ME 115(a): Final Exam Solutions
(Winter Quarter 2015/2016)

Problem 1: (15 points)

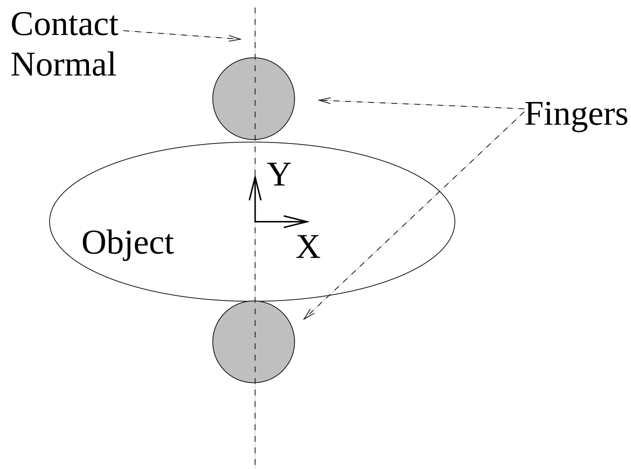

Figure 1: Schematic of two-fingered frictionless grasp

The goal of this problem is to understand the possible motions of the elliptical object grasped by two disk fingers (see Figure 1). To start the analysis, define a fixed reference frame whose origin lies at the midpoint between the two contact points, and whose y-axis aligns with the common line underlying the two contact normals (see Figure 1). Intuitively, the two frictionless fingers can only apply forces along the contact normal axes, which are collinear with the y-axis of the reference frame. Hence, the object should be able to instantaneously slide along the x-axis, as well as rotate about the axis normal to the plane. We can show this formally as follows.

Since the grasped object is restricted to move in the plane, it has 3 degrees of freedom (DOF). All of its instantaneous motions can be expressed as a linear combination of three independent 1-DOF motions. There is not a unique choice of basis vectors for this three dimensional set. Let us choose the basis vectors to be

$$\$_x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\$_x$ represents a unit velocity along the x-axis, $\$_y$ represents a unit velocity along the y-axis and $\$_\theta$ represents a unit angular velocity about the axis orthogonal to the plane. Thus, any velocity can be expressed as

$$V = c_x \$_x + c_y \$_y + c_\theta \$_\theta = \begin{bmatrix} c_x \\ c_y \\ 0 \\ 0 \\ 0 \\ c_\theta \end{bmatrix}$$

The forces that can be applied to the object by the frictionless finger contacts can be expressed as the wrench:

$$W = f_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + f_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 - f_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $f_i > 0$ is the magnitude of the force applied by the i^{th} finger. The applied forces can not stop instantaneous motions which are reciprocal to the applied wrenches. Note that $\$_x$ and $\$_\theta$ are both reciprocal to the wrench of the finger forces. Hence, the fingers can not stop translations along the x-axis and rotations about the axis perpendicular to the plane.

Problem 2: (25 Points)

The "trick" to this problem is how to orient the manipulator in its "home" position in order to make the analysis straightforward. See Fig. 2 for one appropriate way to do this. In this case, it was necessary to redefine the positive direction of the third joint axis.

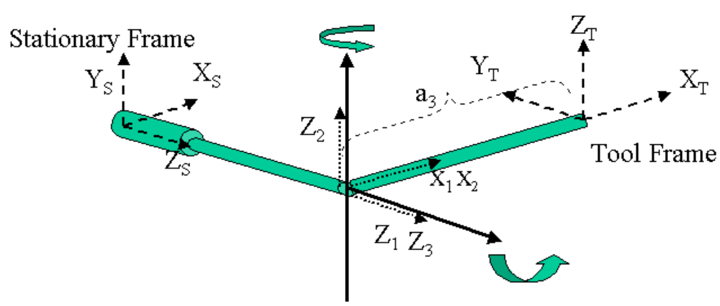

Figure 2: Schematic of PRR Manipulator in "home" position

Part (a) (Denavit-Hartenberg parameters): (5 points) Assuming the definitions shown in Figure 2, the parameters are:

$$
\begin{array}{llll}
a_0 = 0 & \alpha_0 = 0 & d_1 = \text{variable} & \theta_1 = 0 \\
a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{variable} \\
a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_3 = \text{variable} \\
a_3 = L & \alpha_3 = -\frac{\pi}{2} & d_4 = 0 & \theta_4 = 0
\end{array} \tag{1}
$$

Part (b) (forward kinematics): (5 points) Using the Denavit-hartenberg approach:

$$
g_{S1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}
$$

$$
g_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{3T} = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}
$$

The total forward kinematics is:

$$
g_{ST} = g_{S1}\ g_{12}\ g_{23}\ g_{3T} \ == \begin{bmatrix} c_2c_3 & -s_2 & -c_2s_3 & Lc_2c_3 \\ s_3 & 0 & c_3 & Ls_3 \\ -s_2c_3 & -c_2 & s_2s_3 & d_1 - Ls_2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}
$$

where $c_j = \cos\theta_j$, and $s_j = \theta_j$, and length parameter a_3 was set to the constant value L.

Part (c) (Jacobian matrix): (5 points)

The spatial Jacobian has the form:

$$
J = \begin{bmatrix} \vec{\xi}_1 & \vec{\xi}_2' & \vec{\xi}_3' \end{bmatrix} \tag{5}
$$

where

$$
\vec{\xi}_2' = Ad_{e^{d_1\hat{\xi}_1}} \vec{\xi}_2 \tag{6}
$$

$$
\vec{\xi}_3' = Ad_{e^{d_1\hat{\xi}_1}} Ad_{e^{\theta_2\hat{\xi}_2}} \vec{\xi}_3 \tag{7}
$$

Recalling the form of a twist for a prismatic joint, and noting that $d_1 = 0$ in the home position, simple observation of Fig. 2 leads to:

$$
\vec{\xi}_1 = \begin{bmatrix} \vec{z}_S \\ \vec{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{\xi}_2 = \begin{bmatrix} \vec{y}_S \\ \vec{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{\xi}_3 = \begin{bmatrix} \vec{0} \\ \vec{z}_S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{8}
$$

Note that:

$$Ad_{e^{d_1\hat{\xi}_1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -d_1 & 0 \\ 0 & 1 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Ad_{e^{\theta_2\hat{\xi}_2}} = \begin{bmatrix} c_2 & 0 & s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -s_2 & 1 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -s_2 & 0 & c_2 \end{bmatrix} \tag{9}$$

Substitution of Eq.s (9) and (8) into Eq.s (6), (7), and (5) yields:

$$J^s_{ST} = \begin{bmatrix} 0 & -d_1 & 0 \\ 0 & 0 & d_1 \sin\theta_2 \\ 1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta_2 \end{bmatrix} \tag{10}$$

Part (d) (inverse kinematics): (10 points) Let (x_D, y_D, z_D) denote the desired location of the origin of the tool frame, with respect to th origin of the stationary frame. Let's use the algebraic approach for solving inverse kinematics. Of course, to use this method, it assumes that you got the right equations in part (b) we know that:

$$x_D = Lc_2c_3 \tag{11}$$
$$y_D = Ls_3 \tag{12}$$
$$z_D = d_1 - Ls_2c_3. \tag{13}$$

From Eq. (12) we see that $\sin\theta_3 = \frac{y_D}{L}$. Assuming that $|y_D/L| \leq 1$, we have that

$$\theta_3 = \sin^{-1}\left[\frac{y_D}{L}\right]. \tag{14}$$

Two solutions for θ_3 can be obtained from Eq. (14). The second solution, dubbed $\theta_3^{'}$, is $\theta_3^{'} = \pi - \theta_3$. From Eq. (11) we have two θ_2 solutions for each θ_3 solution:

$$\theta_2 = \cos^{-1}\left[\frac{x_D}{L\cos\theta_3}\right]. \tag{15}$$

The second solution is $\theta_2^{'} = -\theta_2$. Finally, from Eq. (13), we can solve for d_1 for each given (θ_2, θ_3) pair:

$$d_1 = z_D + L\sin\theta_2\cos\theta_3.$$

Problem 3: (10 points)

Let $\vec{p} = [p_x \ p_y]^T$ denote the location of the pole of displacement. Let the displacement be described by the homogeneous matrix g:

$$g = \begin{bmatrix} R & \vec{d} \\ \vec{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & \vec{d} \\ \vec{0}^T & 1 \end{bmatrix}. \tag{16}$$

The homogeneous coordinates for the pole are: $\vec{p}^H = [\vec{p}^T \ 1]^T$, where we know that the pole has the special property that $\vec{p} = (I - R)^{-1}\vec{d}$. A simple calculation shows:

$$g\vec{p}^H = \begin{bmatrix} R & \vec{d} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} = \begin{bmatrix} R\vec{p} + \vec{d} \\ 1 \end{bmatrix} = \begin{bmatrix} R(I-R)^{-1}\vec{d} + \vec{d} \\ 1 \end{bmatrix}$$

Let's simplify this term:

$$\begin{aligned} R(I-R)^{-1}\vec{d} + \vec{d} &= [R(I-R)^{-1} + I]\vec{d} \\ &= [R(I-R)^{-1} + I](I-R)(I-R)^{-1}\vec{d} \\ &= [R + I - R](I-R)^{-1}\vec{d} = (I-R)^{-1}\vec{d} \\ &= \vec{p} \end{aligned}$$

Thus,

$$g\vec{p}^H = g\begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} = \vec{p}^H \ ,$$

and we have shown in this way that the pole is an eigenvector with eigenvalue 1.

To understand the properties of the other two eigenvectors/eigenvalues, note that the determinant of g will always be $+1$. Since the determinant is equal to the product of the eigenvalues, and since one eigenvalue has already been determined to be $+1$, it must be true that the product of the remaining eigenvalues is also $+1$. Hence, the remaining eigenvalues are either both real and reciprocal, or they are both complex conjugates. Note also that the trace of g, which is equal to the sum of its eigenvalues, is always $tr(g) = 1 + 2\cos\theta$, where θ is the amount of rotation specified by g. If λ_1 and λ_2 denote these eigenvalues, then:

$$\begin{aligned} \lambda_1 \, \lambda_2 &= 1 \\ \lambda_1 + \lambda_2 &= 2\cos\theta \end{aligned}$$

Hence, the eigenvalues are complex conjugates: $e^{\pm j\theta}$. The associated eigenvectors, which we shall denote by $\vec{e}_1$ and $\vec{e}_2$, will generally be complex and conjugate. As we showed in class for the case of $SO(3)$, we can define two real vectors:

$$\vec{c}_1 = \frac{1}{2}(\vec{e}_1 + \vec{e}_2) \qquad \vec{c}_2 = \frac{j}{2}(\vec{e}_1 - \vec{e}_2)$$

The action of g on these vectors is equivalent to planar rotation.

Problem 4: (20 Points)

Define a reference frame with $\vec{z}$-axis collinear with Screw axis S_1, and with origin of this frame coincident with the point where S_1 intersects the plane. Let the $\vec{x}$-axis of this reference frame point from S_1 to S_2, and the $\vec{y}$-axis be chosen consistent with the right-hand-rule. In this reference frame, the screw coordinates of S_1 and S_2 take the general form:

$$S_i = \begin{bmatrix} \vec{v}_i \\ \vec{\omega}_i \end{bmatrix} = \begin{bmatrix} \vec{\rho}_i \times \vec{\omega}_i + h_i\vec{\omega}_i \\ \vec{\omega}_i \end{bmatrix} \tag{17}$$

where $\vec{\omega}_i$ is a unit vector collinear with the positive direction of the i^{th} screw axis, $\vec{\phi}_i$ is the vector from the origin of the reference frame to the screw axis (intersecting the screw axis perpendicularly), and h_i is the pitch of the i^{th} screw. By the choice of the reference frame given above, $\vec{\rho}_1 = \vec{0}$, $\vec{\rho}_2 = a\vec{x}$, and $\vec{\omega}_1 = \vec{\omega}_2 = \vec{z}$. Since both S_1 and S_2 have zero pitch, their screw coordinates are:

$$S_1 = \begin{bmatrix} \vec{0} \\ \vec{z} \end{bmatrix} \qquad S_2 = \begin{bmatrix} a\vec{x} \times \vec{z} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} -a\vec{y} \\ \vec{z} \end{bmatrix} . \tag{18}$$

All screw axes lying in the plane can be parametrized by

$$\vec{\rho}_R \;=\; d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad \vec{\omega}_R \;=\; \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{19}$$

So that the general equation for any potentially reciprocal screw lying in the plane takes the form:

$$S_R \;=\; \begin{bmatrix} \vec{v}_R \\ \vec{\omega}_R \end{bmatrix} = \begin{bmatrix} \vec{\rho}_R \times \vec{\omega}_R + h_R\vec{\omega}_R \\ \vec{\omega}_R \end{bmatrix} = \begin{bmatrix} -h_R\sin\theta \\ h_R\cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{20}$$

where h_R is the pitch of the potentially reciprocal screw lying in the plane. To enforce reciprocity between S_1 and S_R,

$$0 \;=\; \vec{\omega}_1 \cdot \vec{v}_R + \vec{v}_1 \cdot \vec{\omega}_R = \vec{z} \cdot \vec{v}_R + \vec{0} \cdot \vec{\omega}_R \;=\; d\ . \tag{21}$$

The requirement that $d = 0$ implies that the screw axis of the reciprocal screw in the plane must intersect S_1. To enforce reciprocity between S_2 and S_R,

$$0 \;=\; \vec{\omega}_2 \cdot \vec{v}_R + \vec{v}_2 \cdot \vec{\omega}_R = -a\sin\theta \tag{22}$$

Hence, $\theta = 0$ or $\theta = \pi$ are required by this condition. This implies that the screw axis of the reciprocal screw also intersects S_2. Hence, the screws lying in P which are reciprocal to both S_1 and S_2 intersect both S_1 and S_2, and can have any pitch.

Problem 5: (15 Points)

This problem asked you to look at the represention and manipulation of spatial displacements using the concept of "dual numbers." A dual number, $\tilde{a}$, takes the form:

$$\tilde{a} = a_r + \epsilon\ a_d$$

where a_r is the "real" part of the dual number and a_d is the "dual" or "pure" part of the dual number. The bases for the dual numbers are 1 and ϵ, and they obey the rules:

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 1 \cdot \epsilon &= \epsilon \cdot 1 = \epsilon \\ \epsilon^2 &= 0 \end{aligned}$$

Part (a): (10 points)

1. $\tilde{g}$ will be orthogonal if $\tilde{g}^T\tilde{g} = I$.

$$\begin{aligned}
\tilde{g}^T\tilde{g} &= [R + \epsilon(\hat{p}R)]^T[R + \epsilon(\hat{p}R)] = [R^T - \epsilon R^T\hat{p}][R + \epsilon\hat{p}R] \\
&= R^TR + \epsilon R^T\hat{p}R - \epsilon R^T\hat{p}R - \epsilon^2 R^T\hat{p}^2R \\
&= I + \epsilon R^T(\hat{p} - \hat{p})R = I
\end{aligned}$$

2. Let $\tilde{g}_1 = [R_1 + \epsilon\hat{p}_1R_1]$ and $\tilde{g}_2 = [R_2 + \epsilon\hat{p}_2R_2]$. Then:

$$\begin{aligned}
\tilde{g}_3 = \tilde{g}_1\tilde{g}_2 &= [R_1 + \epsilon\hat{p}_1R_1][R_2 + \epsilon\hat{p}_2R_2] \\
&= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) + \epsilon^2(\hat{p}_1R_1\hat{p}_2R_2) \\
&= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2)
\end{aligned}$$

   Note that $g_3 = g_1g_2$ is given by:

$$g_3 = \begin{bmatrix} R_1R_2 & R_1\vec{p}_2 - \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix}$$

   Hence, $\tilde{g}_3 = R_1R_2 + \epsilon((\widehat{R_1\vec{p}_2 - \vec{p}_1})R_1R_2)$. Note that:

$$\begin{aligned}
((\widehat{R_1\vec{p}_2 - \vec{p}_1})R_1R_2) &= (R_1\hat{p}_2R_1^T - \hat{p}_1)R_1R_2 \\
&= R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2
\end{aligned}$$

   Hence, the two are equivalent.

3. Let $\tilde{\xi}_1 = \vec{\omega}_1 + \epsilon\vec{V}_1$ and $\tilde{\xi}_2 = \vec{\omega}_2 + \epsilon\vec{V}_2$. Then:

$$\begin{aligned}
\tilde{\xi}_1\tilde{\xi}_2 &= (\vec{\omega}_1 + \epsilon\vec{V}_1) \cdot (\vec{\omega}_2 + \epsilon\vec{V}_2) \\
&= \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega_2} + \vec{\omega}_1 \cdot \vec{V}_2) + \epsilon^2(\vec{V}_1 \cdot \vec{V}_2) \\
&= \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega_2} + \vec{\omega}_1 \cdot \vec{V}_2)
\end{aligned}$$

   The dual part is the reciprocal product.

Part(b): (5 points)

1. First note that if transformation g consists of rotation R and displacement $\vec{p}$, then:

$$Ad_g\xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \vec{V} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} R\vec{V} + \hat{p}R\vec{\omega} \\ R\vec{\omega} \end{bmatrix}$$

   The "dual" version of this vector is $R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega})$. But, $\tilde{g}\tilde{\xi} = R + \epsilon(\hat{p}R)$ and $\tilde{\xi} = \vec{\omega} + \epsilon\vec{V}$. Hence:

$$\begin{aligned}
\tilde{g}\tilde{\xi} &= (R + \epsilon(\hat{p}R))(\vec{\omega} + \epsilon\vec{V}) && (23) \\
&= R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega}) + \epsilon^2(\hat{p}R\vec{V}) && (24) \\
&= R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega}) && (25)
\end{aligned}$$

   Thus, the two are equivalent.

2. Let $\xi_1 = [\vec{V}_1^T \ \vec{\omega}_1^T]$ and $\xi_2 = [\vec{V}_2^T \ \vec{\omega}_2^T]$. Then:

$$\begin{aligned}
\tilde{\xi}_1 \cdot \tilde{\xi}_2 &= (\vec{\omega}_1 + \epsilon \vec{V}_1) \cdot (\vec{\omega}_2 + \epsilon \vec{V}_2) = \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2) + \epsilon^2(\vec{V}_1 \cdot \vec{V}_2) \\
&= \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2)
\end{aligned}$$

The dual part of this, $\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2$, is the reciprocal product of ξ_1 and ξ_2.