i = 1, 2, 3$. Prove that the three lines intersect at a common point iff

$$\det \begin{bmatrix} \hat{l}_1 & \hat{l}_2 & \hat{l}_3 \\ p_1 \times \hat{l}_1 & p_2 \times \hat{l}_2 & p_3 \times \hat{l}_3 \end{bmatrix} = 0.$$

Solution: Let the three lines be embedded in the $x_3 = 1$ plane of $I\!R^3$, where (x_1, x_2, x_3) are the coordinates of $I\!R^3$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be the planes through the origin of $I\!R^3$ determined by the three lines, as depicted in Figure 3.1(b). The three lines intersect at a common point in $I\!R^2$ iff $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ intersect along a common line in $I\!R^3$. The latter condition is equivalent to the requirement that the planes' normals be linearly dependent. The pair $\{(p_i, 1), (\hat{l}_i, 0)\}$ is a basis for $\mathcal{P}_i$ $(i = 1, 2, 3)$. Hence the normal to $\mathcal{P}_i$ is given by $(p_i, 1) \times (\hat{l}_i, 0) = (J\hat{l}_i, p_i \times \hat{l}_i)$ $(i = 1, 2, 3)$. The three normals are linearly dependant iff

$$\det \begin{bmatrix} J\hat{l}_1 & J\hat{l}_2 & J\hat{l}_3 \\ p_1 \times \hat{l}_1 & p_2 \times \hat{l}_2 & p_3 \times \hat{l}_3 \end{bmatrix} = -\det \begin{bmatrix} \hat{l}_1 & \hat{l}_2 & \hat{l}_3 \\ p_1 \times \hat{l}_1 & p_2 \times \hat{l}_2 & p_3 \times \hat{l}_3 \end{bmatrix} = 0.$$

Exercise: Determine the screw axis l and the pitch z of the wrench generated by a pure force f acting on $\mathcal{B}$ at a point $x = X_b(q_0)$. Under what condition on the frames $\mathcal{F}_W$ and $\mathcal{F}_B$ the screw line l can be interpreted as the force line through x?

Solution: The wrench generated by a pure force f acting on $\mathcal{B}$ at point$x = X_b(q_0)$ is given by $\boldsymbol{w} = (f, R_0 b \times f)$. Hence its screw axis is the line $l = (\hat{f}, R_0 b \times \hat{f})$. Since $x = R_0 b + d_0$, the frames $\mathcal{F}_W$ and $\mathcal{F}_B$ must share a common origin at q_0. In this case $d_0 = \vec{0}$, and the wrench can e written as $\boldsymbol{w} = (f, x \times f)$. The screw axis of this wrench is the line $l = (\hat{f}, x \times \hat{f})$, which passes through the contact point x along the force direction $\hat{f}$.

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