ME 115(a): Solution to Homework #1
(Winter 2016)

Problem 1: Let the 2×1 vectors ${}^1\vec{v} = \begin{bmatrix} {}^1v_1 & {}^1v_2 \end{bmatrix}^T$ and ${}^2\vec{v} = \begin{bmatrix} {}^2v_1 & {}^2v_2 \end{bmatrix}^T$ have associated complex representations ${}^1\tilde{v} = {}^1v_1 + i\,{}^1v_2$ and ${}^2\tilde{v} = {}^2v_1 + i\,{}^2v_2$ respectively (where $i^2 = -1$). Recall that the goal of this problem is to show that the complex number formula:

$$ {}^1\tilde{v} = \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v}\ . \tag{1} $$

is equivalent to the planar coordinate transformation:

$$ {}^1\vec{v} = \vec{d}_{12} + R(\theta_{12})\ {}^2\vec{v}\ . \tag{2} $$

Let's evaluate the right hand side of expression (1) using the standard rules for multiplication of complex numbers[1]:

$$ \begin{aligned} \tilde{d}_{12} + e^{i\theta_{12}}\ {}^2\tilde{v} &= (x+iy) + (\cos\theta_{12} + i\sin\theta_{12})({}^2v_1 \ + \ i\ {}^2v_2) \\ &= (x + \ {}^2v_1\cos\theta_{12} \ - \ {}^2v_2\sin\theta_{12}) + i(y + \ {}^2v_1\sin\theta_{12} \ + \ {}^2v_2\cos\theta_{12}) \end{aligned} \tag{3} $$

where we have used Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$). Matching the real and complex portions of Equation (3) with the real and complex parts of ${}^1\tilde{v}$ in the left hand side of Equation (1), we see that

$$ {}^1v_1 = x +^2 v_1\cos\theta -^2 v_2\sin\theta \tag{4} $$
$$ {}^1v_2 = y +^2 v_1\sin\theta +^2 v_2\cos\theta\ . \tag{5} $$

These equations are equivalent to

$$ {}^1\vec{v} = \vec{d}_{12} + \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{bmatrix}\ {}^2\vec{v} \tag{6} $$

Problem 2: Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^B\vec{p}$ as seen by an observer in the fixed B frame. After displacement, the observer in the body fixed C frame also sees the pole in his/her coordinates at point ${}^B\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coodinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$$ {}^B\vec{p} = \vec{d}_{12} + R_{12}\ \ {}^B\vec{p}. $$

This equation can be solved to find the pole location:

$$ {}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} $$

[1] If $\tilde{a} = a_1 + ia_2$ and $\tilde{b} = b_1 + ib_2$, then $\tilde{a}\tilde{b} = (a_1b_2 - a_2b_2) + i(a_1b_2 + a_2b_1)$.

Of course, you need to show the fact that $(I - R_{12})$ is invertible. It will always be invertible, except when $R_{12} = I$. In this case, the motion is a pure translation, and the pole is the "pole at infinity."

B) In Frame B, the pole is: ${}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$

C) In Frame C, the vector describing the pole has exactly the same value as seen by the observer in Frame B: ${}^C\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$

A) In Frame A, the expression for the pole vector is obtained by a simple coordinate transformation of the expression in Frame B: ${}^A\vec{p} = d_{01} + R_{01}\ {}^B\vec{p} = d_{01} + R_{01}(I - R_{12})^{-1}\vec{d}_{12}$

Problem 3: To find the pole of the displacement, $D_2 = (x, y, \theta) = (2.0, 3.0, 60.0^o)$, substitute into the above results:

$${}^B\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(60^o) & -\sin(60^o) \\ \sin(60^o) & \cos(60^o) \end{pmatrix}\right]^{-1} \begin{bmatrix} 2.0 \\ 3.0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3\sqrt{3}}{2} \\ \sqrt{3} + \frac{3}{2} \end{bmatrix}$$

You could report this result in Frame B, or transform the results to frame A.

Problem 4: To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by D, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame B. The location of Frame B relative to Frame D is a pure translation of amount $-^1\vec{p}$, and therefore, $D_{DB} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame D, is obtained by a similarity transform $D_{DB}D_{12}D_{DB}^{-1}$:

$$\begin{aligned} D_{DB}D_{12}D_{DB}^{-1} &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} &(7)\\ &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) &(8)\\ &= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) &(9)\\ &= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) &(10) \end{aligned}$$

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{DB}D_{12}D_{DB}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame D, the displacement is a pure rotation by amount R_{12}.

Problem 5: To find the geometry of the moving centrode of the elliptical trammel, place a body fixed reference frame on the moving link so that its origin lies at the mid-point of Points **A** and **B**, and its x-axis point in the direction from point **A** to point **B**. In Figure 1(a) the basis vectors of this moving reference frame are denoted $(\vec{x}_b, \vec{y}_b)$. Let a fixed reference frame (with basis vectors $(\vec{x}_f, \vec{y}_f)$) be placed a the intersection of the two sliding joints.

To solve this problem, one must compute the location of the centrode as seen by an observer

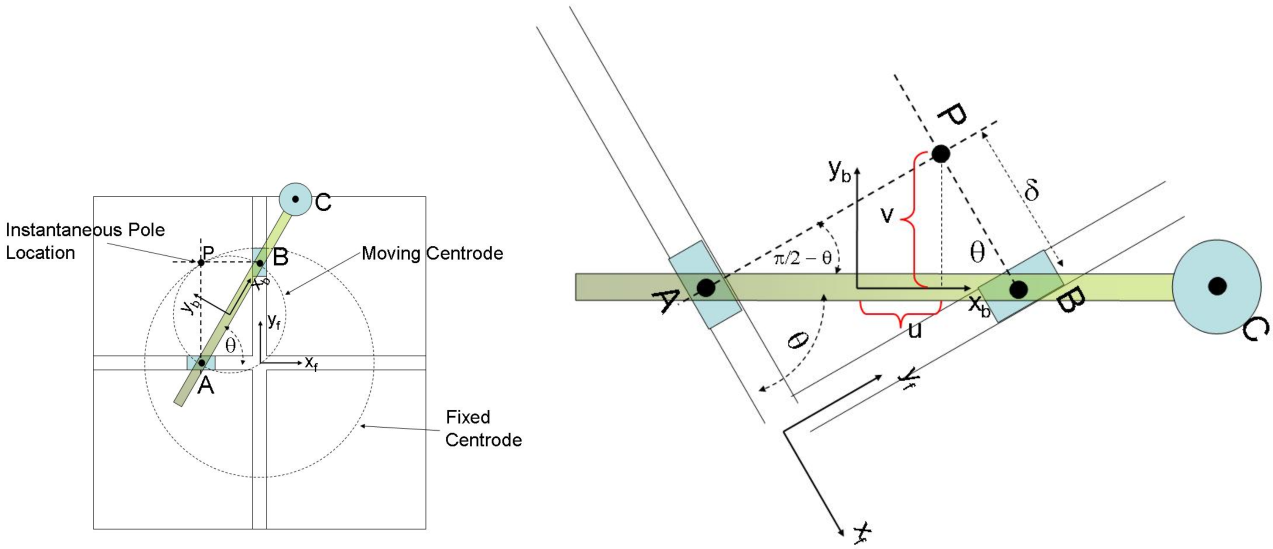

Figure 1: **(a):** Diagram of the Elliptical Trammel. **(b):** Expanded and rotated view of (a), showing the geometry of pole location in the moving coordinate system.

in the moving frame. Let $a = |\mathbf{AB}|$. Let θ denote the angle between the body-fixed x-axis and the x-axis of the fixed reference frame. This angle also defines the angles of the right-handed triangle **ABP**. Using the geometry of Figure 1(b), it can be seen that

$$\delta = |\mathbf{BP}| = a \sin\left(\frac{\pi}{2} - \theta\right) = a \cos\theta \ .$$

Similarly, from this diagram we can deduce that the x-coordinate of the centrode, denoted u, is given by:

$$u = \frac{a}{2} - \delta\cos\theta = \frac{a}{2} - a\cos^2\theta \ .$$

Likewise, the y-coordinate of the centrode in the moving frame, denoted v, is simply:

$$v = \delta\sin\theta = a \ \cos\theta \ \sin\theta \ .$$

Thus, in the moving reference frame:

$$\begin{aligned} u^2 + v^2 &= (a\cos\theta\sin\theta)^2 + \left(\frac{a}{2} - a\cos^2\theta\right)^2 \\ &= a^2(\cos^2\theta\sin^2\theta + \frac{1}{4} + \cos^4\theta - \cos^2\theta) \\ &= a^2(\frac{1}{4} + \cos^2\theta(\sin^2\theta + \cos^2\theta - 1)) \\ &= \left(\frac{a}{2}\right)^2 \end{aligned}$$

Thus, the moving centrode (the set of pole locations in the moving reference frame) is a circle with radius $\frac{a}{2}$ centered at the midpoint of $\vec{\mathbf{AB}}$.

Problem 6: You were to "prove" that a body undergoing spherical motion has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of N particles. Let P_1 denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since P_1 does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle P_2 in the body. Particle P_2 has 3 DOF as a particle. However, it is constrained to lie a fixed distance, d_{12} from particle P_1 due to the fact that P_1 and P_2 are part of the same rigid body. The fixed distance relationship imposes one constraint on P_2. Next consider a point P_3, which lie a fixed distance from P_1 and P_2. Therefore, there are two constraints on its location. Now, consider a particle P_4. Since its must lie a fixed distance from P_1, P_2, and P_3, there are three constraints on its motion. Particles P_5, ..., P_N similarly have 3 constraints.

The total number of degrees of freedom of the N particles are: $3(N-1)+0 = 3N-3$. The total number of constraints on these particles are: $1+2+3(N-3) = 3N-6$. Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: $(3N-3)-(3N-6) = 3$.