Chapter 7

First-Order Immobilizing Grasps

Immobilizing grasps maintain grasp security by preventing any movement of the grasped object with respect to the finger bodies. Immobilizing grasps are universally used in industrial fixturing applications, and they provide the simplest approach for maintaining grasp security during robotic grasping and manipulation tasks. Immobilization theory begins with the observation made in the previous chapter, that object immobilization can only be achieved with *frictionless equilibrium grasps.* Thus consider a rigid object, $\mathcal{B}$, held in a frictionless equilibrium grasp by stationary rigid finger bodies, $\mathcal{O}_1, \ldots, \mathcal{O}_k$. While friction is allowed at the contacts, grasp immobilization seeks to secure the object based on the rigid-body constraints imposed on the object's free motions by the finger bodies.

This chapter focuses on the first-order geometry of grasp immobilization. Section 7.1 describes the *first-order free motions* available to $\mathcal{B}$ with respect to the finger bodies. These free motions are defined in $\mathcal{B}$'s c-space and represent the object's instantaneous motions that break or maintain contact with the finger bodies. Section 7.2 considers the object's first-order free motions at a frictionless equilibrium grasp. In this case the first-order free motions span a subspace of tangent vectors in $\mathcal{B}$'s c-space, and the dimension of this subspace defines the grasp 1^{st}*-order mobility index.* When the 1^{st}-order mobility index is zero, the object is fully immobilized and hence secured by the grasping fingers. Section 7.3 describes a graphical technique for determining the 1^{st}-order mobility index of a given grasp arrangement. The graphical technique will help us recognize important limitations of the 1^{st}-order mobility index.

The notion of first-order immobilization is based on the bodies' first-order geometric properties. However, grasp immobilization can also be achieved by a combination of first and second-order geometric effects. When the bodies' curvature is taken into account, a grasp which has a positive 1^{st}-order mobility index (and is thus mobile according to the first-order theory) can be perfectly immobilizing. This part of grasp mobility theory is considered in Chapter 8. Techniques for synthesizing immobilizing grasps are considered in Chapter 9.

7.1 The First-Order Free Motions

Grasp mobility analysis is based on the *free motions* available to $\mathcal{B}$ at a given grasp arrangement. When $\mathcal{B}$ is contacted by stationary finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$, the finger c-obstacles, $\mathcal{CO}_1, \ldots, \mathcal{CO}_k$, constraint the object's possible motions. The object is free to move within the *free c-space*, $\mathcal{F}$, which is given by

$$\mathcal{F} = \mathbb{R}^m - \cup_{i=1}^{k} \mathsf{int}(\mathcal{CO}_i) \qquad m=3 \text{ in 2D and } m=6 \text{ in 3D},$$

where $\mathsf{int}(\mathcal{CO}_i)$ denotes the interior of $\mathcal{CO}_i$. When $\mathcal{B}$ is held at a configuration q, its *free motions* are those c-space paths which emanate from q and locally lie in $\mathcal{F}$. To analyze the object's free motions in the c-space framework, we will model the finger c-obstacles with the following c-space distance function.

Definition 1. *Let $\mathcal{CO}_i$ be the c-obstacle associated with a finger body $\mathcal{O}_i$, and let $\mathcal{S}_i$ be the boundary of $\mathcal{CO}_i$. The* **c-space distance function,** *$d_i(q) : \mathbb{R}^m \rightarrow \mathbb{R}$, is the signed distance from $\mathcal{S}_i$,*

$$d_i(q) = \begin{cases} -\mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \in \mathsf{int}(\mathcal{CO}_i) \\ 0 & \text{if } q \in \mathcal{S}_i \\ \mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \in \mathbb{R}^m - \mathcal{CO}_i, \end{cases} \tag{7.1}$$

where $\mathtt{dst}(q, \mathcal{S}_i) = \min_{p \in \mathcal{S}_i} \{\|q-p\|\}$ is the minimal Euclidean distance of q from $\mathcal{S}_i$.

The i^{th} finger c-obstacle boundary, $\mathcal{S}_i$, is the zero level set of d_i. We shall assume that $\mathcal{S}_i$ is a smooth $(m-1)$-dimensional manifold. This assumption is met when $\mathcal{B}$ and $\mathcal{O}_i$ have smooth surfaces. In more general cases, for instance when $\mathcal{B}$ is a polygon, $\mathcal{S}_i$ is a piecewise smooth manifold. Such cases can be worked out with the non-smooth analysis tools described in Appendix I (see exercise 1). When $\mathcal{S}_i$ is smooth, ∇d_i is well defined in a neighborhood of $\mathcal{S}_i$. By construction d_i is negative inside $\mathcal{CO}_i$, zero on $\mathcal{S}_i$, and positive outside $\mathcal{CO}_i$. Hence $\nabla d_i(q)$ is collinear with the c-obstacle outward normal, $\eta_i(q)$, at all points $q \in \mathcal{S}_i$. Moreover, d_i is defined in terms of the Euclidean distance, and consequently $\|\nabla d_i(q)\| = 1$ (exercise 2). It follows that $\nabla d_i(q) = \hat{\eta}_i(q)$ at all points $q \in \mathcal{S}_i$, where $\hat{\eta}_i = \eta_i / \|\eta_i\|$.

We now formulate the first-order properties of the free motion curves, which will lead to a *first-order mobility theory.* Let $\alpha(t)$ be a smooth c-space curve such that $\alpha(0) = q \in \mathcal{S}_i$ and $\dot{\alpha}(0) = \dot{q}$. The first-order Taylor expansion of d_i along α is given by

$$d_i(\alpha(t)) = d_i(q) + (\nabla d_i(q) \cdot \dot{q})t + \mathrm{o}(t^2) \qquad t \in (-\epsilon, \epsilon),$$

where $\epsilon > 0$ is a small parameter. When this expression is evaluated at $q \in \mathcal{S}_i$, the first-order approximation of d_i becomes

$$d_i(\alpha(t)) = (\hat{\eta}_i(q) \cdot \dot{q})t + \mathrm{o}(t^2) \qquad t \in (-\epsilon, \epsilon), \tag{7.2}$$

where we substituted $d_i(q) = 0$ and $\nabla d_i(q) = \hat{\eta}_i(q)$. The first-order free motions are characterized in terms of this first-order approximation as follows.

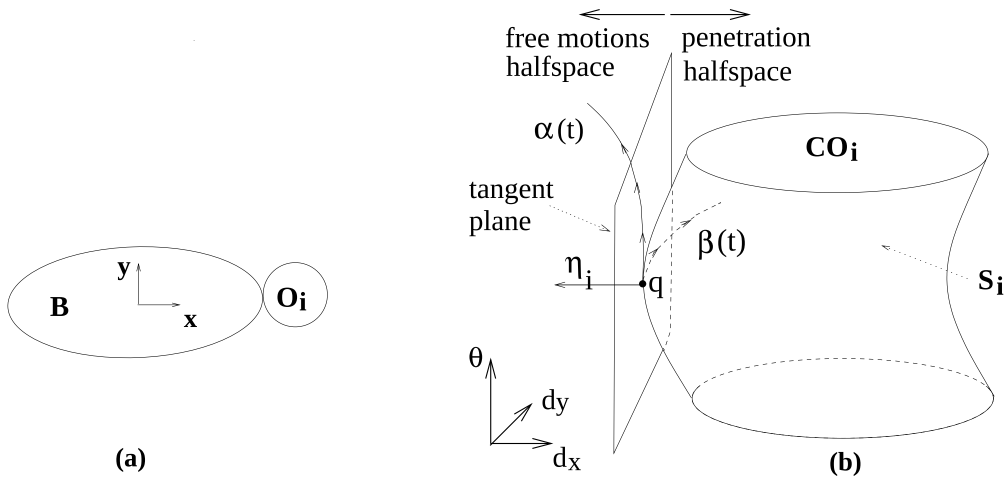

Figure 7.1: (a) An ellipse $\mathcal{B}$ contacted by a disc finger $\mathcal{O}_i$, and (b) the halfspace $M_i(q)$ approximating the exterior of $\mathcal{CO}_i$ at q. While α and β are roll-slide motions to first order, α locally lies in $\mathcal{F}$ while β locally lies in $\mathcal{CO}_i$.

Definition 2 (Single Finger 1'st Order Free Motions). *Let the object $\mathcal{B}$ be contacted by a finger body $\mathcal{O}_i$ at a configuration $q \in \mathcal{S}_i$. The* **first order free motions** *of $\mathcal{B}$ at q is the halfspace of $T_q I\!R^m$,*

$$M_i(q) = \{\dot{q} \in T_q I\!R^m : \hat{\eta}_i(q) \cdot \dot{q} \geq 0\}.$$

The halfspace's boundary, $T_q\mathcal{S}_i = \{\dot{q} \in T_q I\!R^m : \hat{\eta}_i(q) \cdot \dot{q} = 0\}$, [1] *is the set of* **first order roll-slide motions***, while its interior, $\{\dot{q} \in T_q I\!R^m : \hat{\eta}_i(q) \cdot \dot{q} > 0\}$, is the set of* **first order escape motions***.*

Example: Figure 7.1(a) shows an ellipse $\mathcal{B}$ contacted by a disc finger $\mathcal{O}_i$. Figure 7.1(b) shows the c-space geometry of this grasp, with $q \in \mathcal{S}_i$ being the ellipse's contact configuration. The halfspace $M_i(q)$ forms the first-order approximation to the exterior of the finger c-obstacle, $\mathcal{CO}_i$, at q. In particular, the halfspace's boundary coincides with the c-obstacle tangent plane, $T_q\mathcal{S}_i$. Tangent vectors pointing into the interior of $M_i(q)$ represent first-order escape motions, while tangent vectors in $T_q\mathcal{S}_i$ represent first order roll-slide motions. ·

The first-order free motions admit the following geometric interpretation. When $\dot{q} \in M_i(q)$ is a first-order escape motion, $\dot{q}$ points into the interior of $M_i(q)$. In this case any path $\alpha(t)$ with $\alpha(0) = q$ and $\dot{\alpha}(0) = \dot{q}$ *locally lies in the free c-space* for all $t \in [0, \epsilon]$. Motion of $\mathcal{B}$ along this path would cause it to separate from $\mathcal{O}_i$, no matter what the value of the higher derivatives of α. On the other hand, when $\dot{q} \in M_i(q)$ is a first order roll-slide motion, $\dot{q}$ is tangent to the finger c-obstacle boundary. In this case it is *not possible* to determine based on first-order considerations wether α locally lies in $\mathcal{F}$ or penetrates the finger c-obstacle $\mathcal{CO}_i$. This indeterminacy is illustrated in the following example.

Example: Consider the curves α and β depicted in Figure 7.1(b). Both curves start at $q \in \mathcal{S}_i$, and have the same initial tangent vector, $\dot{\alpha}(0) = \dot{\beta}(0) \in T_q\mathcal{S}_i$. Both curves are thus

[1] $T_q\mathcal{S}_i \cong I\!R^{m-1}$ is the tangent space of $\mathcal{S}_i$ at q.

equivalent to first order, and represent the same first order roll-slide motion. Yet α locally lies in the free c-space, while β locally penetrates the finger c-obstacle. As we shall see in Section 7.2, all the free motions available to $\mathcal{B}$ at a frictionless equilibrium grasp are roll-slide to first order, much like the two curves depicted in the figure.·

The definition of the first-order free motions is next extended to multi-finger grasps. When $\mathcal{B}$ is contacted by k finger bodies, its first-order free motions must respect the k halfspace constraints imposed by these fingers, as stated in the following definition.

Definition 3 (Multi-Finger 1'st Order Free Motions). *Let the object $\mathcal{B}$ be contacted by k finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$ at a configuration $q \in \cap_{i=1}^{k} \mathcal{S}_i$. The set of first-order free motions of $\mathcal{B}$ at q, denoted $M_{1\ldots k}(q)$, is given by*

$$M_{1\ldots k}(q) = \cap_{i=1}^{k} M_i(q) = \{\dot{q} \in T_q I\!\!R^m \,:\, \eta_i(q) \cdot \dot{q} \geq 0 \;\; \textit{for } i = 1 \ldots k\}\,.$$

The set $M_{1\ldots k}(q)$ forms a convex cone based at $\mathcal{B}$'s tangent space origin. This property follows from the observation that each halfspace $M_i(q)$ is a convex cone based at $\mathcal{B}$'s tangent space origin, and the general fact that the intersection of convex cones based at the origin is a convex cone based at this point.

The set $M_{1\ldots k}(q)$ is defined in terms of the Euclidean inner product, which is usually *not* preserved by coordinate transformations. Hence we must verify that $M_{1\ldots k}(q)$ is coordinate invariant. Any reasonable coordinate transformation is expected to form a *diffeomorphism*—a differentiable one-to-one and surjective map, whose Jacobian is non-singular at all points of the domain. The proof of the following proposition appears in the appendix.

Proposition 7.1.1 (Coordinate Invariance). *Let q and $\bar{q}$ be two parametrizations of $\mathcal{B}$'s c-space, related by a coordinate transformation $q = h(\bar{q})$. If h is a diffeomorphism, the set of first-order free motions is coordinate invariant*

$$\dot{q} \in M_{1\ldots k}(q) \quad \textit{iff} \quad \dot{\bar{q}} \in \overline{M}_{1\ldots k}(\bar{q}),$$

where $M_{1\ldots k}$ and $\overline{M}_{1\ldots k}$ are the first-order free motion cones in the q and $\bar{q}$ spaces.

In grasp mechanics, different choices of the world and object frames, $\mathcal{F}_W$ and $\mathcal{F}_B$, give different parametrizations of $\mathcal{B}$'s c-space. As verified in exercise 4, the coordinate transformation induced by different frame choices are related by a standard diffeomorphism. The object's first-order free motions can thus be analyzed under any choice of world and object frames.

Physical interpretation of the first-order free motions coordinate invariance: The coordinate invariance of $M_{1\ldots k}(q)$ can be physically justified as follows. The c-obstacle outward normal, $\hat{\eta}_i(q)$, is collinear with the wrench generated by a normal finger force acting on $\mathcal{B}$ at the i^{th} contact. The inner product $\hat{\eta}_i(q) \cdot \dot{q}$ represents the *work* done by the wrench $\hat{\eta}_i(q) \in T_q^* I\!\!R^m$ on tangent vectors $\dot{q} \in T_q I\!\!R^m$. From this perspective, $\hat{\eta}_i(q) \cdot \dot{q} = 0$ and $\hat{\eta}_i(q) \cdot \dot{q} \geq 0$ represent conditions on the *sign* of the work done by $\hat{\eta}_i(q)$ on tangent vectors. This physical notion does not depend on the specific choice of $\mathcal{F}_W$ and $\mathcal{F}_B$.

7.2 The 1^{st}-Order Mobility Index

A grasp *mobility index* is a coordinate invariant integer that measures the object's mobility, or effective number of degrees of freedom, at a frictionless equilibrium grasp. A grasp 1^{st}-order mobility index is based on the object's first-order free motions at the equilibrium grasp. The index is well defined when the fingers form an *essential* equilibrium grasp, which is a generic property of the equilibrium grasps. Let us therefore begin with the notion of essential grasps (which already appeared in Chapter 5).

Definition 4 (**Essential Finger**). *Let the object $\mathcal{B}$ be held in a k-finger equilibrium grasp. A finger $\mathcal{O}_i$ is* **essential** *if its wrench is necessary for maintaining the equilibrium grasp.*

Essential fingers can be identified as follows. A k-finger arrangement forms a feasible equilibrium grasp iff the fingers' net wrench cone contains a full subspace passing through $\mathcal{B}$'s wrench space origin (Proposition **??**). A particular finger is thus essential when the net wrench cone spanned by the remaining $k-1$ fingers is a *pointed cone*[2] in the object's wrench space. The notion of essential fingers is next extended to k-finger grasps.

Definition 5 (**Essential Equilibrium Grasp**). *The object $\mathcal{B}$ is held in an* **essential equilibrium grasp** *by k frictionless fingers under one of the following conditions:*
(i) There are $k \leq m+1$ fingers, and all k fingers are essential for the equilibrium grasp;
(ii) There are $k > m+1$ fingers, and $m+1$ of the k fingers are essential for the equilibrium grasp; where $m = 3$ in the 2D case and $m = 6$ in the 3D case.

The essential equilibrium grasps are generic in the following sense. Recall that *contact c-space* parametrizes the position of the k contacts, while the k-finger frictionless equilibrium grasps form a subset, $\mathcal{E}$, in this space. When a k-finger equilibrium grasp is essential, any local perturbations of the contacts within the equilibrium set $\mathcal{E}$ would give an essential equilibrium grasp (exercise 8). On the other hand, non-essential equilibrium grasps represent special finger arrangements that can be locally perturbed into essential grasps. This property is illustrated in the following example.

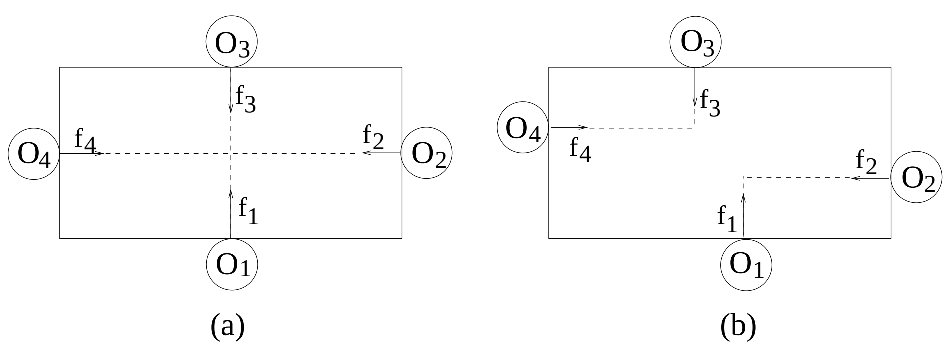

Figure 7.2: (a) A non-essential equilibrium grasp can be perturbed into (b) an essential equilibrium grasp.

[2] I.e., a cone based at the origin which does not contain a full subspace passing through the origin.

Example: Consider the four-finger grasp of a rectangular object shown in Figure 7.2(a). It is a non-essential equilibrium grasp, since the equilibrium can be maintained with one of the antipodal pairs $(\mathcal{O}_1, \mathcal{O}_3)$ or $(\mathcal{O}_2, \mathcal{O}_4)$. However, this grasp can be locally perturbed into an essential equilibrium grasp. As shown in Figure 7.2(b), the grasp resulting from counterclockwise perturbation of $(\mathcal{O}_1, \mathcal{O}_3)$ accompanied by clockwise perturbation of $(\mathcal{O}_2, \mathcal{O}_4)$ is an essential equilibrium grasp. Moreover, the grasp depicted in Figure 7.2(b) remains an essential equilibrium grasp under all sufficiently small perturbations of its contacts.·

When $\mathcal{B}$ is held in an essential equilibrium grasp, its first-order free motions span a subspace tangent to the finger c-obstacles. This fundamental property of the first-order free motions is stated in the following proposition (see appendix for a proof).

Proposition 7.2.1 (Subspace Property). *Let $\mathcal{B}$ be held in an essential k-finger equilibrium grasp at a configuration q_0. The object's first-order free motions, $M_{1\ldots k}(q_0)$, span a* **subspace** *tangent to the finger c-obstacles, whose dimension is given by*

$$\dim\big(M_{1\ldots k}(q_0)\big) = \max\{m-k+1, 0\},$$

where $m=3$ in 2D and $m=6$ in 3D.

Since $M_{1\ldots k}(q_0)$ is tangent to the finger c-obstacles, the only first-order free motions available to $\mathcal{B}$ at a frictionless equilibrium grasp are *roll-slide motions* with respect to each of the finger bodies. The dimension of the subspace $M_{1\ldots k}(q_0)$ defines the object's 1^{st}-order mobility index.

Definition 6 (1^{st}-Order Mobility Index). *Let $\mathcal{B}$ be held in an essential k-finger equilibrium grasp at a configuration q_0. The* **1^{st}-order mobility index** *of $\mathcal{B}$, denoted $m^1_{q_0}$, is the dimension of the subspace spanned by the object's first-order free motions,*

$$m^1_{q_0} = \dim\big(M_{1\ldots k}(q_0)\big) = \max\{m-k+1, 0\}, \tag{7.3}$$

where $m=3$ in 2D and $m=6$ in 3D.

The 1^{st}-order mobility index is *coordinate invariant*, since the set $M_{1\ldots k}(q_0)$ is coordinate invariant according to Proposition 7.1.1. The index can attain values in the range $0 \leq m^1_{q_0} \leq m$, as illustrated in the following examples.

Example: Figure 7.3(a) shows an ellipse held in a frictionless equilibrium grasp by two disc fingers. Figure 7.3(b) shows the c-space geometry of this grasp, with $q_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ being the equilibrium grasp configuration. Substituting $m = 3$ and $k = 2$ into (7.3) gives: $m^1_{q_0} = \max\{2, 0\} = 2$. The subspace $M_{1,2}(q_0)$ is thus two-dimensional, and it forms the tangent plane common to $\mathcal{S}_1$ and $\mathcal{S}_2$ at q_0. Every tangent vector $\dot{q} \in M_{1,2}(q_0)$ represents an instantaneous rotation of $\mathcal{B}$ about the midpoint of the segment connecting the two contacts, combined with an instantaneous translation of $\mathcal{B}$ along the direction perpendicular to this segment. We shall see in the next section that the tangent vectors of $M_{1,2}(q_0)$ can be graphically depicted as a one-parameter family of instantaneous rotations. ·

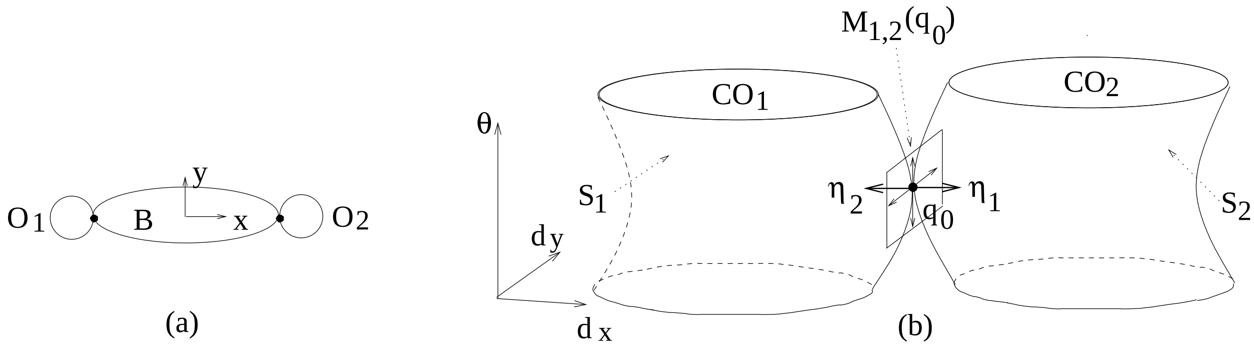

Figure 7.3: (a) An ellipse held in a 2-finger equilibrium grasp. (b) The two-dimensional subspace $M_{1,2}(q_0)$ at the equilibrium grasp.

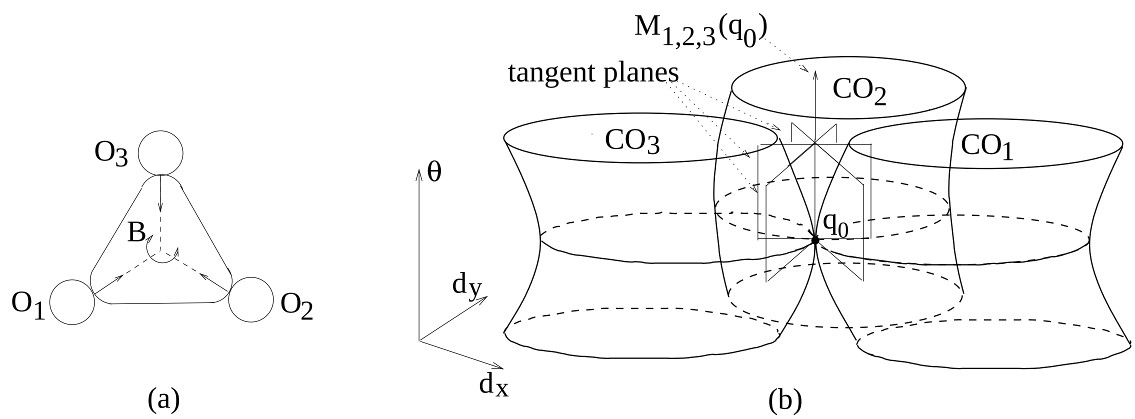

Figure 7.4: (a) A triangular object held in a 3-finger equilibrium grasp. (b) The one-dimensional subspace $M_{1,2,3}(q_0)$ at the equilibrium grasp.

Example: Figure 7.4(a) shows a triangular object held in a frictionless equilibrium grasp by three disc fingers. Figure 7.4(b) shows the c-space geometry of this grasp, with $q_0 \in \cap_{i=1}^{3} \mathcal{S}_i$ being the equilibrium grasp configuration. Substituting $m = 3$ and $k = 3$ into (7.3) gives: $m_{q_0}^1 = \max\{1, 0\} = 1$. The subspace $M_{1,2,3}(q_0)$ is the one-dimensional intersection of the tangent planes to $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ at q_0. Based on a graphical technique discussed in the next section, every tangent vector $\dot{q} \in M_{1,2,3}(q_0)$ represents an instantaneous rotation of $\mathcal{B}$ about the intersection point of the finger contact normals. When the world and object frame origins are located at this point, $M_{1,2,3}(q_0)$ forms the vertical line parallel to the c-space θ axis depicted Figure 7.4(b). ·

Since m attains one of two fixed values, the 1^{st}-order mobility index depends solely on the number of fingers k. As the number of fingers increases, the value of $m_{q_0}^1$ (and hence the amount of first-order free motions available to $\mathcal{B}$) decreases. When $m_{q_0}^1 = 0$, the object has no first-order free motions and should therefore be fully immobilized by the finger bodies. This important means of achieving object immobilization is stated in the following theorem.

Theorem 1 (1'st-Order Immobilization). *Let $\mathcal{B}$ be held in an essential k-finger equilib-*

rium grasp at a configuration q_0. If the 1^{st}-order mobility index satisfies

$$m^1_{q_0} = 0,$$

the object is **fully immobilized** *by the grasping finger bodies.*

Proof: We have to show that the object's equilibrium grasp configuration, q_0, is completely surrounded by the finger c-obstacles. We will prove this fact using the minimum distance function:

$$d_{min}(q) = \min\{d_1(q), \ldots, d_k(q)\} \qquad q \in I\!R^m,$$

where $d_i(q)$ is the signed distance of q from $\mathcal{CO}_i$ as specified in (7.1) ($i=1\ldots k$). While d_{min} is non-differentiable, it is Lipschitz continuous and therefore can be analyzed with the tools described in Appendix A. In particular, d_{min} possesses a *generalized gradient* at q_0, denoted $\partial d_{min}(q_0)$, which is the convex combination of the gradients $\nabla d_i(q_0)$ for $i = 1\ldots k$. Each $\nabla d_i(q_0)$ is collinear with the outward unit normal to $\mathcal{CO}_i$, $\nabla d_i(q_0) = \hat{\eta}_i(q_0)$ for $i=1\ldots k$. The generalized gradient can thus be expressed as the convex combination:

$$\partial d_{min}(q_0) = \sum_{i=1}^{k} \lambda_i \hat{\eta}_i(q_0) \qquad 0 \leq \lambda_i \leq 1 \text{ for } i = 1\ldots k \text{ and } \textstyle\sum_{i=1}^{k} \lambda_i = 1. \tag{7.4}$$

According to Theorem A.**??**, d_{min} has a *strict local maximum* at q_0 when $\partial d_{min}(q_0)$ contains an m-dimensional ball centered at the origin of $T^*_{q_0} I\!R^m$. By definition, $M_{1\ldots k}(q_0) = \{0\}$ when $m^1_{q_0} = 0$. Hence for any $\dot{q} \in T_{q_0} I\!R^m$ there exists $\hat{\eta}_i(q_0)$ such that $\hat{\eta}_i(q_0) \cdot \dot{q} < 0$. The latter inequality can be interpreted as the condition that $\hat{\eta}_i(q_0)$ lies in the interior of the halfspace $H = \{\boldsymbol{w} \in T^*_{q_0} I\!R^m : \boldsymbol{w} \cdot \dot{q} < 0\}$, which is bounded by an $(m-1)$-dimensional plane passing through the origin of $T^*_{q_0} I\!R^m \cong I\!R^m$. Since $\hat{\eta}_i(q_0) \in \partial d_{min}(q_0)$ and $\dot{q}$ freely varies in $T_{q_0} I\!R^m$, $\partial d_{min}(q_0)$ intersects the interior of *every* halfspace H whose boundary passes through the origin. Since $\partial d_{min}(q_0)$ is a convex set in $T^*_{q_0} I\!R^m$, the latter condition can only occur when $\partial d_{min}(q_0)$ contains the origin in its interior (exercise 9). The function d_{min} thus has a strict local maximum at q_0.

Next observe that $d_{min}(q_0) = 0$ (since $q_0 \in \cap_{i=1}^{k} \mathcal{S}_i$). Therefore d_{min} is strictly negative in a small m-dimensional neighborhood centered at q_0. Since $d_{min}(q) = \min\{d_1(q), \ldots, d_k(q)\}$, at each point q in this neighborhood some $d_i(q) < 0$, which implies that q lies in the interior of $\mathcal{CO}_i$ for some $1 \leq i \leq k$. The point q_0 is thus completely surrounded by the finger c-obstacles, which proves that $\mathcal{B}$ is fully immobilized by the finger bodies. □

The theorem provides the following object immobilization technique. Construct a frictionless equilibrium grasp involving at least *four* fingers in 2D and *seven* fingers in 3D. If the fingers are all essential for the equilibrium grasp, the object is fully immobilized by the finger bodies. Moreover, essential equilibrium grasps are invariant under small contact perturbations (exercise 7). The first-order immobilizing grasps are thus *robust* with respect to small finger placement errors. This key property of first-order immobilizing grasps is summarized in the following corollary.

Corollary 7.2.2 (Robust Immobilization). *Let the object $\mathcal{B}$ be held in an essential equilibrium grasp by $k \geq 4$ fingers in 2D and $k \geq 7$ fingers in 3D. The object is* **fully immobilized,** *and the immobilization is* **robust** *with respect to small contact placement errors.*

A robust immobilization of a rectangular object is depicted in Figure 7.2(b). One may argue that under the frictionless point contact model, perfectly secure and robust grasps can always be achieved with *no more than four fingers in 2D,* and *no more than seven fingers in 3D.* These upper bounds are universally applied in critical applications such as high-accuracy fixturing systems, as discussed in the following example.

Missing Example: Discuss the industrial seven-fixel fixturing method.

However, Theorem 1 and its corollary do not preclude the possibility that the object be fully immobilized by a *smaller* number of fingers due to second-order geometric effects. This possibility is captured by the 2^{nd} order mobility index introduced in the next chapter.

7.3 Graphical Interpretation of the 1^{st}-Order Mobility Index

Let us describe a graphical technique for depicting the object's first-order free motions in planar equilibrium grasps. According to Chasles' Theorem (Theorem 2.3), every instantaneous motion of $\mathcal{B}$ can be described as an instantaneous twist. In the case of planar grasps, the object's instantaneous twists are rotations about an axis perpendicular to the plane (exercise 10). It follows that every tangent vector $\dot{q} \in T_{q_0}\mathbb{R}^3$ can be represented as an instantaneous rotation of $\mathcal{B}$ about a point $p \in \mathbb{R}^2$, called the *instantaneous rotation center.* The instantaneous rotation center representing $\dot{q}$ is given by (see exercise 11)

$$p = -\tfrac{1}{\omega} J v \qquad \dot{q} = (v, \omega) \in T_{q_0}\mathbb{R}^3 \text{ and } J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right],$$

where v and ω are $\mathcal{B}$'s linear and angular velocities. Using the identity $J^2 = -I$, the linear velocity v can be written as $v = \omega J p$ for $p \in \mathbb{R}^2$. It follows that the object's tangent space can be parametrized in terms of instantaneous rotation centers by

$$T_{q_0}\mathbb{R}^3 = \left\{ \dot{q} = \omega \begin{pmatrix} Jp \\ 1 \end{pmatrix} : p \in \mathbb{R}^2 \text{ and } \omega \in \mathbb{R} \right\}. \tag{7.5}$$

The parametrization (7.5) contains three free parameters, $p \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$, which is consistent with the fact that $T_{q_0}\mathbb{R}^3$ is a three-dimensional space. In particular, instantaneous rotations about points at infinity represent instantaneous translations of $\mathcal{B}$.

Let us next parametrize the halfspace of first-order free motions associated with a single finger body, $M_i(q_0) = \{\dot{q} \in T_{q_0}\mathbb{R}^3 : \eta_i(q_0) \cdot \dot{q} \geq 0\}$, in terms of instantaneous rotation centers. According to Lemma **??**, $\eta_i(q_0) = (n_i, x_i \times n_i)$ where x_i is the i^{th} contact point and n_i is $\mathcal{B}$'s inward unit normal at x_i. Using the parametrization (7.5), the halfspace $M_i(q_0)$ is

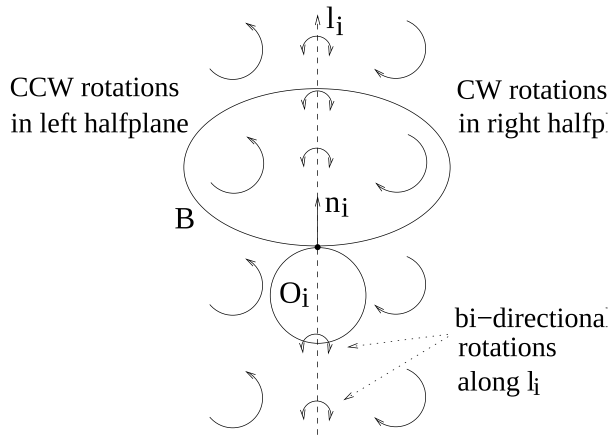

Figure 7.5: The halfspace $M_i(q_0)$ consists of CCW rotations in the left halfplane, CW rotations in the right halfplane, and bi-directional rotations along the contact normal line.

represented by the inequality

$$\eta_i(q_0) \cdot \dot{q} = \omega \begin{pmatrix} n_i \\ x_i \times n_i \end{pmatrix} \cdot \begin{pmatrix} Jp \\ 1 \end{pmatrix} = \omega\,(n_i \times (p - x_i)) \geq 0 \qquad p \in I\!R^2 \text{ and } \omega \in I\!R, \tag{7.6}$$

where $u \times v \in I\!R$ is computed with the formula $u \times v = u^T Jv$. The graphical depiction of the first-order free motions is based on (7.6) as summarized in the following procedure.

Graphical technique for depicting the first-order free motions: First consider the halfspace $M_i(q_0)$. Denote by l_i the directed line passing through x_i along n_i. As shown in Figure 7.5, the object's first-order escape motions are instantaneous counterclockwise rotations on the left side of l_i, and instantaneous clockwise rotations on the right side of l_i. The object's first order roll-slide motions are instantaneous bi-directional rotations about points along l_i. When $\mathcal{B}$ is contacted by finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$, the contact normal lines partition the plane into polygons. When the k contacts agree on the direction of rotation in a particular polygon, the object can escape the fingers along instantaneous motions indicated by this polygon. When $\mathcal{B}$ is held in a frictionless equilibrium grasp, $M_{1,\ldots,k}(q_0)$ forms a subspace according to Proposition 7.2.1. In this case every point in the *intersection* of the contact normal lines $l_1, \ldots, l_k$ is roll-slide to first order with respect to the k fingers. Since this instantaneous rotation can be freely multiplied by $\omega \in I\!R$, each of these points represents a one-dimensional subspace of $M_{1,\ldots,k}(q_0)$.

The following examples apply the graphical technique to several grasp arrangements.

Example: Figure 7.6 shows several two-finger equilibrium grasps of planar objects. In each of these grasps, the contact normal lines lie on common line, say l. According to the graphical technique, the object's first order roll-slide motions with respect to both fingers consist of instantaneous rotations about all points $p \in l$. Moreover, each of these instantaneous rotations is first order roll-slide for any angular velocity $\omega \in I\!R$. As p scans the entire line l and ω varies in $I\!R$, we obtain a two-parameter family of instantaneous rotations. This family represents the two-dimensional subspace of first order roll-slide motions, which is consistent

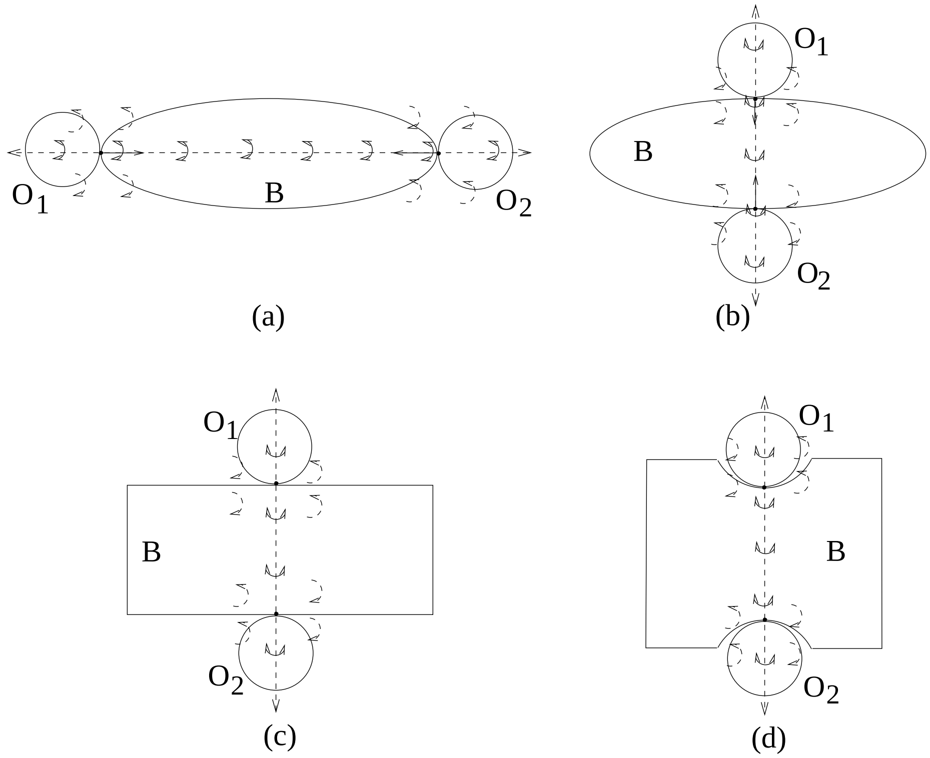

Figure 7.6: All 2-finger equilibrium grasps have the same 1^{st}-order mobility index of $m^1_{q0} = 2$.

with our earlier observation that $m^1_{q0} = 2$ for these grasps. The depicted objects are all first-order *mobile,* since $m^1_{q0} > 0$ in each of these grasps. However, intuition suggests that the grasp in Figure 7.6(a) is the most mobile, while the grasp in Figure 7.6(d) is the least mobile, and is in fact a fully immobilizing grasp. This intuitive observation is made precise using the 2^{nd} order mobility index introduced in the next chapter.·

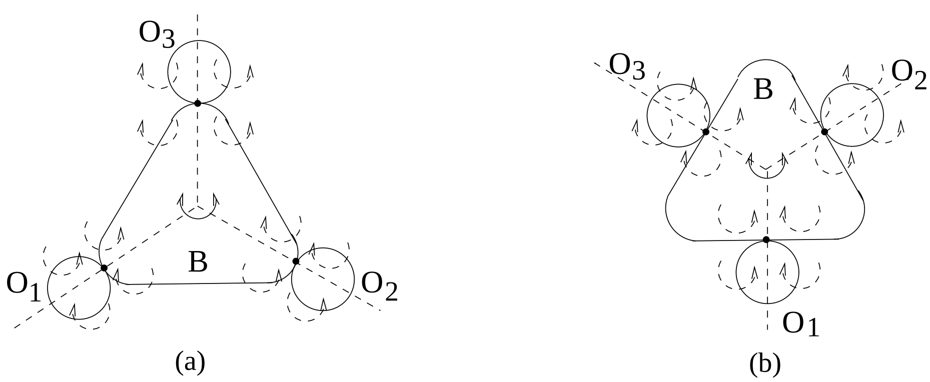

Figure 7.7: All 3-finger equilibrium grasps have the same 1^{st}-order mobility index of $m^1_{q0} = 1$.

Example: Figure 7.7 shows a triangular object held by three disc fingers in two alternative grasp arrangements. These are frictionless equilibrium grasps, since the finger force lines intersect at a common point and positively span the origin of $\mathbb{R}^2$. The contact normal lines intersect at a single point, say p. Based on the graphical technique, every first order roll-slide motion available to $\mathcal{B}$ is an instantaneous rotation about p with some $\omega \in \mathbb{R}$. This one-parameter family represents a one-dimensional subspace of first order roll-slide motions, which is consistent with the fact that $m^1_{q_0} = 1$ for these grasps. Here, too, the triangular object is first-order *mobile* in both grasps. However, intuition suggests that the triangular object is fully immobilized in Figure 7.7(b). This observation is justified when the finger and object curvatures are taken into account, as discused in the next chapter.·

The examples highlight serious limitations of the first-order immobilization approach. First, the number of fingers required to achieve full object immobilization based on first-order geometric effects seems to be excessively high for many grasping applications. While this type of immobilization is justified in high load fixturing applications, it may be too conservative for multi-fingered robot hands that typically operate under medium-to-low load conditions. Second, the 1^{st}-order mobility index is rather crude in its ability to assess object immobility. In particular, it cannot differentiate between equilibrium grasps involving the same number of fingers. The second-order mobility theory described in the next chapter will help us differentiate between different equilibrium grasp choices which are first-order equivalent.

7.4 Bibliographical Notes

In classical screw theory, first order roll-slide motions are represented by *reciprocal screws,* while the first-order escape motions are represented by *repelling screws* [4]. This terminology is associated with the parametrization of $\mathcal{B}$'s c-space in terms of *exponential coordinates* [3]. The notion of first-order free motions is valid under any parametrization of $\mathcal{B}$'s c-space, including exponential coordinates. The need to verify the coordinate invariance of structure containing inner products of tangent vectors (representing instantaneous motions) and cotangent vectors (representing wrenches or generalized forces) is discussed with examples in Ref. [2]. Finally, the theory of first-order immobilization extends to kinematic chains of interconnected rigid bodies. An extension of this theory to hinged polygonal chains is discussed in Ref. [1].

Appendix A: Proof Details

This appendix contains proofs of two key properties of the first-order free motions. The following proposition establishes the coordinate invariance of the first-order free motions.

Proposition 7.1.1. *Let q and $\bar{q}$ be two parametrizations of $\mathcal{B}$'s c-space, related by a coordinate transformation $q = h(\bar{q})$. If h is a diffeomorphism, the set of first-order free motions is coordinate invariant*

$$\dot{q} \in M_{1\ldots k}(q) \quad \textit{iff} \quad \dot{\bar{q}} \in \overline{M}_{1\ldots k}(\bar{q}),$$

where $M_{1\ldots k}$ and $\overline{M}_{1\ldots k}$ are the first-order free motion cones in the q and $\bar{q}$ spaces.

Proof: First consider the coordinate invariance of the halfspaces $M_i(q)$ for $i = 1 \ldots k$. Since $\bar{q}$ and q parametrize the same c-space, h must map $\overline{\mathcal{CO}}_i$ to $\mathcal{CO}_i$ and $\bar{\mathcal{S}}_i$ to $\mathcal{S}_i$ for $i = 1 \ldots k$. Let $\tilde{d}_i(\bar{q})$ be the composition of d_i with h, $\tilde{d}_i(\bar{q}) = d_i(h(\bar{q}))$ (note that $\tilde{d}_i$ is *not* the Euclidean distance in the $\bar{q}$ coordinates). Since $\tilde{d}_i$ is negative in the interior of $\overline{\mathcal{CO}}_i$, zero on $\bar{\mathcal{S}}_i$, and positive outside $\overline{\mathcal{CO}}_i$, $\nabla\tilde{d}_i(\bar{q})$ is the outward normal to $\overline{\mathcal{CO}}_i$ at points $\bar{q} \in \bar{\mathcal{S}}_i$. Let $\bar{\alpha}(t)$ be a path in $\bar{q}$-space such that $\bar{\alpha}(0) = \bar{q}$ and $\dot{\bar{\alpha}}(0) = \dot{\bar{q}}$. This path is mapped by h to a path α in q-space, $\alpha(t) = h(\bar{\alpha}(t))$ such that $\alpha(0) = q$ and $\dot{\alpha}(0) = Dh(\bar{q})\dot{\bar{q}} = \dot{q}$. Application of the chain rule to the identity

$$(d_i \circ \alpha)(t) = \big(d_i \circ (h \circ \bar{\alpha})\big)(t) = \big((d_i \circ h) \circ \bar{\alpha}\big)(t) = (\tilde{d}_i \circ \bar{\alpha})(t),$$

where '∘' denotes function composition, gives

$$\nabla d_i(q) \cdot \dot{q} = \nabla\tilde{d}_i(\bar{q}) \cdot \dot{\bar{q}} \qquad \text{for all } q = h(\bar{q}) \text{ and } \dot{q} = Dh(\bar{q})\dot{\bar{q}}.$$

Therefore $\dot{q} \in M_i(q)$ iff $\dot{\bar{q}} \in \overline{M}_i(\bar{q})$. Next consider the coordinate invariance of the cone $M_{1\ldots k}(q) = \cap_{i=1}^{k} M_i(q)$. Each $M_i(q)$ is a halfspace of $T_q I\!R^m$, and $M_i(q) = Dh(\bar{q})\big(\overline{M}_i(\bar{q})\big)$ for $i = 1 \ldots k$. This implies that

$$M_{1\ldots k}(q) = \cap_{i=1}^{k} M_i(q) = \cap_{i=1}^{k} Dh(\bar{q})\big(\overline{M}_i\big) = Dh(\bar{q})\big(\overline{M}_{1\ldots k}(\bar{q})\big),$$

since, in general, $g(\mathcal{U}_1 \bigcap \mathcal{U}_2) = g(\mathcal{U}_1) \bigcap g(\mathcal{U}_2)$ when g is an invertible function. In our case $g = Dh(\bar{q})$, and $Dh(\bar{q})$ is invertible since h is a diffeomorphism. □

The next proposition establishes that when $\mathcal{B}$ is held in a frictionless equilibrium grasp, its first-order free motions set forms a subspace tangent to the finger c-obstacles.

Proposition 7.2.1. *Let $\mathcal{B}$ be held in an essential k-finger equilibrium grasp at a configuration q_0. The object's first-order free motions, $M_{1\ldots k}(q_0)$, span a* **subspace** *tangent to the finger c-obstacles, whose dimension is given by*

$$\dim\big(M_{1\ldots k}(q_0)\big) = \max\{m - k + 1, 0\},$$

where $m = 3$ in 2D and $m = 6$ in 3D.

Proof: The net wrench cone generated by k frictionless finger wrenches is given by

$$\mathcal{W} = \big\{\boldsymbol{w} \in T^*_{q_0} I\!R^m \,:\, \boldsymbol{w} = \lambda_1\eta_1(q_0) + \cdots + \lambda_k\eta_k(q_0) \quad \text{for } \lambda_1 \ldots \lambda_k \geq 0\big\},$$

where $\eta_i(q_0)$ is the i^{th} finger c-obstacle outward normal for $i = 1 \ldots k$. Let us first show that when when $\mathcal{B}$ is held in a frictionless equilibrium grasp, $\mathcal{W}$ forms a subspace in $T^*_{q_0} I\!R^m$. By definition of essential grasps, either $k \leq m + 1$ and then all fingers are essential, or $k > m + 1$ and then $m + 1$ of the k fingers are essential. Let us therefore focus on k-finger grasp involving $k \leq m + 1$ essential fingers. The k finger wrenches are positively linearly dependent at the equilibrium grasp,

$$\lambda_1\eta_1(q_0) + \cdots + \lambda_k\eta_k(q_0) = \vec{0} \quad \lambda_1, \ldots, \lambda_k \geq 0, \quad k \leq m + 1. \tag{7.7}$$

Since all k fingers are essential, $\lambda_1, \ldots, \lambda_k$ must be strictly positive in (7.7), otherwise the equilibrium can be maintained with a sub-collection of $k-1$ fingers. Since $\lambda_1, \ldots, \lambda_k > 0$, it follows from (7.7) that every negated finger wrench, $-\eta_i(q_0)$, can be expressed as a positive linear combination of the remaining $k-1$ finger wrenches. Therefore, any linear combination of $\eta_1(q_0),\ldots,\eta_k(q_0)$ can be written with positive semi-definite coefficients, which is an element of $\mathcal{W}$. The net wrench cone is thus equal to the subspace spanned by $\eta_1(q_0),\ldots,\eta_k(q_0)$.

To show that $M_{1\ldots k}(q_0)$ spans a subspace, express $\mathcal{W}$ in terms of the negated finger wrenches, $\mathcal{W} = \{\boldsymbol{w} = \lambda_1(-\eta_1(q_0)) + \cdots + \lambda_k(-\eta_k(q_0)) : \lambda_1,\ldots,\lambda_k \geq 0\}$ (this expression is justified since $\mathcal{W}$ is a subspace). The cone *dual* to $\mathcal{W}$ is given by

$$\mathcal{W}^* = \{\dot{q}\in T_q I\!R^m : \boldsymbol{w}\cdot\dot{q} \leq 0 \text{ for all } \boldsymbol{w}\in\mathcal{W}\}.$$

Since $\mathcal{W}$ is a positive linear combination of $\{-\eta_i(q_0),\ldots,-\eta_k(q_0)\}$, its dual cone can be written as

$$\mathcal{W}^* = \{\dot{q}\in T_q I\!R^m : \eta_i(q_0)\cdot\dot{q} \geq 0 \text{ for } i=1\ldots k\} = M_{1\ldots k}(q_0).$$

In general, the cone dual to a subspace is the subspace's orthogonal complement. Hence $M_{1\ldots k}(q_0)$ forms a subspace in $T_{q_0} I\!R^m$, which is the orthogonal complement to $\mathcal{W}$.[3] The subspace $M_{1\ldots k}(q_0)$ is tangent to the finger c-obstacles at q_0, since every $\dot{q}\in M_{1\ldots k}(q_0)$ satisfies $\eta_i(q_0)\cdot\dot{q}=0$ for $i=1\ldots k$.

Finally consider the dimension of $M_{1\ldots k}(q_0)$. Since $M_{1\ldots k}(q_0)$ is the orthogonal complement of $\mathcal{W}$, $\dim\big(M_{1\ldots k}(q_0)\big) = m-\dim(\mathcal{W})$. To determine the dimension of $\mathcal{W}$, consider the $m\times k$ matrix $W = [\eta_1(q_0)\cdots\eta_k(q_0)]$. The rank of W is at most $k-1$, since $\lambda_1\eta_1(q_0)+\cdots+\lambda_k\eta_k(q_0) = \vec{0}$ at the equilibrium grasp. Suppose the rank of W is $k-2$, so that W has a two-dimensional kernel. In this case ker(W) contains the strictly positive vector $(\lambda_1,\ldots,\lambda_k)$, as well as a positive semi-definite vector $(\lambda'_1,\ldots,\lambda'_k)$ such that $\lambda'_i=0$ for some $1\leq i\leq k$. But this contradicts our assumption that all k fingers are essential for the equilibrium grasp. Therefore ker(W) is one-dimensional, the rank of W is $k-1$, and $\dim\big(M_{1\ldots k}(q_0)\big) = m-(k-1) = m-k+1$. □

Exercises

Exercise 7.1: Describe with Using the non-smooth analysis tools described in Appendix I, describe how ∇d_i is generalized to a *generalized gradient,* ∂d_i, at the non-smooth points of $\mathcal{S}_i$.·

Exercise 7.2: Prove that d_i, which is defined in terms of the Euclidean distance, satisfies $\|\nabla d_i(q)\|=1$.·

Exercise 7.3: Sketch the set $M_{1\ldots k}(q)$ for planar grasps involving two and three fingers, in the case where the fingers do *not* form a frictionless equilibrium grasp.·

Exercise 7.4: Let $(\mathcal{F}_W,\mathcal{F}_B)$ and $(\overline{\mathcal{F}}_W,\overline{\mathcal{F}}_B)$ be two choices of world and object frames. Let q and $\bar{q}$ be the parametrization of $\mathcal{B}$'s c-space in terms of these two frame choices. Derive the formula for the coordinate transformation $q=h(\bar{q})$.·

[3] Orthogonality has to be interpreted here as the *zero action* of covectors $\boldsymbol{w}\in\mathcal{W}$ on tangent vectors $\dot{q}\in T_q I\!R^m$.

Exercise 7.5: Prove that the coordinate transformation of the previous exercise is a diffeomorphism.·

Exercise 7.6: Give an example where the inner product between two tangent vectors, $\dot{q}_1, \dot{q}_2 \in T_q I\!R^m$, is *not* preserved by the coordinate transformation associated with different choices of the world and object frames.·

Exercise 7.7: All two-finger equilibrium grasps are essential grasps—true or false? ·

Exercise 7.8: Consider an essential equilibrium grasp involving $k \geq 3$ fingers. Prove that all local perturbations of the contacts within the equilibrium set $\mathcal{E}$ remain essential grasps.·

Exercise 7.9: Let a convex set in $I\!R^m$ intersect the interior of every halfspace whose boundary passes through the origin of $I\!R^m$. Prove that this set contains a small m-dimensional ball centered at the origin (this fact is a key component in the proof of Theorem 1). ·

Exercise 7.10: Verify that the instantaneous twists of a planar body $\mathcal{B}$ are pure rotations about points $p \in I\!R^2$.·

Exercise 7.11: Let a planar object $\mathcal{B}$ be located at a configuration q. Prove that every instantaneous motion of $\mathcal{B}$, $\dot{q} = (v, \omega) \in T_q I\!R^m$, can be represented as an instantaneous rotation about a point $p - \frac{1}{\omega} J v \in I\!R^2$, which is the *instantaneous rotation center* of $\dot{q}$.·

Bibliography

[1] J.-S. Cheong, K. Goldberg, M. H. Overmars, E. Rimon, and A. F. van der Stappen. Immobilizing hinged polygons. *Int. J. of Computational Geometry and Applications*, 17(1), 2007.

[2] J. Duffy. The fallacy of modern hybrid control theory that is based on orthogonal complements of twist and wrench spaces. *J. of Robotics Systems*, 7(2):139–144, 1990.

[3] R. M. Murray, Z. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation.* CRC Press, Boca Raton, FA, 1994.

[4] M. S. Ohwovoriole and B. Roth. An extension of screw theory. *J. of Mechanical Design*, 103:725–735, 1981.