**ME 115(b)**

**Parametrization of a C-obstacle Boundary**
(revised version II)

# 1 Background

We have already looked at the problem of how to symbolically describe a portion (or "patch") of a c-obstacle boundary corresponding to a EV contact between a planar polygonal robot, $\mathcal{A}$, and a planar polygonal obstacle, $\mathcal{O}$. The goal of this handout handout is to choose a parametrization of the robot and obstacle geometries which can then be used to derive a concrete formula that describes the boundary of a c-obstacle due to an EV contact.

# 2 Parametrization

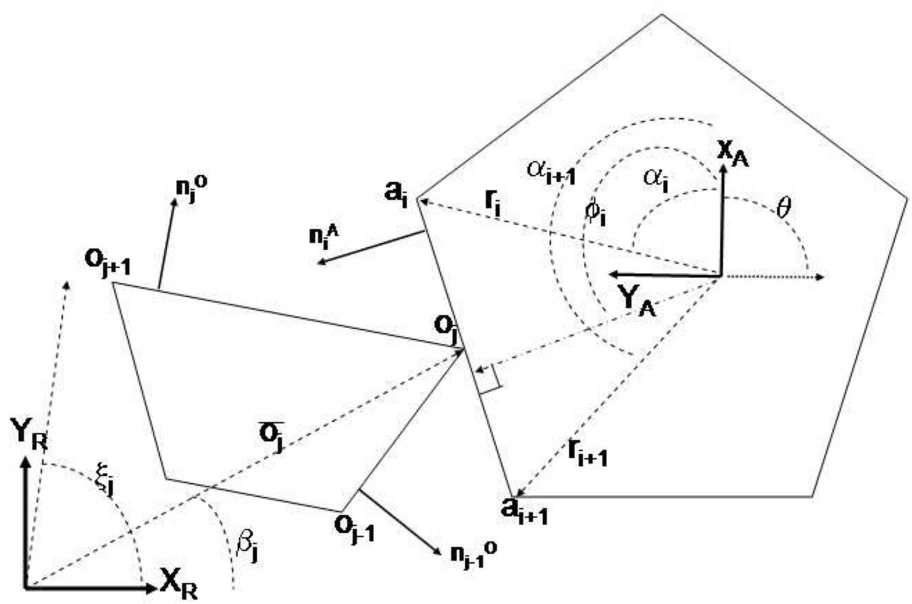


Figure 1: Parametrization of polygonal obstacle and polygonal robot

Figure 1 describes a parametrization of the robot and an obstacle. Note that one must choose a fixed observing reference frame, whose basis vectors are subscripted by $R$, and a reference frame fixed to the body of the moving robot, whose basis vectors are subscripted by $A$. We choose a parametrization with the following variables

- $\vec{r}_i$ is a vector from the origin of $\mathcal{A}$'s body fixed frame to the $i^{th}$ vertex of $\mathcal{A}$, $a_i$.

- $||\vec{r}_i||$ is the Euclidean length of $\vec{r}_i$.
- By abuse of notation, let $\vec{o}_j$ be a vector from the origin of the fixed observing frame to the $j^{th}$ vertex of $\mathcal{O}$, $o_j$.
- $||\vec{o}_i||$ is the Euclidean length of $\vec{o}_i$.
- $\alpha_i$ is the angle between $\vec{x}_A$, the $x$-axis of the robot's body fixed frame and the vector $\vec{r}_i$.
- $\phi_i$ is the angle from $\vec{x}_A$ to $\vec{n}_i^A$, the normal to the $i^{th}$ edge of $\mathcal{A}$, $E_i^A$.
- $\beta_j$ is the angle between $\vec{x}_R$ (the $x$-axis of the fixed observing reference frame) and $\vec{o}_j$.
- $\xi_j$ is the angle between $\vec{x}_R$ and $\vec{n}_j^O$, the normal to the $j^{th}$ edge of $\mathcal{O}$, $E_i^O$.

With these definitions, the basic vectors that are involved in the constraint equations are:

$$\vec{o}_j = ||\vec{o}_j|| \begin{bmatrix} \cos(\beta_j) \\ \sin(\beta_j) \end{bmatrix} \qquad \vec{r}_i = ||\vec{r}_i|| \begin{bmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{bmatrix} \tag{1}$$

$$\vec{n}_i^A(q) = \begin{bmatrix} \cos(\phi_i + \theta) \\ \sin(\phi_i + \theta) \end{bmatrix} \qquad \vec{n}_j^O = \begin{bmatrix} \cos(\xi_j) \\ \sin(\xi_j) \end{bmatrix} \tag{2}$$

# 3 The Constraint Equations in Parametrized Form

$$a_i(q) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{r}_i = \begin{bmatrix} x + ||\vec{r}_i|| \cos(\alpha_i + \theta) \\ y + ||\vec{r}_i|| \sin(\alpha_i + \theta) \end{bmatrix} \tag{3}$$

**First constraint.** First consider the constraint which ensures that vertex $o_j$ lies on the line underlying the $i^{th}$ edge of $\mathcal{A}$:

$$\vec{n}_i^A(q) \cdot (o_j - a_i(a)) = 0. \tag{4}$$

Substituting in the variables from above, and performing some algebra results in the equation:

$$0 = -x\cos(\phi_i + \theta) - y\sin(\phi_i + \theta) + ||\vec{o}_j|| \cos(\phi_i + \theta - \beta_j) - ||\vec{r}_i|| \cos(\phi_i - \alpha_i) \ . \tag{5}$$

This equation has the form:

$$A(\theta)\ x \ + \ B(\theta)\ y \ + C(\theta) \ = \ 0 \ . \tag{6}$$

For a constant orientation (i.e., when the value of $\theta$ is fixed), Equation (6) represents a straight line in the $x$-$y$ plane (i.e., a straight line in the contant orientation slice of c-space at level $\theta$). Thus, the local "patch" of the configuration-space obstacle boundary is a *ruled* *surface*, since this equation shows that the surface is bounded by a line whose orientation changes as a function of $\theta$.

**Second Pair of Constraints.** Next we consider the pair of inequality constraints that insure that the robot and obstacle don't overlap:

$$\vec{n}_i^A(q) \cdot (\vec{o}_{j-1} - \vec{o}_j) \;\; \geq 0 \tag{7}$$
$$\vec{n}_i^A(q) \cdot (\vec{o}_{j+1} - \vec{o}_j) \;\; \geq 0 \tag{8}$$

Using the observation that:

$$(\vec{o}_{j-1} - \vec{o}_j) = ||E_{j-1}^O|| \begin{bmatrix} \cos(\xi_{j-1} - \pi/2) \\ \sin(\xi_{j-1} - \pi/2) \end{bmatrix} \tag{9}$$

Substituting the parametrized terms into Equation (7), and simplifying yields the equivalent constraint:

$$\cos(\phi_i + \theta - \xi_{j-1} + \pi/2) \geq 0 \ . \tag{10}$$

In general, for an angle $\gamma$ to satisfy the equation $\cos\gamma \geq 0$, we require that $-\frac{\pi}{2} \leq \gamma \; (mod\; 2\pi) \leq \frac{\pi}{2}$. Hence, Equation (10) is equivalent to

$$-\pi \leq \phi_i + \theta - \xi_{j-1} \leq 0 \qquad (mod\; 2\pi). \tag{11}$$

Note, for this equation, only the *lower* bound is physically meaningful for the geometry shown in Figure 1. Thus, constraint Equation (7) reduces to:

$$\xi_{j-1} \;\; - \;\; \phi_i - \pi \;\; \leq \;\; \theta \tag{12}$$

Similarly, using the observation that

$$(\vec{o}_{j+1} - \vec{o}_j) = ||E_j^O|| \begin{bmatrix} \cos(\xi_j + \pi/2) \\ \sin(\xi_j + \pi/2) \end{bmatrix} \tag{13}$$

Equation (8) can be rewritten as

$$\cos(\phi_i + \theta - \xi_j - \pi/2) \geq 0 \tag{14}$$

which is equivalent to

$$0 \leq \phi_i + \theta - \xi_j \leq \pi \qquad (mod\; 2\pi).. \tag{15}$$

For this equation, only the *upper* bound is physically meaningful, and thus constraint Equation (8) reduces to:

$$\theta \leq \pi + \xi_j - \phi_i \tag{16}$$

These two constraints can then be summarized as:

$$\theta \;\in\; [(\xi_{j-1} - \phi_i - \pi), (\xi_j - \phi_i + \pi)] \quad (mod\; 2\pi) \tag{17}$$

Thus, these constraints bound the range of $\theta$ over which the local "patch" is defined. Note that the "*mod* $2\pi$" modification applies to each of the upper and lower bounds.

**Third pair of constraints.** The final pair of inequality constraints bounds the vertex $o_j$ to lie within the line segment $E_i^A$:

$$0 \leq (o_j - a_i(q)) \cdot (a_{i+1}(q) - a_i(q)) \leq ||E_i^A||^2 \ . \tag{18}$$

Substituting the parametrized expressions for $o_j$, $a_i(q)$, and $a_{i+1}(q)$ into this equation yields:

$$\begin{aligned}
0 \ \leq \ & x\,[||\vec{r}_i||\cos(\alpha_i+\theta) - ||\vec{r}_{i+1}||\cos(\alpha_{i+1}+\theta)] && (19)\\
& +y\,[||\vec{r}_i||\sin(\alpha_i+\theta) - ||\vec{r}_{i+1}||\sin(\alpha_{i+1}+\theta)] && (20)\\
& +||\vec{o}_j||||\vec{r}_{i+1}||\cos(\theta+\alpha_{i+1}-\beta_j) - ||\vec{o}_j||||\vec{r}_i||\cos(\theta+\alpha_i-\beta_j) && (21)\\
& -||\vec{r}_{i+1}||||\vec{r}_i||\cos(\alpha_i-\alpha_{i+1}) + ||\vec{r}_i||^2 \leq ||E_i^A||^2 \ . && (22)
\end{aligned}$$

These equations have the form:

$$0 \leq D(\theta)\ x \ + \ E(\theta)\ y \ + \ \ F(\theta) \ \leq ||E_i^A||^2 \tag{23}$$

where:

$$\begin{aligned}
D(\theta) &= [||\vec{r}_i||\cos(\alpha_i+\theta) - ||\vec{r}_{i+1}||\cos(\alpha_{i+1}+\theta)]\\
E(\theta) &= [||\vec{r}_i||\sin(\alpha_i+\theta) - ||\vec{r}_{i+1}||\sin(\alpha_{i+1}+\theta)]\\
F(\theta) &= +||\vec{o}_j||||\vec{r}_{i+1}||\cos(\theta+\alpha_{i+1}-\beta_j) - ||\vec{o}_j||||\vec{r}_i||\cos(\theta+\alpha_i-\beta_j)\\
&\quad -||\vec{r}_{i+1}||||\vec{r}_i||\cos(\alpha_i-\alpha_{i+1}) + ||\vec{r}_i||^2
\end{aligned}$$

## 3.1 Summary

The c-obstacle bounday patch defined by these constraint equations can thus be viewed as a ruled surface formed by sweeping a line segment (whose underlying line is given by Equation (6)) through the $\theta$-range defined by Equation (5). The end points of the line segment can be determined as follows. One end of the line segment (for a given $\theta$) occurs at the lower equality of Equation (23). Thus, this point can be found as the solution of the two linear equations

$$\begin{aligned}
0 &= Ax + By + C\\
0 &= Dx + Ey + F
\end{aligned}$$

Similarly, the other end-point of the line segment (again, for a given $\theta$) can be found from the upper inequality of Equation (23). That is, the other point (for fixed $\theta$) is found by solving the linear equations:

$$\begin{aligned}
0 &= Ax + By + C\\
||E_i^A||^2 &= Dx + Ey + F
\end{aligned}$$