ME 115(a): Homework #1

(Due Wednesday, January 15, 2016)

Problem 1: (10 points) Let $\mathcal{F}_1$ denote a fixed reference frame in the plane, with orthonormal basis vectors $\vec{x}_1$ and $\vec{y}_1$. Similarly, consider a second reference frame $\mathcal{F}_2$ with orthonormal basis vectors $\vec{x}_2$ and $\vec{y}_2$. Let $d_{12} = \begin{bmatrix} x & y \end{bmatrix}^T$ be the vector pointing from the origin of $\mathcal{F}_1$ to the origin of $\mathcal{F}_2$. Let θ_{12} denote the relative orientation of the two reference frames: θ_{12} is the angle between $\vec{x}_1$ and $\vec{x}_2$. Let ${}^2\vec{v} = \begin{bmatrix} {}^2v_x & {}^2v_y \end{bmatrix}^T$ denote the coordinates of a point, P, as seen by an observer in $\mathcal{F}_2$. In class we developed a formula for the coordinate transformation of P to its representation in $\mathcal{F}_1$:

$$ {}^1\vec{v} = \vec{d}_{12} + R(\theta_{12})\ {}^2\vec{v} \tag{1} $$

where $R(\theta_{12})$ is the 2×2 rotation matrix:

$$ R(\theta_{12}) = \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{bmatrix} $$

For computational purposes, it is sometimes convenient to use different representations of coordinates, vectors, and rotations. For example, consider complex numbers such that if $\vec{w}$ is a 2×1 vector $\vec{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ in the plane, then $\tilde{w} = w_1 + iw_2$ where i is the complex number such that $i \cdot i = -1$. Show that if ${}^2\tilde{v}$ is the complex representation of ${}^2\vec{v}$, then the complex representation of the coordinate transform in Equation (1) is:

$$ {}^1\tilde{v} = \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v} $$

Problem 2: (10 points) Every planar rigid body displacement is *equivalent* to a rotation about a unique point in the plane, known as the pole (see Figure 1).

Let A be a fixed reference frame. A rigid body, L, which has local frame B attached to it, is located relative to reference frame A by $D_1 = (\vec{d}_{01}, R_{01},)$. Body L moves to position C, where the displacement to location C, as measured by an observer in frame B, is given by $D_2 = (\vec{d}_{12}, R_{12})$. Where is the *pole* of the body displacement from position B to position C, as a function of R_{01}, R_{12}, $\vec{d}_{01}$, and $\vec{d}_{12}$?

a. As measured in Frame A

b. As measured in Frame B

c. As measured in Frame C

Problem 3: (5 points) In the above problem, suppose $D_1 = (x, y, \theta) = (1.0, 2.0, 30.0^o)$ and $D_2 = (x, y, \theta) = (2.0, 2.0, 45^o)$. Where is the pole of the displacement from B to C in this case?

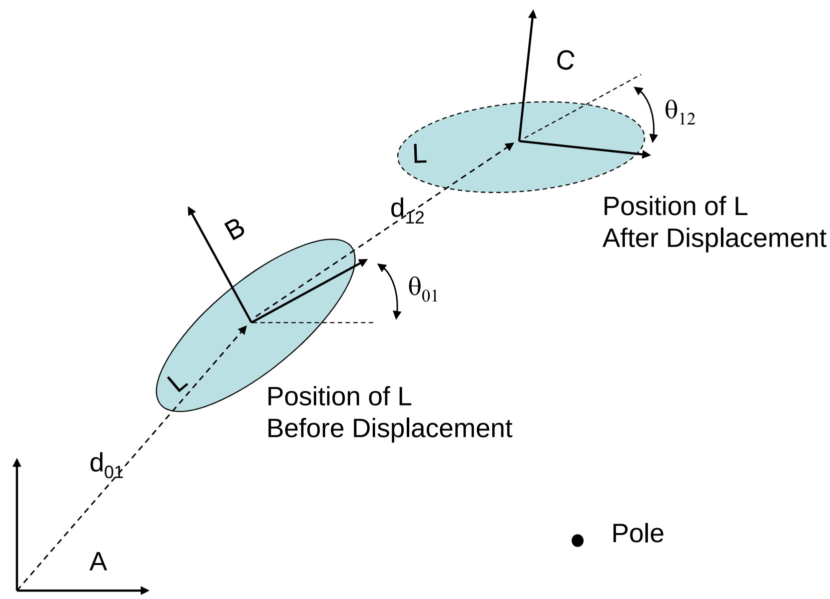

Figure 1: Geometry of planar displacement

Problem 4: (15 points) Using the set up of Problem 2, pick a coordinate system whose origin is located at the pole of the displacement, and show that in this coordinate system, the displacement of the body from B to C is a pure rotation.

Problem 5: (15 points) Consider again the *Elliptical Trammel* that was analyzed in class, and whose diagram is repeated in Figure **??**. As in the handout on the Elliptical Trammel, let **A** and **B** denote the points on the moving rigid body that coincide with the revolute joint axes, and let **C** denote that point on the moving body that traces a path. Define the following distances:

$$a = |\mathbf{AC}| \qquad b = |\mathbf{BC}| \qquad c = |\mathbf{AB}| \ . \tag{2}$$

In class we showed that the *fixed centrode* (the locus of poles, as seen by a fixed observer) was a circle of diameter c. Show that the *moving centrode* (the local of the poles as seen by an observer positioned on the moving bar) is a circle of diameter $c/2$.

Problem 6: (10 points) In class we used the particle nature of rigid bodies to "prove" that a planar rigid body has three degrees of freedom. Use the same idea to spherical motion has three degrees of freedom. (Hint: use the definition of a rigid body as a set of particles, and then calculate the total net degrees of freedom of the rigid body based on the particles and their constraints).