ME 115(b): Homework #1 Solution

Problem 1: (Singularities of an RPR Manipulator).

Using either the Denavit-Hartenberg or Product of Exponentials approach, you should have derived the forward kinematics function which describes the location of the tool frame origin, $\vec{P}$ as:

$$\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} (d_2 + L\cos\theta_3)\cos\theta_1 \\ (d_2 + L\cos\theta_3)\sin\theta_1 \\ d_1 - L\sin\theta_3 \end{bmatrix} \tag{1}$$

Part (a): As discussed in class, the hybrid Jacobian matrix can be calculated as:

$$J^h_{ST} = \begin{bmatrix} \frac{\partial \vec{P}}{\partial \theta_1} & \frac{\partial \vec{P}}{\partial d_2} & \frac{\partial \vec{P}}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -(d_2 + L\cos\theta_3)\sin\theta_1 & \cos\theta_1 & -L\cos\theta_1\sin\theta_3 \\ (d_2 + L\cos\theta_3)\cos\theta_1 & \sin\theta_1 & -L\sin\theta_1\sin\theta_3 \\ 0 & 0 & -L\cos\theta_3 \end{bmatrix} \tag{2}$$

Part (b): The singularities can be found by taking the determinant of the hybrid Jacobian matrix and setting it to zero:

$$det(J^h_{ST}) = -L\cos\theta_3(d_2 + L\cos\theta_3) = 0 \tag{3}$$

Hence, one set of singularities will occur when $\cos\theta_3 = 0$, which is equivalent to $\theta_3 = \pm\frac{\pi}{2}$. In these configurations, the third link is pointing straight up and down, and the manipulator cannot making any verticle motions. Another set of singularities occurs when $d_2 + L\cos\theta_3 = 0$. In these configurations, the tool frame origin lies along the first joint axis, and the manipulator cannot make any motions in the x-y plane in the direction orthogonal to the plane passing through the first two joint axes.

Problem 2: (Special Configurations of a planar linkages).

Part (a): Let's establish a coordinate system with origin at joint A, with the x-axis colinear with $\vec{AB}$ and the y-axis normal to the x-axis. The z-axis points out of the plane of the mechanism, forming a right handed coordinate system.

Part(b): While the device is planar, for convenience let's express the twist coordinates for each joint in the full 3-D space. Let joint axis 1 correspond to the revolute joint at A, joint axis 2 be the revolute joint at C, joint axis 3 be the prismatic joint, and joint axis 4 be the revolute joint at B. Let θ_1 denote the angle between $\vec{AB}$ and $\vec{AC}$. Similarly, let θ_4 denote the angle between the x-direction and the line $\vec{BC}$.

The twist coordinates for revolute joints #1 and #2 are straightforward:

$$\begin{aligned} \xi_1 &= [0\ 0\ 0\ 0\ 0\ 1]^T \\ \xi_2 &= [(l_1 s_1)\ \ -(l_1 c_1)\ 0\ 0\ 0\ 1]^T \end{aligned} \tag{4}$$

where s_1 and c_1 are respectively shorthand for $\sin\theta_1$, and l_1 is the dimension $|AC|$. The third joint is prismatic, and thus an infinite pitch joint. Hence, its twist is:

$$\xi_3 = [c_4 \; s_4 \; 0 \; 0 \; 0 \; 0]^T \tag{5}$$

The fourth joint is again a revolute joint. Its screw coordinates are simply:

$$\xi_4 = [-0 \; -1 \; 0 \; 0 \; 0 \; 1]^T \tag{6}$$

Part (b):

A stationary configuration of joint 4 (the joint at point B) will occur when:

$$det \begin{bmatrix} \xi_1 \cdot \xi_1 & \xi_1 \cdot \xi_2 & \xi_1 \cdot \xi_3 \\ \xi_2 \cdot \xi_1 & \xi_2 \cdot \xi_2 & \xi_2 \cdot \xi_3 \\ \xi_3 \cdot \xi_1 & \xi_3 \cdot \xi_2 & \xi_3 \cdot \xi_3 \end{bmatrix} = 0 \; . \tag{7}$$

For this linkage:

$$\begin{array}{ll} \xi_1 \cdot \xi_1 & = \xi_1 \cdot \xi_2 \;=\; \xi_2 \cdot \xi_1 \;=\; 1 \\ \xi_1 \cdot \xi_3 & = \xi_3 \cdot \xi_1 = 0 \\ \xi_2 \cdot \xi_2 & = l_1^2 \\ \xi_2 \cdot \xi_3 & = \xi_3 \cdot \xi_2 = l_1 \sin(\theta_1 + \theta_4) \\ \xi_3 \cdot \xi_3 & = 1 \end{array} \; . \tag{8}$$

Substituting Equation (8) into Equation (7) and taking the determinant:

$$det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 + l_1^2 & l_1 \sin(\theta_1 + \theta_4) \\ 0 & l_1 \sin(\theta_1 + \theta_4) & 1 \end{bmatrix} = l_1^2(1 - \sin^2(\theta_1 + \theta_4)) = l_1^2 \cos^2(\theta_1 + \theta_4) \tag{9}$$

Discarding the trivial case where $l_1 = 0$, joint B will have a special configuration when $(\cos\theta_1 + \cos\theta_4) = 0$. For this to occur, lne $\vec{AC}$ must intersect line $\vec{BC}$ orthogonally. This can only occur if $l_1 \leq |AB| = 1$

Part (c): The prismatic joint will have two special configurations at the extremes of its travel along $\vec{BD}$.

Problem 2: Problem 21(a,b,c), Chapter 3 of MLS.

Part (a): We can think of this mechanism as consisting of 8 bodies (noticing that each prismatic joint can be thought of as two bodies). The total number of *relative* degrees of freedom possed by this collection of bodies is 3(8-1) = 21. This collection of bodies is held together by 6 revolute joints and 3 prismatic joints. Thus, the number of constraints is $9 \cdot 2 = 18$. Therefore, the mechanism as 21-18=3 internal degrees of freedom.

Part (b): The structure equations can be derived either using the **P**roduct **O**f **E**xponentials (POE) method, or the Denavit-Hartenberg (DH) convention. Let's consider the POE approach. In this approach, the structure equations take the form:

$$e^{\xi_{11}\alpha_1} e^{\xi_{12} d_1} e^{\xi_{13}\beta_1} g_{bt}(0) = e^{\xi_{21}\alpha_2} e^{\xi_{22} d_2} e^{\xi_{23}\beta_2} g_{bt}(0) = e^{\xi_{31}\alpha_3} e^{\xi_{32} d_3} e^{\xi_{33}\beta_3} g_{bt}(0)$$

Using the obvious "home position," (the one showed in the left hand diagram of the figure which goes along with problem 3.21 in the MLS text) the twists are:

$$\begin{aligned}
\xi_{11} &= \begin{bmatrix} h/2 \\ 0 \\ 1 \end{bmatrix}; \qquad \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad \xi_{13} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{21} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \qquad \xi_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad \xi_{23} = \begin{bmatrix} -h/2 \\ -w \\ 1 \end{bmatrix} \\
\xi_{31} &= \begin{bmatrix} -h/2 \\ 0 \\ 1 \end{bmatrix}; \qquad \xi_{32} = \frac{1}{\sqrt{w^2+b^2}} \begin{bmatrix} w \\ h \\ 0 \end{bmatrix}; \qquad \xi_{33} = \begin{bmatrix} h/2 \\ -w \\ 1 \end{bmatrix}
\end{aligned} \tag{10}$$

where α_j is the angle made between the axis of the j^{st} prismatic joint axis and the x-axis of the stationary, or base, frame. I.e., this is the angle made the first passive revolute joint in the j^{th} serial chain.

Note that the angles α_i and β_i can be found using the law of cosines. For example:

$$\cos\alpha_1 = \frac{h^2 + d_1^2 - d_3^2}{2hd_1}$$

Part (c): Using the notation in the figure above, we can see that the location of the points $P_1, \cdots, P_4$ are:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \qquad P_2 = \begin{bmatrix} 0 \\ -h/2 \end{bmatrix}; \\
P_3 &= \begin{bmatrix} x \\ y \end{bmatrix}; \qquad +R_\phi \begin{bmatrix} 0 \\ h/2 \end{bmatrix}; \qquad P_4 = \begin{bmatrix} x \\ y \end{bmatrix}; \qquad +R_\phi \begin{bmatrix} 0 \\ -h/2 \end{bmatrix};
\end{aligned} \tag{11}$$

The actuator lengths can be derived as:

$$\begin{aligned}
d_1 &= ||P_3 - P_1|| \\
d_2 &= ||P_4 - P_2|| \\
d_3 &= ||P_3 - P_2||
\end{aligned} \tag{12}$$