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Caging Planar Objects with a Three-Finger One-Parameter Gripper

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Abstract

This paper extends the caging theory of Rimon and Blake [11] for one-parameter two-finger grippers to one-parameter three-finger grippers. The caging theory describes how the fingers of a robot gripper can be placed around an object so that it cannot escape. As the fingers close from such a configuration, the freedom of the object to move is gradually restricted, until, in the absence of friction, it is completely immobilized by the fingers at an immobilizing grasp. The extension of the caging theory to three-fingered grippers is important because convex objects cannot be caged by two-finger grippers. The computation of the set of caging formations requires the identification of both the two- and three-finger frictionless grasps. These grasps correspond to critical points *of the opening parameter in the gripper's configuration space. There are two main problems here. Firstly, a method is needed to compute the critical points of the opening parameter in the three-finger contact space. This is complicated by the fact that this space may have more that one connected component. The second problem is how to associate the immobilizing grasps with punctures. In this paper we solve both these problems, presenting efficient algorithmic solutions.*

1 Introduction

In this paper a method is described for grasping planar objects in an error-tolerant fashion using a three-finger one-parameter gripper mounted on a robot arm. An image of the object and its immediate surroundings is provided by a camera, mounted directly above the object. It is assumed that the mapping from camera coordinates to real-world coordinates is known, and that the apparent contour of the object can be automatically extracted as a closed B-spline curve $\mathbf{r}(s)$, for example using an active contour technique [1, 5] (see figure 1). The fingers of the robot gripper are modelled as identical cylinders with their axis perpendicular to the image plane. Thus the fingers appear as discs in the plane. Without loss of generality we can assume point fingers as the original object contour can be replaced in the usual manner, via a Minkowski sum with the disc. The geometry of the object has been previously used to find optimal point grasps [9, 4, 2, 10]. A novel method of computing optimal three-finger grasps with a one-parameter gripper is presented here. We also present a method of computing an approach of the gripper which is guaranteed to reach the chosen optimal grasp. This succeeds even if the object is displaced from its observed configuration either by mechanical vibration or by measurement error. The technique is somewhat similar to those employed in *part feeder design*, where the objective is to design fixed obstacles to automatically orient parts on a conveyor belt into a stable known configuration [3]. Kinematic systems in which energy dissipates over time tend to reach configurations where the potential energy is at a local minimum, which correspond to stable poses of the object [7]. By choosing the right approach, the set of configurations which inevitably lead to a particular stable pose can be maximised, producing a highly error-tolerant system.

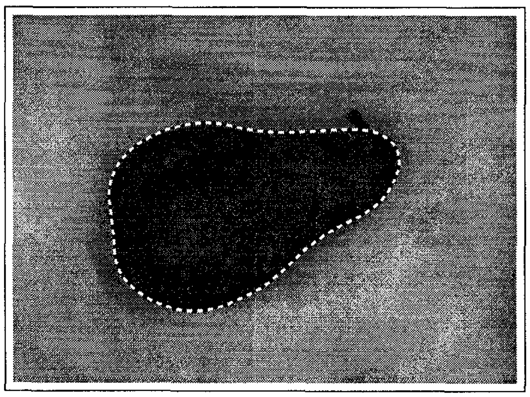

Figure 1: **A quadratic B-spline is automatically fitted to the apparent contour of a pear.**

The *caging problem* is to fix the fingers of a robot gripper around a stationary object in such a way that though the object may have some freedom to move, it cannot be removed completely. This is a *caging configuration*, and the problem of finding the *maxi-*

mal caging set of caging configurations was solved in the case of two-finger one-parameter grippers by Rimon and Blake [11]. This caging set has the additional property that closing the fingers from any configuration in the set is guaranteed to reach a particular immobilizing grasp. Here, we extend caging theory to the case of one-parameter three-finger grippers.

In the following, gripper configurations are thought of as points in configuration space, or *C-space*, denoted $\mathcal{C}$, which is four-dimensional. The four dimensions are the (x, y) translation of the arm in the plane, the rotation θ relative to the plane, and the opening parameter of the gripper σ (which is strictly positive). The function $\sigma(\mathbf{x})$ projects configurations $\mathbf{x} = (x, y, \theta, \sigma)$ of the robot gripper onto the opening parameter σ. The subset of $\mathcal{C}$ for which none of the fingers intersects the stationary object is *freespace*, denoted $\mathcal{F}$.

1.1 The caging set

A caging set $\mathcal{K} \subset \mathcal{F}$ is defined such that, given $\mathbf{x} \in \mathcal{K}$, any configuration connected to $\mathbf{x}$ by a path in freespace on which σ is non-increasing is also in $\mathcal{K}$. Thus, closing the fingers on the object from a caging configuration in $\mathcal{K}$ leads only to other caging configurations in $\mathcal{K}$. The set $\mathcal{K}$ is defined as follows. Consider the set of configurations

$$\mathcal{F}_c = \{\mathbf{x} \in \mathcal{F} \ : \ \sigma(\mathbf{x}) \leq c\}.$$

Let $\mathcal{K}_c$ be the connected component of $\mathcal{F}_c$ containing the immobilizing grasp $\mathcal{I}$, and let $\sigma(\mathcal{I}) = \sigma_0$. Then $\mathcal{K}_{\sigma_0}$ contains only the single point $\mathcal{I}$. As c increases from σ_0, $\mathcal{K}_c$ grows as a region and is the set of configurations reachable from $\mathcal{I}$ without the opening parameter exceeding c. At a certain critical value of $c = \sigma_1$, this region joins another component of $\mathcal{F}_c$ and the critical point where this happens is termed a *puncture point.* Thus any set $\mathcal{K}_c$ with $c < \sigma_1$ is a caging set and the maximal caging set $\mathcal{K}$ is

$$\mathcal{K} = \bigcup_{c<\sigma_1} \mathcal{K}_c.$$

Caging theory uses the mathematical tool *Stratified Morse Theory* [6] to prove that puncture points are frictionless equilibrium grasps which are critical points of σ with Morse index 1. (A critical point of a Morse function f on a manifold $\mathcal{M}$ is a point where $\nabla f = \mathbf{0}$ [6]. The Morse index of a critical point is the number of negative eigenvalues of the Hessian.) The puncture point of the caging set $\mathcal{K}$ is the puncture point with least σ value that also passes the *topological check*, a test which associates puncture points with immobilizing grasps. This gives rise to the algorithm in [11] for a two fingered gripper, where the set of all critical points is calculated and tested by the topological check in order of increasing σ-value. In order to use this algorithm, all the puncture points must be accounted for. In the case of the three-finger gripper, the punctures may include three-finger contact points, an example of which is illustrated in figure 2. Later, the alternate possibility of a two-finger puncture for a three-finger grasp is demonstrated, and it is shown that the three-finger-contact space can be complicated, even for a simple object.

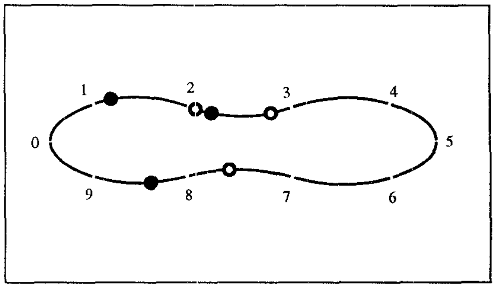

Figure 2: *A three-finger puncture point (filled circles) associated with a three-finger immobilizing grasp (unfilled circles), for a gripper whose fingers form the vertices of an equilateral triangle.*

2 The three-finger gripper

For three-finger grippers, the following problems must be solved. First, a method for efficiently searching $\mathcal{F}$ for all the critical points of σ is required. Second, these critical points must be classified as puncture points, immobilizing grasps and *false critical points* (*i.e.* critical points which are not grasps). Third, the topological check needs to be extended to deal with both two-finger and three-finger contacts.

One restriction is made on the choice of one-parameter three-finger gripper, namely that the Euclidean distance between any two of the fingers in the plane $|\mathbf{f}_i - \mathbf{f}_{i+1}|$ (where index arithmetic is modulo 3) is a strictly increasing function of σ. This restriction is acceptable as all standard three-finger grippers satisfy this property. It is important because, if the distance between fingers i and $i+1$ varies monotonically with σ, then knowing the finger positions $\mathbf{f}_i$ and $\mathbf{f}_{i+1}$ uniquely determines the position of the third finger, so we can write $\mathbf{f}_{i+2} = \mathbf{f}_{i+2}(\mathbf{f}_i, \mathbf{f}_{i+1})$. This defines three functions $\mathbf{f}_1(\mathbf{f}_2, \mathbf{f}_3)$, $\mathbf{f}_2(\mathbf{f}_3, \mathbf{f}_1)$, and $\mathbf{f}_3(\mathbf{f}_1, \mathbf{f}_2)$ which fully describe the geometry of the gripper.

This allows the definition of the *contact C-space* $\mathcal{X}_{ij} := \mathcal{S} \times \mathcal{S}$ for any two fingers i and $j = i+1$ (modulo 3), where $\mathcal{S}$ is the periodic interval $[0, L)$, L is the number of B-spline spans, and $(s_i, s_j) \in \mathcal{X}_{ij}$ corresponds to a two-finger grasp at $\mathbf{f}_i = \mathbf{r}(s_i)$ and $\mathbf{f}_j = \mathbf{r}(s_j)$. Since the location of the third finger $k = i + 2$ is uniquely defined by $\mathbf{f}_k(\mathbf{r}(s_i), \mathbf{r}(s_j))$, every two-finger contact involving fingers i and j is represented by a point in $\mathcal{X}_{ij}$. This implies that every three-finger contact is represented by a point in each of the $\mathcal{X}_{ij}$.

3 Searching for critical points

The freespace $\mathcal{F}$ is a stratified set, consisting of a collection of manifolds called *strata*. In this section, the strata are determined, and then searched for critical points.

3.1 The strata of $\mathcal{F}$

The freespace $\mathcal{F}$ is formed by the removal of three *finger C-obstacles* from $\mathcal{C}$. These are the configurations where each finger intersects with the object. It follows that points on the boundary of each finger C-obstacle are configurations where that finger touches the object. $\mathcal{F}$ can therefore be partitioned into the following eight strata:

- One four-dimensional manifold, the interior of freespace, $\mathcal{F}_0 = I(\mathcal{F})$. This is the no-finger contact stratum and contains no critical points of $\sigma(\mathbf{x})$.
- Three three-dimensional manifolds, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$. These are the one-finger contact strata, containing configurations where exactly one finger touches the object. These strata also do not contain any critical points of $\sigma(\mathbf{x})$.
- Three two-dimensional manifolds, $\mathcal{F}_{12}$, $\mathcal{F}_{23}$, and $\mathcal{F}_{31}$. These are the two-finger contact strata, containing configurations of the gripper where exactly two fingers touch the object. These strata do contain critical points of $\sigma(\mathbf{x})$.
- One one-dimensional manifold, $\mathcal{F}_{123}$. This is the three-finger contact stratum. This stratum also contains critical points of $\sigma(\mathbf{x})$.

These sets form a disjoint partition of $\mathcal{F}$, and satisfy $\overline{\mathcal{F}_{ij}} = \overline{\mathcal{F}_i} \cap \overline{\mathcal{F}_j}$ and $\overline{\mathcal{F}_{123}} = \overline{\mathcal{F}_1} \cap \overline{\mathcal{F}_2} \cap \overline{\mathcal{F}_3}$, where $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$.

Let $\phi_{ij} : \mathcal{F}_{ij} \rightarrow \mathcal{X}_{ij}$ be the smooth 1-1 function, mapping configurations of the gripper in which fingers i and j are touching the object and k is outside the object, to the corresponding contact configurations. Consider $\phi_{ij}(\mathcal{F}_{123})$; this is the set of three-finger contacts expressed as a subset of $\mathcal{X}_{ij}$. Since ϕ_{ij} is 1-1 we can write

$$\mathcal{F}_{123} = \phi_{ij}^{-1}(\{(s_i, s_j) \in \mathcal{X}_{ij} \ : \ D(\mathbf{f}_k(\mathbf{r}(s_i), \mathbf{r}(s_j))) = 0\})$$

where D is the minimum Euclidean distance of a point in the plane from the object. Since all these functions are continuous, it follows that $\mathcal{F}_{123}$ is the level set of a continuous real-valued function on a two-dimensional space. Therefore it consists of closed loops and arcs which terminate at $\sigma = 0$.

3.2 The two-finger contact strata

Since ϕ_{ij} is smooth, the critical points of $\sigma(\mathbf{x})$ on $\mathcal{F}_{ij}$ can be sought in $\mathcal{X}_{ij}$ as the critical points of $\sigma(s_i, s_j)$ just as in the case of the two-finger gripper, by tracking along the antisymmetry set searching for intersections of the symmetry set [2]. For the two-finger gripper, $\phi_{ij}(\mathcal{F}_{ij})$ fills the entire contact C-space $\mathcal{X}_{ij}$ except for the axis $s_i = s_j$, where $\sigma = 0$. In the case of the three-finger gripper, it is not necessarily true that all three fingers meet at $\sigma = 0$. Hence $\mathcal{X}_{ij}$ may contain regions which do not correspond to any gripper configuration.

In addition, $\mathcal{X}_{ij}$ contains points where the third finger lies inside the object. Such configurations are not members of $\mathcal{F}$, and form ***forbidden regions*** in $\mathcal{X}_{ij}$ which correspond to the third finger C-obstacle. Critical points in forbidden regions are ignored for this reason. The edges of forbidden regions are made up of points on the boundary of the third finger C-obstacle, which implies that they correspond to points in $\mathcal{F}_{123}$, the three finger contact stratum.

3.3 The three-finger contact stratum

$\mathcal{F}_{123}$ is a one-dimensional set. When projected into $\mathcal{X}_{ij}$, it consists of a number of closed loops (which surround forbidden regions) and a number of curved arcs which, together with the curve $\sigma = 0$, bound forbidden regions (see figure 4). The critical points of $\sigma(\mathbf{x})$ could be easily found by tracking along these curves. Unfortunately, there is no obvious way of calculating the number of connected components in this stratum, and so we cannot easily obtain sample points on every component from which to begin tracking.

Therefore, the following alternative strategy is adopted. Consider the space of all three-finger contacts, $\mathcal{S} \times \mathcal{S} \times \mathcal{S}$. Fixing one of the fingers at a particular value, say $s_1 = t$, is equivalent to intersecting this space with a plane, containing points of the form (t, s_2, s_3). The position of fingers 1 and 2 uniquely determines the position of finger 3, $\mathbf{f}_3(\mathbf{r}(s_1), \mathbf{r}(s_2))$. Therefore, any three-finger contact in the plane (t, s_2, s_3) is also a solution of the equation

$$D(\mathbf{f}_3(\mathbf{r}(t), \mathbf{r}(s_2))) = 0 \qquad (1)$$

where as before, $D(\mathbf{x})$ is the minimum distance of a point from the object. By considering single spans of the B-spline, this equation reduces to a polynomial in

s_2. The degree of this polynomial is kept to a minimum by using quadratic B-splines, and by using a gripper of simple geometry. In any case, for higher order polynomials, all the roots can be found by a technique such as homotopy continuation [8]. Values of s_1 are sampled at N points per span, on L spans of the B-spline in $O(NL^3)$ time. This is then repeated, sampling in the s_2 and then s_3 directions. In this way, the space of three-finger contacts $\mathcal{S} \times \mathcal{S} \times \mathcal{S}$ is effectively divided into cubes of side $1/N$, and all the intersections of the surface of these cubes with $\mathcal{F}_{123}$ are calculated. As N becomes large, there will be at most 2 such intersections per cube, which can be linked to produce connected chains of points (see figure 3).

compute 3-finger contact stratum

1 **For** $f = 1, 2, 3$ *; each finger*
  For $i = 0, ..., L-1$ *; for s_f in each span*
   For $t = 0, ..., N-1$ *; N samples of s_f per span*
    For $j = 0, ..., L-1$ *; for s_{f+1} in each span*
     For $k = 0, ..., L-1$ *; for s_{f+2} in each span*
      Set $s_f = i + t/N$
      For each root s of
       $D_k(\mathbf{f}_{f+2}(\mathbf{r}(s_f), \mathbf{r}_j(s))) = 0$
      If ($s \in I\!R$ and $s \in (0,1)$) **Then**
       Set $s_{f+1} = j + s$
       Solve $\mathbf{r}_k(r) = \mathbf{f}_{f+2}(\mathbf{r}(s_f), \mathbf{r}(s_{f+1}))$ for r
       If ($r \in I\!R$ and $r \in (0,1)$) **Then**
        Set $s_{f+2} = k + r$
        Append (s_1, s_2, s_3) to $PLIST$
2 **For** $i = 1, ..., |PLIST|$
  Calculate the centre-points (c_{11}, c_{12}, c_{13}), (c_{21}, c_{22}, c_{23}) of the two cubes of side $1/N$ to which $PLIST[i]$ belongs
  Append the vectors $(c_{11}, c_{12}, c_{13}, i)$ and $(c_{21}, c_{22}, c_{23}, i)$ to QLIST
3 **Sort** QLIST lexicographically
4 **For** $i = 1, ..., |QLIST|$
  If $QLIST[i]_j = QLIST[i+1]_j$ for $j = 1, 2, 3$ **Then**
   Mark $QLIST[i]_4$ connected to $QLIST[i+1]_4$

Figure 3: **Algorithm for computing the three-finger contact stratum.**

A demonstration of the output of the algorithm is shown in figure 4, for a gripper whose fingers form the vertices of an equilateral triangle. In this case, the functions $\mathbf{f}_{i+2}(\mathbf{f}_i, \mathbf{f}_{i+1})$ for $i = 1, 2, 3$ are linear and so, for a quadratic B-spline, the function $D(\mathbf{f}_{i+2}(\mathbf{r}(t), \mathbf{r}(s_{i+1})))$ is only quartic in s_{i+1}. As a result, the roots can be calculated quickly.

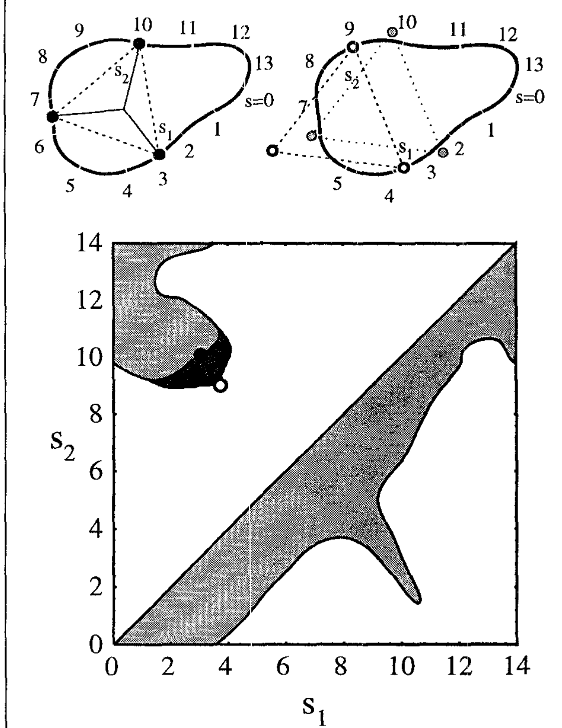

Figure 4: **Grasping a pear.** *(Top left) A three-finger immobilizing grasp. (Top right) The puncture grasp of the cage, and (grey circles) a configuration inside the cage. (Bottom) The contact C-space $\mathcal{X}_{12}$ (black curve). The cage is shaded dark. Light grey regions are forbidden regions, and the black curve boundary is the three finger contact space. The immobilizing grasp and the puncture point are shown.*

Having found all of the closed loops and arcs which make up $\mathcal{F}_{123}$, the critical points on this stratum can be determined as the local minima and maxima of σ along the curves of $\mathcal{F}_{123}$.

4 Classification of the critical points

The classification of two-finger-contact critical points is exactly as in the two-finger gripper case [11]. Local minima (index 0) give immobilizing grasps; saddles (index 1) give puncture points; local maxima (index 2) are ignored. False critical points must be pruned. The three-finger critical points are straightforward; local minima (index 0) give immobilizing grasps, and local maxima (index 1) give puncture points. All that remains is to prune false critical points, using the following standard test for a three-finger equilibrium-grasp.

At a frictionless grasp, contact forces of the three fingers are in the direction of the inward-pointing unit normals, $\mathbf{n}_i$. For equilibrium, the forces must sum to zero. Therefore, there must exist scalars $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 + \mathbf{n}_3 = 0$. In addition, these forces must produce no net moment. This gives the equation $\lambda_1(\mathbf{r}_1 - \mathbf{r}_3) \cdot \mathbf{t}_1 + \lambda_2(\mathbf{r}_2 - \mathbf{r}_3) \cdot \mathbf{t}_2 = 0$ where $\mathbf{t}_i$ is the tangent to the curve. A false critical point will fail to satisfy one of these equations because it will require one of the λ_i to be negative.

5 The topological check

The purpose of the topological check is to associate puncture points with immobilizing grasps by demonstrating a path on which σ is monotonic decreasing. The key to implementing the topological check is to keep track of which fingers are in contact with the object. The path is then just gradient descent on σ *in the relevant stratum* until either (a) a local minimum of $\sigma(\mathbf{x})$ on that stratum is reached, or (b) a free finger comes into contact with the object (detected by a change in sign of the distance of the finger from the B-spline). The track stops if an immobilization is reached. Otherwise, the local minimum is false and the track continues in case (a) by allowing one finger to break contact with the object. In case (b) the trace continues in a different stratum by maintaining the new contact. See figure 5 for the algorithm.

6 The parallel-jaw gripper

A particular gripper for which the three-finger contact space can be easily computed is the three-finger parallel-jaw gripper. It consists of two fingers ($\mathbf{f}_1$ and $\mathbf{f}_2$) and an opposing thumb ($\mathbf{f}_3$). The thumb moves along the perpendicular bisector of the line segment joining the two fingers, and the two fingers remain a fixed distance apart: $|\mathbf{f}_1 - \mathbf{f}_2| = c$. Note that this technically breaks the assumption that the distance between any two fingers varies strictly monotonically with σ. In this special case, the stratum $\mathcal{F}_{12}$ cannot be searched but contains no critical points anyway. The strata $\mathcal{F}_{23}$ and $\mathcal{F}_{31}$ can be searched in the usual way as these fingers do satisfy the monotonicity assumption. Therefore, the problem reduces to calculating the three-finger contact stratum $\mathcal{F}_{123}$.

For a given contact point of either finger 1 or 2, three-finger contacts can be constructed as follows. Consider $\mathbf{f}_1 = \mathbf{r}(s_1)$ for some fixed s_1. Calculate all intersections of the circle centred on $\mathbf{f}_1$ having radius c with the object outline $\mathbf{r}(s_2)$. These are the possible positions of the other finger, $\mathbf{f}_2$. Now construct the perpendicular bisector of the line segment joining $\mathbf{f}_1$ and $\mathbf{f}_2$ and calculate its intersections with the object outline to locate the thumb, $\mathbf{f}_3$. This set of points

find puncture associated with $\mathcal{I}$

1 **Calculate** the set $\mathcal{A} = \{A_1, ..., A_a\}$ of possible puncture points (critical points of σ in freespace with index 1 which are also grasps and satisfy $\sigma > \sigma(\mathcal{I})$)

2 **Sort** the A_i by increasing σ-value

3 **For** $i = 1, ..., a$

Calculate the negative eigenvalue λ of the Hessian at $\mathcal{A}_i$ and the corresponding eigenvector $\mathbf{e}$

For $j = 0, 1$

If (A_i is a two-finger puncture) **Then**

Set $t_0 = A_i + (-1)^j \sqrt{\frac{2\varepsilon}{\lambda}}\mathbf{e}$ (where ε is a small constant)

If (A_i is a three-finger puncture) **Then**

Set t_0 to be the previous ($j = 0$) or next ($j = 1$) three-finger contact in the chain

While (t_0 is not an immobilizing grasp and $\sigma(t_0) > \sigma(\mathcal{I})$)

For $n = 0, ...$ **Until** (t_n is a critical point or a free finger comes into contact with the object)

Set $t_{n+1} = t_n - (\Delta\sigma)\nabla\sigma(t_n)$ (where $\nabla\sigma$ is the gradient of the function σ restricted to the current stratum and $\Delta\sigma$ is chosen so that σ is monotonic decreasing on the line segment (t_n, t_{n+1}))

If (free finger f has touched) **Then**

Set t_0 to the point nearest to t_n on the stratum in which finger f is in contact

If (t is a false critical point) **Then**

Set t_0 to the point nearest to t_n on a stratum in which continued descent is possible

If ($t_n = \mathcal{I}$) **Then Return** $\mathcal{A}_i$

Figure 5: **Algorithm for finding the puncture.**

PLIST can be turned into a chain in a similar way to the algorithm in figure 3, calculating the centre-points of the squares of side $1/N$ in (s_1, s_2) space to which the points belong and inserting the actual s_3 value. Nearest neighbours in the s_3 direction in each square are connected to form chains. See figure 6 for an example of this gripper.

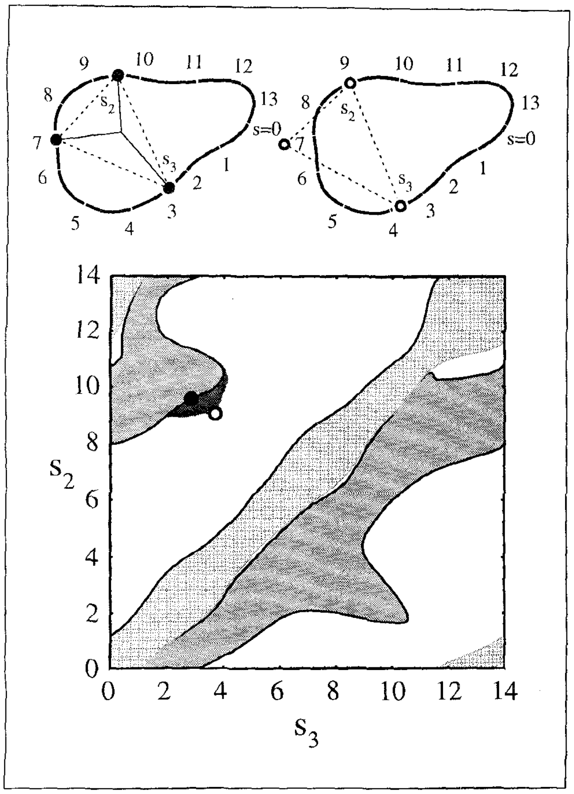

Figure 6: **A three-finger parallel-jaw gripper grasping a pear.** *(Top Left) The 3-finger immobilizing grasp. (Top Right) The puncture point, a two-finger saddle. (Bottom) The two-finger contact C-space* $\mathcal{X}_{23}$ *showing the forbidden regions in light grey and* $\mathcal{F}_{123}$ *as black curves. The hatched regions do not correspond to gripper configurations. The white region is freespace and the dark region is the caging set, with the immobilizing grasp and the puncture on its boundary.*

7 Conclusion

We have shown how to construct a cage with a one-parameter three-finger gripper. Closing the fingers on the object from any configuration in the caging set gradually restricts the possible movements of the object until it is completely immobilized. The object is thus held in a known frictionless immobilizing grasp. This paper has also proposed algorithms to efficiently compute the caging set, with run times typically under 10 seconds.

We made the assumption that the distance between any two of the fingers varies strictly monotonically with σ. We also assumed that the geometry of the gripper permits equation (1) to be written as a polynomial. However, we also demonstrated the solution in the special case where two of the fingers are a constant distance apart, breaking these assumptions. This suggests that the assumption can be relaxed to include a larger class of grippers. Another extension may be to deal with multiple degree of freedom grippers. Industrially, one-parameter grippers are preferred for their simplicity. However, it is possible that the problem of grasping curved objects with more complicated grippers may be greatly simplified by caging theory.

Finally, although it is easy to construct configurations in the caging set, it is not clear how to find a configuration in the caging set that maximises the clearance of the three fingers from the object. Such a configuration would be the ideal target to aim for in an automatic gripping system.

8 Acknowledgements

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References

[1] A. Blake and M. Isard. *Active contours.* Springer, 1998.

[2] A. Blake and M. Taylor. Planning planar grasps of smooth contours. In *Proc. IEEE Int. Conf. Robotics and Automation*, volume 2, pages 834–839, 1993.

[3] M. Brokowski, M. Peshkin, and K. Goldberg. Curved fences for part alignment. In *Proc. IEEE Int. Conf. Robotics and Automation*, volume 3, pages 467–473, 1993.

[4] I.M. Chen and J.W. Burdick. Finding antipodal point grasps on irregularly shaped objects. In *Proc. IEEE Int. Conf. Robotics and Automation*, pages 2278–2283, 1992.

[5] C. Davidson and A. Blake. Error-tolerant visual planning of planar grasp. In *Proc. 6th Int. Conf. on Computer Vision*, pages 911–916, 1998.

[6] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Springer-Verlag, 1988.

[7] D.J. Kriegman. Let them fall where they may: Capture regions of curved objects and polyhedra. Int. J. Robotics Research, June 1997.

[8] A. Morgan. Computing all solutions to polynomial systems. *Applied Mathematics and Computation*, 24:115–138, 1987.

[9] V.D. Nguyen. Constructing force-closure grasps. *Int. J. Robotics Research*, 7(3):3–16, 1988.

[10] J. Ponce, J.W. Burdick, and E. Rimon. Computing the immobilizing three-finger grasps of planar objects. In *Proc. of the 1995 Workshop on Computational Kinematics*, pages 281–300, 1995.

[11] E. Rimon and A. Blake. Caging 2D bodies by one-parameter, two-fingered gripping systems. In *Proc. IEEE Int. Conf. Robotics and Automation*, pages 1458–1464. IEEE Press, 1996.